Lecture 7 material

- **Required reading:**
  - Elton et al., Chapter 16

- **Supplementary reading:**
  - Luenberger, Chapter 13
  - Alexander et al., Chapter 12
Lecture 7: Checklist

- By the end of this lecture you should:
  - Understand the derivation of the APT
  - Understand the differences between the CAPM and the APT
  - Be able to establish the existence of an arbitrage opportunity in the APT framework
  - Be familiar with the different ways in which the APT can be tested

Introduction (1)

- The CAPM is derived from assumptions about investors’ utility functions
- Utility is assumed to be a function of expected return and return variance alone
- Securities are evaluated in terms of their marginal contribution to the market portfolio
- This is determined by their non-diversifiable risk (which is measured by beta), not their total risk
- This implies that actual returns are generated by a single systematic factor - the market return
Introduction (2)

- The Arbitrage Pricing Theory (APT) starts by assuming that actual returns are generated by a number of systematic factors.
- A security’s risk is measured by its sensitivity to each of these factors.
- From this, we can derive an equilibrium relationship between expected return and risk.
- The APT and CAPM may have a similar features, but they have very different foundations.

APT: A heuristic derivation (1)

- Suppose that the following two-factor model describes actual returns:
  $$r_{it} = \alpha_i + \beta_{1i}F_{1t} + \beta_{2i}F_{2t} + \epsilon_{it}$$
- If an investor holds a well-diversified portfolio, residual risk will be eliminated and the only source of risk will be the systematic component of a stock’s risk, which is determined by its sensitivity coefficients $\beta_{1i}$ and $\beta_{2i}$.
- Thus, the only characteristics of a portfolio or asset that an investor need to consider are $E(r_i)$, $\beta_{1i}$, and $\beta_{2i}$.
APT: A heuristic derivation (2)

- Suppose we observe the following three widely diversified portfolios

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$E(r_i)$</th>
<th>$\beta_{1i}$</th>
<th>$\beta_{2i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>15</td>
<td>1.0</td>
<td>0.6</td>
</tr>
<tr>
<td>B</td>
<td>14</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>0.3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

- We start by asserting that these portfolios, in equilibrium, must lie on a plane in $E(r_i) \times \beta_{1i} \times \beta_{2i}$ space.

- The general formula for such a plane would be

$$E(r_i) = \lambda_0 + \lambda_1 \beta_{1i} + \lambda_2 \beta_{2i}$$

APT: A heuristic derivation (3)

- We have three points on this plane (the portfolios A, B and C) and so we can easily deduce the formula for the plane to be

$$E(r_i) = 7.75 + 5.00 \cdot \beta_{1i} + 3.75 \cdot \beta_{2i}$$

(Check yourself!)

- This is the equation of the APT in this two-factor world.
APT: A heuristic derivation (4)

- Why, in equilibrium, must the portfolios lie on a plane? The answer is that if they did not, there would be an arbitrage opportunity.

- This arbitrage opportunity would be exploited by investors which would ultimately ensure that all portfolios lie on the plane.

APT: A heuristic derivation (5)

- Suppose for example we observed the following portfolio:

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>E(r_i)</th>
<th>β_{1i}</th>
<th>β_{2i}</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>15</td>
<td>0.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

- By combining portfolios A, B and C, we could construct another portfolio, E, that had the same β_{1i} and β_{2i} as portfolio D, but a lower expected return.

- Portfolio E, in this case, is given by

\[ \frac{1}{3} \cdot A + \frac{1}{3} \cdot B + \frac{1}{3} \cdot C \]
APT: A heuristic derivation (6)

- By the law of one price, two portfolios that have the same risk (measured by $\beta_{1i}$ and $\beta_{2i}$) must have the same expected return (or equivalently the same price).
- In this situation, there will be excess demand from arbitrageurs for portfolio D.
- This will push its price up and its expected return down, until it lies on the plane.

