# DYNAMIC LIMIT PRICING*

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Abstract. This paper studies a simple multi-period model of limit pricing under one-sided incomplete information. I characterize pooling and separating equilibria, determine conditions under which the latter exist and study under which conditions on the primitives the equilibria involve limit pricing. The results are compared to a static benchmark in the spirit of Milgrom and Roberts (1982). I identify two regimes that depend on the primitives of the model, namely a monopoly price regime and a limit price regime. In the former, the unique reasonable equilibrium is separating and both types of incumbent set their respective monopoly prices. In the latter, both reasonable pooling and separating equilibria exist and both involve limit pricing. In the monopoly price regime, the set of limit price equilibria expands as the horizon recedes, but the reasonable equilibrium remains unchanged. In the limit price regime, equilibrium existence becomes easier to obtain but these equilibria may, if the players are sufficiently patient, involve prices that confer infinitely large losses in order for separation to be incentive compatible. Last, I consider a perturbed version of the model and find that the equilibria of this game, separating as well as pooling, correspond in a natural way to the equilibria of the static benchmark game.

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## 1. Introduction

Since Bain's (1949) pioneering work, limit pricing has been a staple of industrial organization. In a nutshell, limit pricing is the practice by which an incumbent firm (or cartel) deters potential entry to an industry by pricing below the profit maximizing price level. Early work on the subject took its cue from the observation that in some industries, firms price below the myopic profit maximizing price level on a persistent basis. This observation lead to the notion that by doing so, incumbent firms could somehow discourage potential entry which would otherwise have occurred, in effect by sacrificing profits in the short run in return for a maintenance of the monopoly position.

The present work revisits received wisdom on equilibrium limit pricing in dynamic contexts by way of two dynamic extensions of a simple static model of one-sided incomplete information. It is argued that while the main predictions of a standard (static) analysis are preserved qualitatively when moving to dynamic contexts, the quantitative results may radically differ.

Bain (1949) identifies two possible channels through which current prices may deter entry: (i) a low current price may signal to potential entrants that existing and future

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market conditions are unfavorable to entry and that (ii) a low current price may signal to potential entrants something about the incumbent's response to entry.

The early literature focused almost exclusively on explanation (ii) and featured models that were fully dynamic in nature, an approach which is suitable for the study of ongoing relationships between competing firms. Nonetheless, most contributions had the unsatisfactory feature that entrants' decisions were not the outcome of rational deliberations, but rather mechanistic (assumed) responses to the incumbent's pricing behavior.[1] Furthermore, incumbents in these models were endowed with perfect ability to commit to future (i.e. post-entry) pricing behavior. This critique, first articulated by Friedman (1979), cast serious doubt on the received wisdom from Bain's insights. Milgrom and Roberts (1982) confronted this challenge by reformulating the situation as one of incomplete information and drew on both explanations (i) and (ii). In doing so, they succeeded in validating Bain's insights. Unlike the early theory on limit pricing, the Milgrom-Roberts analysis is essentially static in nature (since signaling through prices occurs once and for all). This raises the question as to how robust their findings are to dynamic extensions.

In this paper, I consider two multi-period repeated version of a canonical model of limit pricing with one-sided incomplete information. I do this for two reasons. First, it is of interest purely as a matter of business strategy under which assumptions on the primitives it is possible to deter entry through limit pricing. Second, the answer to this question may have interesting policy implications. Unlike instances of predatory pricing, limit pricing does by its very nature not leave behind a visible victim. This means that it is very difficult to determine empirically if limit pricing has been attempted and if so, to which extent it has been successful. This means that in understanding limit pricing, economists as well as courts must rely heavily on the insights gained through formal analysis. This fact makes it all the more important to check the received wisdom from economic theory.

In the first dynamic version of the model, the basic entry stage game is repeated as long as entry has not occurred. In this model, I identify two distinct regimes, a *monopoly price regime* and a *limit price regime*. In the monopoly price regime, limit price equilibria exist but all such equilibria are ruled out by the forward induction criterion. The unique reasonable equilibrium is one of separation on monopoly prices. In the limit price regime, both reasonable pooling and separating equilibria exist and these involve limit pricing.

I find that in the limit price regime, the basic logic of separating equilibria of the static setting carries over to the separating equilibria of the dynamic setting. In particular, I find that by sacrificing enough in early stages of the game, the efficient incumbent may credibly convey his identity to the entrant and thus maintain his monopoly position. The main difference to the static setting is that in the static setting, the benefits from deterring entry are bounded while this is no longer the case in the dynamic setting. When players are sufficiently patient and the horizon long enough, the benefits from entry deterrence become arbitrarily large and consequently only infinitely large sacrifices incurred by the efficient incumbent can credibly convey the information that the entrant should not enter.

In the second dynamic version of the model, I let the incumbent's type evolve stochastically over time. This may be regarded as a natural perturbation of the first dynamic version of the model which bypasses several difficult issues about post separation beliefs. In this

[1] A model in which entry is an increasing deterministic function of pre-entry price is that of Gaskins (1971). In Berck and Perloff (1988), the rate of entry is proportional to future profitability. Models featuring stochastic entry which is increasing in the pre-entry price are those of Kamien and Schwartz (1971), Baron (1973), De Bondt (1976) and Lippman (1980). Flaherty (1980) takes a different approach and assumes increasing returns to scale in production, while Judd and Petersen (1986) consider the possibility that entrants must fund operations by means of internal finance. See Carlton and Perloff (2004) for a thorough review and further references.

version, both separating and pooling equilibria are similar to those in the static setting in a sense to be made precise. This finding indicates that the main insights from the static analysis of equilibrium limit pricing are robust.

