Span. Econ. Rev. 2, 65–71 (2000)

Spanish
Economic
Review


# The possibility of mixed-strategy equilibria with constant-returns-to-scale technology under Bertrand competition

**Todd R. Kaplan, David Wettstein**

Department of Economics, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva, Israel 84105 (e-mail: wettstn@bgumail.bgu.ac.il)

**Abstract.** We analyze the Nash equilibria of a standard Bertrand model. We show that in addition to the marginal-cost pricing equilibrium there is a possibility for mixed-strategy equilibria yielding positive profit levels. We characterize these equilibria and find that having unbounded revenues is the necessary and sufficient condition for their existence. Hence, we demonstrate that under realistic assumptions the only equilibrium is marginal-cost pricing.

**JEL classification:** C72, L13

**Key words:** Bertrand, price competition

## 1 Introduction

In 1883, Bertrand wrote a review of Cournot's book (1838). In the review, he criticized Cournot's model of competition by offering another model where firms compete with prices rather than in quantities. The use of prices in the Bertrand model leads to undercutting which results in marginal-cost pricing where each of the two identical firms earns zero profits.[1] In these works, the possibility of mixed-strategies was not considered. We will show that there is indeed a possibility of mixed-strategy equilibria, but only in a narrow class of the standard Bertrand environment.

Mixed-strategy equilibria have been previously examined in other Bertrand settings. Spulber (1995) obtained Bertrand equilibria in which firms choose prices above marginal costs when faced with uncertainty about rivals' costs. Sharkey

We wish to thank two referees and a co-editor for valuable suggestions that have significantly improved this paper. We also gratefully acknowledge the support from the Kreitman Foundation and the Monaster Center for Economic Research. This paper is a revised version of Kaplan and Wettstein [1997].

[1] See Shapiro (1989) for an exellent exposition of the two competing models.

and Sibley (1993) find mixed-strategies in Bertrand games with fixed costs, while Marquez (1997) finds such equilibria with asymmetric fixed costs.

In contrast to the natural occurrence of mixed-strategy equilibria in the aforementioned works, the emergence of mixed-strategy equilibria in our environment of complete information, no fixed costs, and constant marginal costs occurs only under the restrictive condition of unbounded revenue. In a paper written concurrently, Baye and Morgan (1997) examine a more general environment and find sufficient conditions for the existence and uniqueness of the Bertrand paradox equilibrium (firms make zero profits). These conditions, which differ from our conditions, are that the profit (of a firm without competition as a function of price) must be bounded, lower semi-continuous and have a break-even price where all prices below this price do not yield profit. While our environment is more specific (causing the Bertrand paradox equilibrium to always exist), we are able to obtain a necessary and sufficient (as opposed to only sufficient) condition that eliminates the possibility of a mixed-strategy equilibrium.

In our environment, which is the typical Bertrand model, there are two firms that produce a homogeneous good at marginal cost $c$ facing a decreasing demand curve $q = D(p)$ and there exists positive demand at some price exceeding costs. They simultaneously announce prices $p^1$ and $p^2$. The firm announcing the lowest price fulfills the entire demand. If the two firms announce the same price, the demand is evenly split between them.

The pure-strategy equilibria of this model is marginal-cost pricing $p^1 = p^2 = c$. Neither firm has incentive to deviate from this price since a firm raising its price will lose all sales and a firm lowering its price will lose money. To see this is the only pure-strategy equilibrium consider the following cases: $p^1 = c$ and $p^2 > c$, $p^2 > p^1 > c$, and $p^1 = p^2 > c$ (without loss of generality we assume $p^2 \geq p^1$). In the first two cases, firm 1 could increase its profits by raising its price. In the third case, either firm can increase its profits by undercutting its rival. Therefore, there is a unique pure-strategy equilibrium where firms earn zero profits.