APT: A more rigorous derivation (1)

Assume that actual returns are generated by a multi-factor model:

$$r_i = \alpha_i + \beta_{1i}F_1 + \ldots + \beta_{Ki}F_K + \epsilon_i$$

where

- $r_i$ is the actual return on security $I$.
- $F_k$ is the $k$-th zero-mean factor that influences $r_i$.
- $\beta_{ki}$ is the sensitivity of security $i$ to the $k$-th factor.
- $\epsilon_i$ is a zero-mean term that is uncorrelated across securities.
- $\alpha_i$ is the expected return on the stock when all factors take the value zero.
APT: A more rigorous derivation (2)

- The expected return on security \( i \) can be found by taking expectations of the equation for actual returns

\[
E(r_i) = \alpha_i + \beta_{i1} E(F_1) + \ldots + \beta_{ik} E(F_k)
\]

- The expected return on a security is equal to some constant plus the expected value of each of \( K \) different factors times the sensitivity of the security’s return to those factors

APT: A more rigorous derivation (3)

- Subtracting from the equation for actual returns yields the following expression

\[
r_i = E(r_i) + \beta_{i1} [F_1 - E(F_1)] + \ldots + \beta_{ik} [F_k - E(F_k)] + \epsilon_i
\]

- The actual return on a security is equal to the expected return, plus the weighted sum of \( K \) unanticipated factors (the weights being equal to the appropriate sensitivities), plus a purely random component

- This relationship holds by definition
APT: A more rigorous derivation (4)

- Suppose that there are a sufficient number of securities, \( i = 1, \ldots, N \), that we can construct a portfolio \( P \), with proportions \( w_i \) in each security \( i \), that has the following properties
  
  i. Zero net investment \( \sum_{i=1}^{N} w_i = 0 \)
  
  ii. Zero systematic risk \( \sum_{i=1}^{N} w_i \beta_{ki} = 0 \) for all \( k \)
  
  iii. Zero non-systematic risk \( \sum_{i=1}^{N} w_i \epsilon_i = 0 \)

- This is called an arbitrage portfolio

APT: A more rigorous derivation (5)

- It requires no investment (since the portfolio weights sum to zero and hence long positions are financed by short positions), it has no systematic risk, and it has no idiosyncratic risk

- Consequently, by the law of one price or the no-arbitrage condition, this portfolio must earn an expected return of zero

\[
\sum_{i=1}^{N} w_i E(r_i) = 0
\]
APT: A more rigorous derivation (6)

- Condition (i) implies that the vector of portfolio weights is orthogonal (i.e. the product is zero, or equivalently they are not linearly related) to a vector of ones
- Condition (ii) implies that the vector of portfolio weights is also orthogonal to each of the $K$ vectors of betas
- The result of the no-arbitrage condition implies that the vector of portfolio weights is orthogonal to the vector of expected returns

APT: A more rigorous derivation (7)

- The no-arbitrage condition can therefore be restated in the following way: If a portfolio is constructed such that its weights are orthogonal both to a vector of ones and each of the $K$ vectors of betas, then the portfolio weights must also be orthogonal to the vector of expected returns
- In order for this result to hold, it must be the case that the vector of expected returns is spanned by the unit vector and the $K$ vectors of betas
APT: A more rigorous derivation (8)

• In other words the vector of expected returns is a linear combination of the vector of ones and each of the $K$ vectors of betas
• This gives the following equilibrium condition
  $$r_i = \lambda_0 + \lambda_1 \beta_{i1} + \ldots + \lambda_K \beta_{iK}$$
• It gives the equilibrium expected return on a security as a linear function of its sensitivity to each of the $K$ factors that determine its actual return
• This is known as the Arbitrage Pricing Theory (APT)
• In equilibrium, this relationship must hold for all securities and portfolios of securities

APT: A more rigorous derivation (9)

• Each of the coefficients $\lambda_k$ can be interpreted as the market price of risk of factor $k$, with $\lambda_0$ being the expected return on a security that had zero sensitivity to all $K$ factors, or in other words the risk free rate
• As in the CAPM, the expected return of a security is a function of its systematic risk, not its idiosyncratic risk
• This is because in a sufficiently large portfolio, idiosyncratic risk can be completely diversified away, and so there is no compensation for bearing idiosyncratic risk
• The CAPM can be derived as a special case of the APT by assuming that the only systematic factor that generates returns is the market return
The CAPM vs. the APT

- CAPM derived from utility maximisation argument
- APT derived from profit maximisation argument
- CAPM states that all economy-wide factors that affect actual returns can be condensed into a single factor, namely the market return
- APT states that there may be a number of factors that have different effects on different stocks
- CAPM requires us to specify the market portfolio
- APT does not tell us how many factors there are, nor what they are