The Milgrom-Roberts analysis and the insights derived from it have greatly shaped the way economists think about limit pricing. As such, it is important to determine to what extent the lessons are robust to variations in the modeling approach. I am not the first to check the robustness of the Milgrom-Roberts insights against variations in the setup. Harrington (1986) considers a variation of the basic model in which the entrant is uncertain of his own costs, which are in turn correlated with those of the incumbent. This modeling approach means that a high pre-entry price may signal that the entrant's costs are likely to be high, thereby making entry less appealing. In turn, this may imply that in equilibrium, the incumbent charges a price higher than the monopoly price and also deters entry.[2] Jun and Park (2005) consider a dynamic setup where the incumbent faces a sequence of entrants that can be either weak or strong, of which only the former can be deterred. Rather than having a strong opponent enter, the incumbent may wish to appear strong by charging a price higher than the monopoly price, thereby encouraging entry by weak entrants. This conclusion is also at odds with that gained from the Milgrom-Roberts analysis. Closest to my analysis is the work of Kaya (2005), who studies repeated signaling in a reduced form setup with dynamic limit pricing as an example. She assumes two-sided asymmetric information and focuses on separating equilibria. Her work complements the current analysis, focusing on somewhat different issues. In unpublished work, Saloner (1984) extends the Matthews and Mirman (1983) setup of noisy signaling to multi-period settings.

The remainder of the paper is structured as follows. In Section 2, I introduce the basic static model that will constitute the stage game of the dynamic analysis. I characterize separating and pooling equilibria, determine sufficient conditions under which the former exist and last I perform equilibrium selection analysis. In Section 3, I analyze a finite horizon setting and compare the outcomes of this analysis to the static setting. Last, I perform comparative statics analysis with respect to the length of the horizon and discuss equilibrium existence further. In Section 5, I consider an alternative formulation of the dynamic game in which the incumbent's type evolves randomly over time. In this setting, I characterize separating and pooling equilibria and discuss equilibrium existence and selection. In Section 6, I discuss the results and indicate possible avenues for further research.

## 2. THE STATIC MODEL

Consider an incumbent monopolist $I$ and a potential entrant $E$. The monopolist serves a market with demand $Q(p)$ and the entrant can enter the market at cost $K > 0$. The monopolist can be one of two types, high cost ($H$) or low cost ($L$), with probability $\mu$ and $(1-\mu)$ respectively. The incumbent knows his type, but his type is unknown to the entrant. Let $C_H(q)$ and $C_L(q)$ be the cost functions of $H$ and $L$ respectively. Let $\pi_i(p)$ be the profit function of the incumbent of type $i = H, L$ when he sets price $p$. These profits are given by

$$\pi_i(p) = pQ(p) - C_i(Q(p)), \quad i = H, L \tag{1}$$

Let $D_i$ be the duopoly profit of the incumbent of type $i = H, L$ when competing against $E$. Last, let $D_E(i)$ be the duopoly profits of $E$ when competing against the incumbent of type $i = H, L$.

Denote by $p_H^M$ and $p_L^M$ the monopoly prices under the technologies $C_H(\cdot)$ and $C_L(\cdot)$ respectively.

[2]Matthews and Mirman (1983) consider the possibility that the incumbent's price only provide noisy information to the entrant about the profitability of entry. Under certain conditions, they find that limit pricing can be successfully employed by the incumbent to limit entry.

Throughout, I make the following assumptions:

**Assumptions**

**0** $C_i(q)$, $i = H, L$ and $Q(p)$ are differentiable, for $q > 0$ and $p > 0$ respectively.

**1** $C'_H(q) > C'_L(q), \forall q \in \mathbb{R}_+$, with $C_H(0) \geq C_L(0)$.

**2** $Q'(p) < 0, \forall p \geq 0$.

**3** $D_E(L) - K < 0$.

**4** $D_E(H) - K > 0$.

**5** $\pi_i(p)$ is strictly increasing for $p \leq p_i^M$ and strictly decreasing for $p \geq p_i^M$, $i = H, L$.

**6** $\pi_i(p_i^M) > D_i$, $i = H, L$.

Assumption 1 implies that $\pi_i(p)$ has the single crossing property, i.e. that $\pi_H(p)$ and $\pi_L(p)$ cross only once. Assumption 2 means that demand is downward sloping. Assumptions (3)-(4) imply that $E$ will not enter if he knows that $I$ is of type $L$ while he will enter if he knows that $I$ is of type $H$.

Although the main focus will be on the multi-period problem, I start out by analyzing the static problem. I do this for two reasons. First, the static model will serve as the benchmark against which the dynamic model will be compared. Second, the analysis of the static problem plays an important role in the analysis of the equilibria of the dynamic game.