## 2 A mixed strategy equilibrium: An example

There may exist a mixed-strategy equilibrium of this model where prices announced always exceed marginal cost. Assume demand is given by $D(p) = p^{-\eta}$ where $0 < \eta < 1$. Assume the firms choose prices according to the distribution function $F(p) = 1 - \frac{m^{-\eta}(m-c)}{p^{-\eta}(p-c)}$ for $p \geq m$, $F(p) = 0$ otherwise, and $m > c$. To show that this is an equilibrium we note that for either firm the expected payoff for submitting any price $p \geq m$ is $p^{-\eta}(p-c)(1-F(p)) = m^{-\eta}(m-c)$. Taking the opponent's strategy as given, any pure strategy choice $p' < m$ will result in lower expected profits for the firm. Hence, neither firm has incentive to deviate from its strategy choice and this is indeed a Nash equilibrium.

In this Bertrand environment there exists a continuum of mixed strategy equilibria.

## 3 Characterization of equilibria

We now proceed to characterize the environments for which similar mixed-strategy equilibria exist. To that effect, we prove several properties satisfied by any equilibrium strategy. We recall that the support of a distribution function $F$ is defined as the set of all points $x$ that satisfy for every $\varepsilon > 0$ , $F(x + \varepsilon) - F(x - \varepsilon) > 0$. We denote by $A_1$ and $A_2$ the supports of the equilibrium strategies of firms 1 and 2 respectively.

**Lemma 1.** *No price strictly lower than $c$ belongs to the equilibrium support of either firm.*

*Proof.* Assume by way of contradiction that $q \in A_1$ and $q < c$. Firm 1 could always guarantee itself zero expected profits by choosing $c$. Hence if, in equilibrium, it attaches positive probability to an interval where it makes strictly negative profits if it wins, it should have a zero probability of winning on that interval. This would be the case only if $A_2$ was to the left of that interval, but then firm 2 would win at least part of the time and could not be playing an equilibrium strategy. Hence, no $q < c$ is in $A_1$ or in $A_2$. □

**Lemma 2.** *If $c$ belongs to the support of the equilibrium distribution of one of the firms, then $c$ is the only point in the support of the equilibrium distribution of both firms.*

*Proof.* Without loss of generality assume that $c$ belongs to $A_1$. At prices arbitrarily close to $c$, firm 1 is making profits arbitrarily close to zero. If there is a point $q$ strictly greater than $c$ in $A_2$, then firm 1 can earn strictly positive expected profits by choosing a point strictly between $c$ and $q$. These profits are greater than the expected profits for prices arbitrarily close to $c$ and contradict the fact that $c$ is in $A_1$. Thus, any point $q$ in $A_2$ must satisfy $q \leq c$. By Lemma 1, we then have $A_2 = \{c\}$. Thus, we have shown that if $c$ belongs to $A_1$, then $A_2 = \{c\}$; similarly, since $c$ is then in $A_2$, it must be that $A_1$ also equals $\{c\}$. Therefore, both firms supports are equal to $\{c\}$. □

**Lemma 3.** *At equilibrium the supports of the two distributions must be the same.*

*Proof.* If $c$ belongs to the equilibrium support of either firm, the Lemma holds. Assume then that $c$ is not in the equilibrium support of either firm. Hence, there exists a $\theta > 0$ such that the equilibrium supports of both firms are contained in $[2\theta, \infty)$ (by Lemma 1, prices less than $c$ cannot be in the equilibrium support of either firm).

Assume there is a point $x$ in $A_1$ but not in $A_2$. From using the definition of support, there must exist an $\varepsilon > 0$ such that $F^1(x + \varepsilon) - F^1(x - \varepsilon) > 0$ and $[x - 2\epsilon, x + 2\epsilon]$ is not in $A_2$. If no $q \geq x + 2\epsilon$ belongs to $A_2$, firm 1 has a zero probability of winning on $[x - \varepsilon, x + \varepsilon]$. In this case, firm 1 could strictly increase its expected profits by moving the probability assigned to $[x - \varepsilon, x + \varepsilon]$ to the price $c + \theta$. This is in contradiction to firm 1 playing an equilibrium strategy.