Estimating and testing the APT (1)

- APT assumes a model of actual returns
  \[ r_i = \alpha_i + \beta_{i1}F_1 + \ldots + \beta_{ik}F_k + \varepsilon_i \]  
  (1)
- It gives a model of expected returns
  \[ E(r_i) = \lambda_0 + \lambda_i\beta_{i1} + \ldots + \lambda_k\beta_{ik} \]
  (2)
where
- \( r_i \) is the actual return on security \( i \)
- \( F_k \) is the \( k \)-th zero-mean factor that influences \( r_i \)
- \( \beta_{ki} \) is the sensitivity of security \( i \) to the \( k \)-th factor
- \( \varepsilon_i \) is a zero-mean term that is uncorrelated across securities
- \( \alpha_i \) is the expected return on the stock when all factors take the value zero
- \( \lambda_k \) is the extra expected return required because of a security’s sensitivity to the \( k \)-th factor
Estimating and testing the APT (2)

- We are interested in testing (2), but this requires data on $\beta_{ki}$, which in turn requires that we estimate (1)
- However, the APT gives no indication of what the factors $F_k$ are
- There are three possible approaches:
  i. Simultaneously estimate $F_k$ and $\beta_{ki}$
  ii. Arbitrarily specify $F_k$ and estimate $\beta_{ki}$ and $\lambda_k$
  iii. Arbitrarily specify $\beta_{ki}$ and estimate $\lambda_k$

Simultaneous estimation of $F_k$ and $\beta_{ki}$ (1)

- A statistical technique called factor analysis allows us to simultaneously estimate $F_k$ and $\beta_{ki}$ in (1)
- Assume that each factor, $F_k$, can be represented by a portfolio of securities that have high sensitivity to that factor and low sensitivity to all other factors
- Pre-specify the number of factors, e.g. two
- Factor analysis then finds the two portfolios, $F_k$, and corresponding sensitivities, $\beta_{ki}$, which best explain the covariance of $r_i$ in (1) over time, i.e. which minimises $\text{cov}(\varepsilon_i, \varepsilon_j)$
Simultaneous estimation of $F_k$ and $\beta_{ki}$ (2)

- Increase the number of factors by one and re-estimate $F_k$ and $\beta_{ki}$
- Continue increasing the number of factors until the last factor offers no significant improvement in explanatory power
- Estimate the cross-section regression (2) using the sensitivity coefficients from factor analysis and see how many factors are significant

Simultaneous estimation of $F_k$ and $\beta_{ki}$ (3)

- Roll and Ross (1980) find that five factors are generally sufficient for explaining the covariance of daily returns 1962-72 in 42 groups of 30 stocks
- They find that only three of the five estimated sensitivities are significant in explaining the cross-section of security returns
- The estimated risk premia, $\lambda_k$, tend to be similar for different groups of securities
Simultaneous estimation of $F_k$ and $\beta_{ki}$ (4)

- Residual variance tends to be insignificant in explaining the cross-section of security returns
- Factor analysis generally finds that APT explains and predicts returns better than the CAPM
- Problems with factor analysis:
  i. Can only be used for a small number of securities
  ii. No economic interpretation of factors or sensitivities

Arbitrary specification of the factors $F_k$ (1)

- We could alternatively specify the factors
- First pass regression to estimate $\beta_{ki}$ in (1)
- Second pass regression to estimate $\lambda_k$ in (2) and test APT
- Chen, Roll and Ross (1986) consider the following four factors: inflation, term structure, risk premium, industrial production
- $\beta_{ki}$ for all factors are significant in explaining returns and have ‘correct’ signs
- CAPM beta insignificant in multivariate regression
Arbitrary specification of the factors $F_k$ (2)

- Burmeister and McElroy (1988) consider the following four factors:
  - default risk
  - time premium
  - deflation
  - change in expected sales

- All factors significant in explaining time series of returns, and all sensitivities significant in explaining cross-section of returns

Arbitrary specification of the factor sensitivities $\beta_{ki}$

- Finally, we could consider firm specific characteristics as proxies for the factor sensitivities
- Sharpe (1982) considers the following measures of factor sensitivity:
  - CAPM beta with securities
  - CAPM beta with bonds
  - CAPM alpha
  - Dividend yield
  - Firm size
  - Industry dummies

- All characteristics were found to be statistically significant