The game between $I$ and $E$ is played in three stages. At the first stage, $I$ sets a price that will serve as a signal for $E$ of $I$'s type. After observing the price set by $I$, $E$ decides at the second stage whether or not to enter (incurring the entry cost $K$). Denote $E$'s entry decision by $s_E \in \{0, 1\}$, where $s_E = 0$ stands for *stay out* and $s_E = 1$ stands for *enter*. At the third stage, if $E$ enters he will learn $I$'s type and compete against him in complete information fashion. Both incumbent and entrant discount the future by a factor $\delta \in [0, 1]$.[3]

The payoff to $E$ is given by

$$f_E(p) = \begin{cases} 0 & \text{if } s_E = 0 \\ D_E(H) - K & \text{if } s_E = 1, \quad i = H \\ D_E(L) - K & \text{if } s_E = 1, \quad i = L \end{cases} \tag{2}$$

A strategy for $I$ is a price for each of his two types, $p_H$ or $p_L$, at the first stage, a price at the second stage if the entrant stays out and a quantity or price to set at the third stage if the entrant enters (depending on the mode of competition), both as functions of his type and the decisions made at the first stage. A strategy for $E$ is a decision rule to enter or not as a function of the price set by $I$ at the first stage and a quantity or price to set at the third stage in case he enters (again, depending on the mode of competition).

If $E$ enters, then at the second stage $I$ and $E$ play a duopoly game of complete information. Hence in any subgame perfect equilibria of the game after $E$'s entry, $I$'s equilibrium payoffs in the second period are $D_H$ or $D_L$. If $E$ stays out, then $I$'s equilibrium payoffs in the second period are $\pi_H(p_H^M)$ or $\pi_L(p_L^M)$, depending on his type. That is, the payoffs to the incumbent of type $i = H, L$ are given by

$$f_i(p) = \begin{cases} \pi_H(p) + \delta\pi_H(p_H^M) & \text{if } s_E = 0 \quad i = H \\ \pi_H(p) + \delta D_H & \text{if } s_E = 1 \quad i = H \\ \pi_L(p) + \delta\pi_L(p_L^M) & \text{if } s_E = 0 \quad i = L \\ \pi_L(p) + \delta D_L & \text{if } s_E = 1 \quad i = L \end{cases} \tag{3}$$

[3] I assume that there is no discounting between stages two and three, or alternatively, that the entry cost is borne at the third stage.

Let $\sigma = (p_L, p_H)$ be a pure strategy of the game. Denote by $p_H^*$ and $p_L^*$ the equilibrium prices.

**Definition 1.** *$\sigma$ is a separating equilibrium if $p_H^* \neq p_L^*$ and a pooling equilibrium if $p_H^* = p_L^*$.*

**Definition 2.** *$\sigma$ is a limit price equilibrium if either $p_H^* < p_H^M$ or $p_L^* < p_L^M$.*

The aim is to analyze separating and pooling limit price equilibria of the game. Before continuing with the analysis, I will first state and prove the following useful result:

**Lemma 3.** *(i) $\pi_L(p) - \pi_H(p)$ is decreasing in $p$. In particular, $p_1 < p_2 \Rightarrow \pi_L(p_2) - \pi_L(p_1) < \pi_H(p_2) - \pi_H(p_1)$. (ii) $p_H^M \geq p_L^M$.*

**Proof.** (i) First, note that $\pi_L(p) - \pi_H(p) = C_H(Q(p)) - C_L(Q(p))$. Next, note that

$$\frac{\partial}{\partial p}\left[\pi_L(p) - \pi_H(p)\right] = Q'(p)\left[C_H'(Q(p)) - C_L'(Q(p))\right] \tag{4}$$

By Assumption 2, $Q'(p) < 0$. Thus, by Assumption 1 it follows that $\frac{\partial}{\partial p}\left[\pi_L(p) - \pi_H(p)\right] < 0$.

(ii) By the definition of monopoly prices, it follows that

$$p_L^M Q_L^M - C_L(Q_L^M) \geq p_H^M Q_H^M - C_L(Q_H^M) \tag{5}$$
$$p_H^M Q_H^M - C_H(Q_H^M) \geq p_L^M Q_L^M - C_H(Q_L^M) \tag{6}$$

Adding these inequalities, I obtain

$$C_H(Q_L^M) - C_L(Q_L^M) \geq C_H(Q_H^M) - C_L(Q_H^M) \tag{7}$$

Hence, by Assumption 1, $Q_L^M \geq Q_H^M$ and by Assumption 2, $p_L^M \leq p_H^M$ ■

To make the problem interesting, I will henceforth assume that $p_H^M > p_L^M$, i.e. I exclude the case $p_H^M = p_L^M$.

### 2.1. Separating Limit Price Equilibria.

**Characterization.** Assume that $p_H^* \neq p_L^*$. The best reply strategy of $E$ in this case is to enter if $p = p_H^*$ and to stay out if $p = p_L^*$, i.e. $s_E(p_H^*) = 1$ and $s_E(p_L^*) = 0$. Therefore the $H$ type incumbent is best off setting $p = p_H^M$, knowing that entry will occur in the second period, so $s_E(p_H^M) = 1$. Hence

$$p_H^* = p_H^M \tag{8}$$

To obtain a limit price equilibrium, it is thus required that

$$p_L^* \neq p_L^M \tag{9}$$

I confine the analysis to the following monotone second-stage decision rule for $E$:

$$s_E(p) = \begin{cases} 1 \text{ if } & p > \overline{p} \\ 0 \text{ if } & p \leq \overline{p} \end{cases} \tag{10}$$

for some $\overline{p}$ to be determined by $E$.