Assume then there is a point $q \geq x + 2\epsilon$ in $A_2$. The profit at $q$ if firm 2 wins must be strictly greater than the profit on $[x - \varepsilon, x + \varepsilon]$. If it were not, firm 2 could move some probability mass into $[x-\varepsilon, x+\varepsilon]$. This would increase its probability of winning without giving up profits and generate higher expected profits, in contradiction to firm 2 playing an equilibrium strategy. Let $\widehat{q}$ be the lowest price in $A_2$ to the right of $x+2\epsilon$ (note that $\widehat{q}$ exists since we assumed there are points in the support of firm 2 to the right of $x+2\epsilon$, and the support is a closed set). Since demand is decreasing and price changes continuously there cannot be a discrete upward jump in revenue as price increases. Thus, since winning profits are strictly higher at $\widehat{q}$ than on $[x-\varepsilon, x+\varepsilon]$, there must exist a point between $x+\epsilon$ and $\widehat{q}$ that yields higher profits than any point in $[x-\varepsilon, x+\varepsilon]$. Firm 1 could increase its expected profits by moving the probability assigned to $[x-\varepsilon, x+\varepsilon]$ to this point, in contradiction to firm 1 playing an equilibrium strategy. Thus, $A_1$ is contained in $A_2$. Similarly, $A_2$ is contained in $A_1$ and hence we have shown that in equilibrium both supports coincide. □

**Lemma 4.** *An equilibrium distribution, whose support does not contain $c$, cannot possess any atoms.*

*Proof.* Assume that firm 1 placed a probability mass of $\delta > 0$ on a point $q$ $(q > c)$. Since $q \in A_1$, from Lemma 3 it must also be that $q \in A_2$. We will now show that firm 2 cannot place a positive mass $\delta'$ on $q$, and that prices in $(q, q+\epsilon)$ do not belong to $A_2$ for a small enough $\epsilon > 0$.

First, we show that firm 2 does not place a positive mass on $q$. Since demand is decreasing and price changes continuously for any $\varepsilon' > 0$, firm 2 could choose a price $q'$ lower than $q$ such that the winning profits at $q'$ is no more than $\varepsilon'$ less than the winning profits at $q$ $(D(q)(q-c) - D(q')(q'-c)) < \varepsilon')$. The probability of winning will be higher at $q'$ than $q$ by at least $\delta/2$. The expected profit at $q'$ minus the expected profit at $q$ is equal to $\Pr(q')\cdot\pi(q') - \Pr(q)\cdot\pi(q) = (\Pr(q') - \Pr(q))\pi(q') - \Pr(q)(\pi(q) - \pi(q'))$ where Pr is the probability of winning and $\pi$ is the winning profit. Since $\Pr(q') - \Pr(q) \geq \frac{\delta}{2}$ and $\pi(q) - \pi(q') < \varepsilon'$, this difference is greater than $\pi(q')\cdot\frac{\delta}{2} - \Pr(q)\cdot\varepsilon'$. Since $\pi(q') > 0$, there is a small enough $\varepsilon'$ such that this is strictly positive. Thus, firm 2 can increase its expected profits by moving the probability mass $\delta'$ to a point slightly less than $q$, in contradiction to firm 2 playing an equilibrium strategy.

Second, we show that prices in $(q, q+\epsilon)$, for a small enough $\epsilon > 0$, do not belong to $A_2$. Since demand is decreasing and price changes continuously, for any $\epsilon'' > 0$ there exists an $\epsilon$ such that $p \in (q, q+\epsilon)$ implies $D(p)(p-c) - D(q)(q-c)) < \varepsilon''$. The probability of winning will be higher at $q$ than $p$ by at least $\delta/2$. The expected profit at $q$ minus the expected profit at $p$ is equal to $\Pr(q)\cdot\pi(q) - \Pr(p)\cdot\pi(p) = (\Pr(q) - \Pr(p))\pi(q) - \Pr(p)(\pi(p) - \pi(q))$ where Pr is the probability of winning and $\pi$ is the winning profit. Since $\Pr(q) - \Pr(p) \geq \frac{\delta}{2}$ and $\pi(p) - \pi(q) < \varepsilon''$, this difference is greater than $\pi(q)\cdot\frac{\delta}{2} - \Pr(p)\cdot\varepsilon''$. Since $\pi(q) > 0$, there is a small enough $\varepsilon''$ such that this is strictly positive. Thus, firm 2 can increase its expected profits by moving probability mass from around $p$ to $q$, in contradiction to firm 2 playing an equilibrium strategy.