Throughout the paper, the out of equilibrium beliefs of $E$ will be assumed to have the following monotone structure: For any observed price above the $L$ type's equilibrium price, the entrant will assign probability one to the incumbent being of type $H$. For prices below the $L$ type's equilibrium price, the entrant will assign probability one to the incumbent being of type $L$.

**Lemma 4.** $\overline{p} = p_L^*$ *and* $\overline{p} < p_L^M$.

**Proof.** Suppose to the contrary that $\overline{p} \geq p_L^M$. Then $s_E(p_L^M) = 0$ and $L$ is therefore best off switching from $p_L^*$ to $p_L^M$, contradicting (9). Next, observe that in a separating equilibrium, $s_E(p_L^*) = 0$ ($E$ knows that $L$ set $p_L^*$) and hence $p_L^* \leq \overline{p}$. Suppose that $p_L^* < \overline{p}$. Since $\overline{p} < p_L^M$, I have by Assumption 5 that $L$ is better off by increasing his price from $p_L^*$ to $\overline{p}$, which is a contradiction. Thus, $\overline{p} = p_L^*$ ■

**Corollary 5.** *(i) In any separating limit price equilibrium,* $p_L^* < p_L^M$. *(ii) In any separating equilibrium, either* $p_L^* = p_L^M \leq \overline{p}$ *or* $p_L^* = \overline{p} < p_L^M$.

**The Incentive Compatibility Constraints.** Since $p_H^* = p_H^M$, the following incentive compatibility constraint for $H$ should hold:

$$f_H(p_H^M) \geq f_H(p), \quad \forall p \tag{11}$$

Clearly, (11) holds for $p > \overline{p}$ since in this case, $E$ enters and $I$ can do no better than to set the monopoly price. Consider $p$ such that $p \leq \overline{p}$. By Corollary 5, in a separating limit pricing equilibrium $\overline{p} = p_L^*$ and hence by Assumption 5 and Lemma 3(ii), it follows that $p \leq \overline{p} = p_L^* < p_L^M < p_H^M$ and thus it is sufficient to consider the following inequality:

$$f_H(p_H^M) \geq f_H(p_L^*) \tag{12}$$

By the definition of $f_H$ given in (3), (12) is equivalent to

$$\pi_H(p_L^*) \leq (1-\delta)\pi_H(p_H^M) + \delta D_H \tag{13}$$

To write (13) in terms of prices, first define the set

$$A_H \equiv \left\{p : \pi_H(p) = (1-\delta)\pi_H(p_H^M) + \delta D_H\right\} \tag{14}$$

Since $D_H = \pi_H(p)$ for some $p$, then by Assumptions 5 and 6 the set $A_H$ is non-empty and contains at most two points. Next, define

$$\widehat{p} \equiv \min A_H, \quad \widehat{q} \equiv \max A_H \tag{15}$$

where $\widehat{p} < \infty$ and $\widehat{q} \leq \infty$.

Hence, according to (13), $p_L^*$ must satisfy

$$p_L^* \leq \widehat{p} \text{ or } p_L^* \geq \widehat{q} \tag{16}$$

Also, note that by definition,

$$\pi_H(\widehat{p}) = \delta D_H + (1-\delta)\pi_H(p_H^M) = \pi_H(\widehat{q}) \tag{17}$$

Observe that $p_L^* < p_L^M < p_H^M < \widehat{q}$. Hence it must be that

$$p_L^* \leq \widehat{p} \tag{18}$$

Let us now return to $L$. The incentive compatibility constraint for $L$ is given by

$$f_L(p_L^*) \geq f_L(p), \quad \forall p \tag{19}$$

But the relevant $p$ is only $p = p_L^M$ (since deterring entry is only optimal if it yields higher payoffs than setting the monopoly price in the first period and accommodating entry). Hence (19) becomes

$$f_L(p_L^*) \geq f_L(p_L^M) \tag{20}$$

By the definition of $f_L$ given by (3), inequality (20) is equivalent to

$$\pi_L(p_L^*) + \delta\pi_L(p_L^M) \geq \pi_L(p_L^M) + \delta D_L \tag{21}$$

Consequently,

$$\pi_L(p_L^*) \geq (1-\delta)\pi_L(p_L^M) + \delta D_L \tag{22}$$

is the relevant incentive compatibility constraint for $L$. Define the set

$$A_L \equiv \left\{p : \pi_L(p) = (1-\delta)\pi_L(p_L^M) + \delta D_L\right\} \tag{23}$$

Since $D_L = \pi_L(p)$ for some $p$, then by Assumptions 5 and 6 the set $A_L$ is non-empty and contains at most two points. Let

$$p_0 \equiv \min A_L, \quad q_0 \equiv \max A_L \tag{24}$$

where $p_0 < \infty$ and $q_0 \leq \infty$.

In terms of prices, (22) means that $p_L^*$ should satisfy

$$p_0 \leq p_L^* \leq q_0 \tag{25}$$

where, by definition, I have that

$$\pi_L(p_0) = \delta D_L + (1-\delta)\pi_L(p_L^M) = \pi_L(q_0) \tag{26}$$

I can summarize the previous results as follows:

**Lemma 6.** *Any separating limit price equilibrium is a triple $(p_H^*, p_L^*, \overline{p})$ such that (i) $p_H^* = p_H^M$, (ii) $\overline{p} = p_L^*$, (iii) $p_0 \leq p_L^* \leq \widehat{p}$ and (iv) $p_L^* < p_L^M$.*

Hence, to show existence of a separating limit price equilibrium, I need to show that $p_0 < \widehat{p}$. Before stating the conditions which are sufficient to guarantee that $p_0 < \widehat{p}$, let us characterize the set of separating equilibria without the limit price requirement.

**Separating Equilibria.** If I eliminate the limit pricing requirement, then in addition to the set

$$\{(p_H^*, p_L^*, \overline{p}) : \text{ (i), (ii), (iii), (iv) satisfied}\} \tag{27}$$

there are other separating equilibrium points if $\widehat{p} \geq p_L^M$. In particular, the following result holds:

**Lemma 7.** *Any separating equilibrium is a triple $(p_H^*, p_L^*, \overline{p})$ such that (i) $p_H^* = p_H^M$, (ii) $\overline{p} = p_L^*$, (iii) $p_0 \leq p_L^* \leq \min\{\widehat{p}, p_L^M\}$.*

**Existence of Separating Limit Price Equilibria.**

**Lemma 8.** *Suppose that*

$$\pi_L(p_L^M) - D_L > \pi_H(p_H^M) - D_H \tag{28}$$

*Then $\widehat{p} > p_0$ and the set of separating limit pricing equilibria is non-empty.*

**Proof.** From (28), (17) and (26), I have that

$$\pi_L(p_L^M) - \pi_H(p_H^M) > \pi_L(p_0) - \pi_H(\widehat{p}) \tag{29}$$

Adding and subtracting $\pi_H(p_L^M)$ yields

$$\pi_L(p_L^M) + \pi_H(p_L^M) + \left[\pi_H(p_L^M) - \pi_H(p_H^M)\right] > \pi_L(p_0) - \pi_H(\widehat{p}) \tag{30}$$

By the definition of $p_H^M$, it follows that $\pi_H(p_L^M) - \pi_H(p_H^M) \leq 0$. It thus follows from (30) that

$$\pi_L(p_L^M) + \pi_H(p_L^M) > \pi_L(p_0) - \pi_H(\widehat{p}) \tag{31}$$

Since $p_0 \leq p_L^M$, I have by Lemma 1 that

$$\pi_L(p_0) - \pi_H(p_0) \geq \pi_L(p_L^M) - \pi_H(p_L^M) \tag{32}$$

Combined with (31), this implies that $\pi_H(p_0) < \pi_H(\widehat{p})$. Finally, $p_0 \leq p_H^M$ and $\widehat{p} \leq p_H^M$ and therefore it follows by Assumption 5 that $p_0 < \widehat{p}$ ■

It can be shown that condition (28) holds for the special cases of Cournot competition with linear demand and fixed marginal costs and Bertrand competition with product differentiation (see Tirole, 1988).

**Equilibrium Selection.** As seen above, perfection fails in uniquely determining $L$'s equilibrium price $p_L^*$. It will now be argued that only one price satisfies the forward induction criterion. In particular, I have:

**Lemma 9.** *(i) Suppose that $p_0 < \widehat{p} \leq p_L^M$. Then only $p_L^* = \widehat{p}$ satisfies the forward induction criterion. (ii) Suppose that $p_0 < p_L^M \leq \widehat{p}$. Then only $p_L^* = p_L^M$ satisfies the forward induction criterion.*

**Proof.** (i) Suppose that $p_0 < \widehat{p} \leq p_L^M$ and let $p'$ satisfy $p_L^* < p' < \widehat{p}$. Whichever strategy $E$ picks, it is a strictly dominated strategy for $H$ to choose $p'$. If $s_E(p') = 1$, then since $p' < \widehat{p} \leq p_L^M \leq p_H^M$ it follows that

$$\pi_H(p') + \delta D_H < \pi_H(\widehat{p}) + \delta D_H \tag{33}$$

If in turn $s_E(p') = 0$, then

$$\pi_H(p') + \delta\pi_H(p_H^M) < \pi_H(\widehat{p}) + \delta\pi_H(p_H^M) = \pi_H(p_H^M) + \delta D_H \tag{34}$$

Hence, even if $H$ fools $E$ to believe that he is $L$, he will obtain less than $\pi_H(p_H^M) + \delta D_H$ which he would obtain under the equilibrium strategy $p_H^* = p_H^M$. In the game obtained after eliminating the strategy $p'$ from $H$'s strategy set, $E$ must play $s_E(p') = 0$ since $p'$ can have been set only by $L$ and thus by backward induction staying out at the price $p'$ is a best response for $E$. But in the new reduced game, $L$ can profitably deviate from $p_L^*$ to $p'$ and obtain $\pi_L(p') - \pi_L(p_L^*) > 0$, which follows from Assumption 5 and the fact that $p' \leq p_L^M$. The proof of (ii) follows similar steps as that of (i) ■

The price selected by the forward induction is known as the least-cost separating equilibrium price, as it's the equilibrium price which involves the lowest possible cost for the $L$ type in terms of foregone profits. In other words, it's the price largest price (i.e. the price closest to the monopoly price) consistent with the incentive compatibility constraints. This outcome is known in the literature as the *Riley outcome*.