If no price $q' > q$ belongs to $A_2$, then firm 1 has zero probability of winning at $q$ and can increase its expected profits by moving the probability mass $\delta$ from

$q$ to a price greater than $c$ but still below any point in the current $A_1$ (which equals $A_2$). If there is a $q' \in A_2$ with $q' > q + \epsilon$, we proceed as in Lemma 3 to show that firm 1 could increase its expected profits by moving the mass $\delta$ currently at $q$ to a higher price. Thus in any case, we reach a contradiction to firm 1 playing an equilibrium strategy.

Hence, the equilibrium distribution of firm 1 cannot contain any atoms and similarly for firm 2. □

The above Lemmata lead to the following proposition.

**Proposition.** *There exist equilibria other than marginal-cost pricing if and only if* $\lim_{p\to\infty} p \cdot D(p) = \infty$. *Furthermore, these equilibria are characterized by the following family of distribution functions (indexed by* $m > c$*):*

$$F^1(p) = F^2(p) = 1 - \frac{Max_{m' \le m} D(m')(m' - c)}{Max_{p' \le p} D(p')(p' - c)} \text{ for } p \ge m, \tag{1}$$

*where* $F^1$ *and* $F^2$ *represent the firms' price choices.*[2]

*Proof*. We now know that an equilibrium can only be either marginal-cost pricing or a pair of non-atomic mixed strategies starting at points strictly above $c$. Hence, the only equilibria other than marginal-cost pricing can be represented by distribution functions $G^1(p^1)$ and $G^2(p^2)$. By Lemma 3, the supports of $G^1$ and $G^2$ must be the same. Let $\underline{m}$ denote the infimum of the supports and $\overline{m}$ denote the supremum of the supports. By definition $G^i(\underline{m}) = 0$ and $G^i(\overline{m}) = 1$. These distributions would constitute an equilibrium if and only if the following two conditions are satisfied. First, a firm must earn the same expected profit at each point in the support. Second, this profit must be weakly greater than the profit at points not on this support. Denote the expected profit of choosing a price in the support for firm 1 by $\alpha^1(\alpha^2$ for firm 2). To satisfy the first condition, it must be the case that

$$\alpha^1 = D(p^1)(p^1 - c)\left(1 - G^2(p^1)\right) \text{ for all } p^1 \text{ in the support.} \tag{2}$$

Therefore by the boundary conditions for $G^2$ (recall that the support of both distributions has the same infimum),

$$\alpha^1 = D(\underline{m})(\underline{m} - c) \tag{3}$$

By Eqs. 2 and 3,

$$G^2(p) = 1 - \frac{D(\underline{m})(\underline{m} - c)}{D(p)(p - c)} \text{ for all } p \text{ in the support.}$$

By a similar process, we can show similar equalities hold for $\alpha^2$ and $G^1$. Thus, any equilibria must satisfy Eq. 1.

[2] The notation used in the proposition captures the idea that players concentrate their probability only on those regions where the winning profit is increasing in price.

These constructs will be legitimate distribution functions and thus yield strategies if and only if $D(p)(p - c)$ approaches infinity as $p$ approaches $\overline{m}$ (since demand is decreasing $D(p)(p - c)$ approaches infinity is equivalent to $p \cdot D(p)$ approaching infinity) yielding that $F^i(\overline{m}) = 1$.[3] Since demand is non-increasing in price, $\overline{m}$ must be infinite.

To conclude this proof we will now show that $G^1$ and $G^2$ are indeed equilibrium strategies. The first condition for equilibrium is satisfied by construction. The second condition for equilibrium is satisfied since $G^i(\underline{m}) = 0$ and $D(\underline{m})(\underline{m} - c) \geq D(p)(p - c)$ for all $p \leq \underline{m}$ imply that no firm could do better by submitting a price outside the support. Hence, the strategies constructed indeed constitute an equilibrium. These distribution functions $G^i$ coincide with the proposed distribution functions $F^i$.