**2.2. Pooling Limit Price Equilibria.** In a pooling equilibrium, I have

$$p_L^* = p_H^* = p^* \tag{35}$$

Observe first that if

$$\mu D_E(H) + (1-\mu) D_E(L) - K > 0 \tag{36}$$

then pooling equilibria cannot exist, since the expected profit of $E$ when he cannot distinguish between the incumbent's types is positive and he thus enters. By backward induction, each type of incumbent is better off setting his monopoly price. Since $p_H^M > p_L^M$, I thus have that $p_L^* \neq p_H^*$, contradicting the supposition that the types pool. I thus assume that

$$\mu D_E(H) + (1-\mu) D_E(L) - K < 0 \tag{37}$$

so that the entrant expects to make negative profits against the incumbent if he cannot distinguish the incumbent's type.

As before, I consider the following first period decision rule for $E$:

$$s_E(p) = \begin{cases} 1 \text{ if } & p > \overline{p} \\ 0 \text{ if } & p \leq \overline{p} \end{cases} \tag{38}$$

for some $\overline{p}$ to be determined by $E$.

**Characterization.**

**Lemma 10.** *$\overline{p} = p^*$ and $p^* \leq p_L^M$.*

**Proof.** Clearly $\overline{p} \geq p^*$. Otherwise, $E$'s decision rule dictates entry if $p^*$ is charged. That is, $s_E(p^*) = 1$ if $p^* > \overline{p}$ and thus each type of incumbent would benefit from deviating to their respective monopoly prices, contradicting (35). Next observe that if $p^* > p_L^M$, then $L$ is best off setting the price $p_L^M$ and entry will still be deterred (i.e. $s_E(p_L^M) = 0$). Consequently, $p^* \leq p_L^M$ as claimed. Finally, suppose to the contrary that $\overline{p} > p^*$. Since $p^* \leq p_L^M < p_H^M$, it follows by Assumption 5 that the $H$ type is better off increasing his price slightly above $p^*$ to increase profits while still deterring $E$'s entry. Therefore $\overline{p} = p^*$ must hold as claimed ■

**The Incentive Compatibility Constraints.** The incentive compatibility constraints for the $H$ type and $L$ type are given by

$$\pi_H(p^*) + \delta \pi_H(p_H^M) \geq \pi_H(p_H^M) + \delta D_H \tag{39}$$

$$\pi_L(p^*) + \delta \pi_L(p_L^M) \geq \pi_L(p_L^M) + \delta D_L, \quad p^* < p_L^M \tag{40}$$

Note that for each type, the best alternative strategy to choosing the entry deterring pooling price is to set the monopoly price and inviting entry. Also note that if $p^* = p_L^M$, then there is no incentive compatibility constraint for the $L$ type.[4] The two incentive constraints (39)-(40) reduce to

$$\pi_H(p^*) \geq (1-\delta)\pi_H(p_H^M) + \delta D_H \tag{41}$$

$$\pi_L(p^*) \geq (1-\delta)\pi_L(p_L^M) + \delta D_L, \quad p^* < p_L^M \tag{42}$$

Inequality (41) holds if and only if $\widehat{p} \leq p^* \leq \widehat{q}$ while (42) holds for $p^* \geq p_0$ as long as $p^* < p_L^M$. Combining these constraints, I obtain

$$\max\{p_0, \widehat{p}\} \leq p^* \leq p_L^M < p_H^M \tag{43}$$

Note that a pooling equilibrium necessarily involves limit pricing.

[4]Throughout the paper, the qualifier $p^* < p_L^M$ will reappear in connection with constraints on pooling prices. It will henceforth be implicit that if $p_t^* = p_L^M$ in some period $t$, then there is no incentive compatibility constraint for the $L$ type in that period.

**Equilibrium Selection.**

**Lemma 11.** *The only pooling equilibrium that satisfies the forward induction criterion is* $p^* = p_L^M$.

**Proof.** The set of pooling equilibrium outcomes is the set

$$\left\{p^* : \max\left\{p_0, \widehat{p}\right\} \leq p^* \leq p_L^M\right\} \tag{44}$$

Suppose that $p^* < p_L^M$. First note that $s_E(p_L^M) = 1$, for otherwise the $L$ type is better off switching from $p^*$ to $p_L^M$. Thus it is a strictly inferior strategy for $H$ to select $p_L^M$ or $p_H^M$. Indeed, by $H$'s incentive compatibility constraint (39) I have

$$\pi_H(p^*) + \delta\pi_H(p_H^M) \geq \pi_H(p_H^M) + \delta D_H > \pi_H(p_L^M) + \delta D_H \tag{45}$$

Consider the new reduced game which is obtained from the original game by eliminating $p_L^M$ from $H$'s strategy set. In the equilibrium of the new game, $s_E(p_L^M) = 0$, since this price can only have been set by the $L$ type. Hence $L$, in the new game, is better off deviating from $p^*$ to $p_L^M$ ■

As was the case with the selected separating limit price equilibrium, the forward induction criterion selects the least-cost pooling limit price equilibrium.