Note that if $p \cdot D(p)$ does not tend to infinity, these equilibria cannot exist and hence the "only if" part. □

## 4 Summary

We have demonstrated that standard Bertrand competition may lead to Nash equilibria with positive profit levels. This was done by constructing mixed strategy equilibria where firms' strategies are price distribution functions that entail prices above marginal cost. These strategies are equilibrium strategies only for environments with demand curves that yield unbounded revenues.

It is interesting to note that these equilibria co-exist with the standard marginal-cost-pricing equilibria. Furthermore, similar to the usual case, the existence of Bertrand competition serves to potentially reduce the firms' profits compared to the monopoly situation. In environments where such equilibria exist, the firms' profits in these equilibria are finite (given by $\alpha$ in the proposition) and involve strictly positive production levels. However, it should be remarked that those equilibria can be ranked and if the firms were to coordinate on their best equilibrium outcome, profits would tend to infinity. A monopoly, in contrast, would always make infinite profits by producing infinitesimal amounts.

We considered the case of two identical firms. If there are $n$ identical firms ($n > 2$), the basic results remain the same. Lemma 1 will be slightly different in that any equilibrium in which the winning price is $c$ there is the possibility of a price higher than $c$ being chosen by $n - 2$ or less firms. Lemmata 2, 3 and 4 hold as well as the Proposition. However, the equilibrium distribution functions constructed in the Proposition while still yielding positive levels of profits, would be slightly different.

If the two firms have different marginal costs, the usual, pure strategy, Bertrand scenario leads to the low cost firm serving all the market at a price equal to the marginal cost of the high cost firm. Mixed equilibria again exist if

[3] Notice that the numerator in Eq. 1 is always strictly greater than zero. Demand is decreasing and there is positive demand at some price exceeding marginal costs therefore $Max_{m' \leq m} D(m')(m' - c) > 0$ for all $m > c$.

and only if revenues are unbounded. However, they are harder to characterize and the Lemmata don't necessarily hold. Detailed analysis of this scenario is a topic of future research.

We examine an environment with several assumptions about the demand curve and firms costs under Bertrand competition and determine for this environment the necessary and sufficient condition for mixed-strategy equilibria. Our environment is quite general, since environmental assumptions about the demand curve are not strong. First, a demand that isn't decreasing would be inconsistent with the assumption that the lower price firm receives all the demand. Second, a demand that is zero at all prices above costs is uninteresting since all firms either produce nothing or charge marginal costs. However, the necessary and sufficient condition we find for equilibria other than marginal-cost pricing (unbounded revenue) is unreasonable. In any conceivable market, the revenue function is bounded. Hence, realistic market environments will satisfy our environmental assumptions about the demand curve but violate the necessary condition for mixed-strategy equilibria. Thus for realistic markets, we have shown that marginal-cost pricing is the outcome of the Bertrand pricing game even when allowing for mixed-strategies.

## References

Baye, M.R., Morgan, J. (1997) *Necessary and sufficient conditions for existence and uniqueness of Bertrand paradox outcomes.* Princeton Woodrow Wilson School, Discussion Paper in Economics 186

Bertrand, J. (1883) Book review of theorie mathematique de la richesse sociale and of recherches sur les principles mathematiques de la theorie des richesses. *Journal de Savants* 67: 499–508

Cournot, A. (1838) *Recherches sur les principles mathematiques de la theorie des richesses.* Kelly, New York

Kaplan, T.R., Wettstein, D. (1997) Mixed strategy equilibria with constant-returns-to-scale technology. *Monaster Center Discussion Paper* 97–4

Marquez, R. (1997) A note on Bertrand competition with asymmetric fixed costs. *Economics Letters* 57: 87–96

Shapiro, C. (1989) Theories of oligopoly behavior. In: Schmalensee, R., Willig, R.D. (eds.) *Handbook of industrial organization vol. I.* North-Holland, Amsterdam, Chap. 6, pp. 329–414

Sharkey, W., Sibley, D. (1993) A Bertrand model of pricing and entry. *Economics Letters* 41: 199–206

Spulber, D.F. (1995) Bertrand competition when rivals' costs are unknown. *Journal of Industrial Economics* 43: 1–11