Last, note the following:

**Lemma 12.** *If* $\widehat{p} \geq p_L^M$, *then essentially no pooling equilibrium satisfying the forward induction criterion exists.*

**Proof:** First note that $\widehat{p} \geq p_L^M$ if and only if

$$(1-\delta)\pi_H(p_H^M) + \delta D_H \geq \pi_H(p_L^M) \tag{46}$$

To see this, note that from (17) it follows that

$$\delta D_H = \pi_H(\widehat{p}) - (1-\delta)\pi_H(p_H^M) \tag{47}$$

Substituting this in (46) yields

$$\pi_H(\widehat{p}) \leq \pi_H(p_L^M) \tag{48}$$

Since $\widehat{p} < p_H^M$ and $p_L^M < p_H^M$, the result then follows from Assumption 5. Next, recall that for pooling on $p^* = p_L^M$ to be incentive compatible, I need by (41) that

$$\pi_H(p_L^M) \geq (1-\delta)\pi_H(p_H^M) + \delta D_H \tag{49}$$

The result then follows immediately[5] ■

This is a point I will return to in the dynamic version of the model.

## 2.3. Existence of Sensible Limit Price Equilibria.

Before continuing the analysis, some comments on the existence of sensible limit price equilibria are in order. Note that the above existence result concerns itself only with the existence of limit price equilibria and *not* with the existence of *sensible* limit price equilibria. After performing equilibrium selection, the set of equilibria can, if non-empty, be divided into two distinct regimes, namely a *limit price regime* and a *monopoly price regime*. The former obtains if $\widehat{p} < p_L^M$ and the latter if $\widehat{p} \geq p_L^M$. These regimes will reappear in an important way in the dynamic game. In the monopoly price regime, the unique sensible equilibrium is characterized by both firms separating by setting their respective monopoly prices while in the limit price regime, both sensible pooling and sensible separating equilibria can coexist.

[5] The qualifier *essentially* is due to the fact that for the case $\pi_H(p_L^M) = (1-\delta)\pi_H(p_H^M) + \delta D_H$, pooling on $p^* = p_L^M$ is incentive compatible.

## 3. The Dynamic Model

Now consider the model in which the basic setting is repeated $T-1$ times as long as entry hasn't occurred (so that period $T$ is the last period and period $T-1$ is the last period in which entry may occur). Note that this is not a repeated game as entry can only occur once and thus the stage game is not unvarying across periods.

In a dynamic setting like this, care needs to be taken when specifying strategies and beliefs. The theory of dynamic or repeated signaling is still in its infancy and so there is still little consensus as to how to appropriately treat such environments (but see Kaya, 2005 for new developments). The central question in a repeated signaling game is how to treat post-separation strategies and beliefs. At first, it may be tempting to treat the post-separation game as one of perfect information. This is the approach taken by e.g. LeBlanc (1992).[6] After all, when the two types of incumbent reveal their types by separating on different equilibrium prices, one may think that treating the continuation game as one of perfect information is innocuous. This is not the case however. Such an approach is not inherent in the employed equilibrium concept, is basically ad-hoc and furthermore rules out interesting dynamics with repeated costly signaling. Madrigal et al. (1987) and Noldeke and van Damme (1990) discuss the treatment of such degenerate posteriors in depth.

Rather than formally develop a framework for repeated signaling games, I take a pragmatic approach and focus on an intuitively appealing set of strategies which I term *probationary strategies*. In short, the entrant will give the $L$ type the benefit of the doubt only after he has successfully completed a number of periods of probation during which he is expected to behave as would the $L$ type. I will analyze the simplest special case of such strategies in which signaling is required only once.

I restrict attention to the following monotone decision rule for the entrant, for $t = 1, ..., T-1$, if entry has not occurred by time $t$:

$$s_E(p_t) = \begin{cases} 1 \text{ if } & p_t > \overline{p}_t \\ 0 \text{ if } & p_t \leq \overline{p}_t \end{cases} \tag{50}$$

for a sequence $\{\overline{p}_t\}_{1=1}^{T-1}$ to be determined by $E$.

I will proceed by analyzing the case in which, if separation occurs in period $t = 1, ..., T-2$, then $\overline{p}_s = p_L^M$ for $s \geq t+1$. This means that after separation occurs, the entrant continues to treat the incumbent as if he were of type $L$ as long as he observes prices no higher than $L$'s monopoly price. In turn, this means that the $H$ type must, if he wishes to mimic the $L$ type off the equilibrium path, also charge no more than $p_L^M$ after separation has occurred.

**3.1. Separating Limit Price Equilibria.** The magnitude of the entry costs in the dynamic setting plays a more delicate role than it does in the static setting. A necessary condition for a separating limit price equilibrium with separation in any period $t = 1, ..., T-1$ to exist is that

$$D_E(L) < \left(\frac{1-\delta}{1-\delta^{T-t+1}}\right) K < D_E(H), \quad t = 2, ..., T \tag{51}$$

Since the coefficient on the entry cost $K$ is decreasing in the number of remaining periods, the condition may fail to hold for some $t$.[7] In order to do away with time varying necessary conditions, I instead impose the following condition which ensures that separation is feasible in an arbitrary period $t = 1, ..., T-1$:

[6] See also Kaya (2005) for a review of papers imposing support restrictions.

[7] In particular, it may be the case that the necessary condition for a separating limit price equilibrium to be feasible is that the remaining number of periods be small. Since I eventually want to study the infinite horizon limit of the game, I disregard this possibility.

**Assumptions**

**3*** $D_E(L) < \left(\frac{1-\delta}{1-\delta^T}\right) K$

**4*** $D_E(H) > K$

It should be pointed out that these conditions are stronger than Assumptions 3-4.[8]

Next, the entrant's strategy can be characterized as follows:

**Lemma 13.** *Consider the equilibrium price sequence* $(p_1^*, ..., p_{t-1}^*, p_{t,L}^*, p_L^M, ..., p_L^M)$. *Then (i)* $\overline{p}_s = p_{s,L}^*$ *and* $\overline{p}_s < p_L^M, \quad s = 1, ..., t$ *and (ii)* $\overline{p}_s = p_{s,L}^* = p_L^M, \quad s = t+1, ..., T-1$.

**Proof:** See proof in static analysis ■

**The Incentive Compatibility Constraints.** Last, note that as in the static setting, the best alternative for the $L$ type to setting the separating equilibrium price is to set his monopoly price. In contrast, the best alternative for the $H$ type to setting the separating equilibrium price, i.e. his monopoly price, is to mimic the $L$ type's equilibrium price price.

**Lemma 14.** *For the price sequence* $(p_1^*, ..., p_{t-1}^*, p_{t,L}^*, p_L^M, ..., p_L^M)$ *to constitute a separating limit price equilibrium, it must satisfy*

$$\pi_L(p_s^*) \geq (1-\delta)\pi(p_L^M) + \delta D_L, \quad p_s^* < p_L^M, \quad s = 1, ..., t-1 \tag{52}$$

$$\pi_L(p_{s,L}^*) \geq \left(1 - \frac{\delta - \delta^{T-s+1}}{1-\delta}\right) \pi_L(p_L^M) + \left(\frac{\delta - \delta^{T-s+1}}{1-\delta}\right) D_L, \quad s = t \tag{53}$$

**Proof:** See Appendix A ■

Define the following set:

$$A_L(T,t) \equiv \left\{ p : \pi_L(p) = \left(1 - \frac{\delta - \delta^{T-t+1}}{1-\delta}\right) \pi_L(p_L^M) + \left(\frac{\delta - \delta^{T-t+1}}{1-\delta}\right) D_L \right\} \tag{54}$$

Since $\pi_L(p) = D_L$ for some $p$, then by Assumptions 5 and 6 I know that the set $A_L(T,t)$ is non-empty and contains at most two points. Let

$$p_0(T,t) \equiv \min A_L(T,t), \quad q_0(T,t) \equiv \max A_L(T,t) \tag{55}$$

where $p_0(T,t) < \infty$ and $q_0(T,t) \leq \infty$.

In terms of prices, the $L$ type's incentive compatibility constraint requires that

$$p_0(T,t) \leq p_{t,L}^* \leq q_0(T,t) \tag{56}$$

For later use, note that by definition I have that

$$\pi_L(p_0(T,t)) = \left(1 - \frac{\delta - \delta^{T-t+1}}{1-\delta}\right) \pi_L(p_L^M) + \left(\frac{\delta - \delta^{T-t+1}}{1-\delta}\right) D_L = \pi_L(q_0(T,t)) \tag{57}$$

I next consider the incentive compatibility constraints for the $H$ type. These are slightly more complicated, due to the fact that the $H$ type may in general wish to mimic the behavior of the $L$ type for an arbitrary number of periods after the $L$ type has chosen to separate. It turns out that the exact amount of mimicking undertaken by the $H$ type out of equilibrium depends in a simple way on parameter values.

[8] Note that in the static setting, Assumptions 3-4 can be replaced by the assumptions that $D_E(L) < 0$ and $D_E(H) > 0$ (and that $\mu D_E(H) + (1-\mu) D_E(L) < 0$ for a pooling equilibrium to exist). See e.g. Tirole (1988) for such a setup. In the dynamic setting however, these assumptions would mean that an entrant discovering that he has entered against the $L$ type would immediately leave the market. The $L$ type may in turn find it optimal to allow entry in the first period, knowing that $E$ will subsequently leave the market.

**Lemma 15.** *(i) Suppose that (129) is satisfied. For the price sequence $(p_1^*, ..., p_{t-1}^*, p_{t,L}^*)$ to constitute a separating limit price equilibrium, it must satisfy*

$$\begin{aligned} \pi_H(p_s^*) &\geq (1-\delta)\pi_H(p_H^M) + \delta D_H, \quad p_s^* < p_L^M, \quad s = 1, ..., t-1 & (58) \\ \pi_H(p_{s,L}^*) &\leq (1-\delta)\pi_H(p_H^M) + \delta D_H, \quad s = t & (59) \end{aligned}$$

*(ii) Suppose that (129) is violated. For the price sequence $(p_1^*, ..., p_{t-1}^*, p_{t,L}^*)$ to constitute a separating limit price equilibrium, it must satisfy*

$$\begin{aligned} \pi_H(p_s^*) &\geq (1-\delta)\pi_H(p_H^M) + \delta D_H, \quad p_s^* < p_L^M, \quad s = 1, ..., t-1 & (60) \\ \pi_H(p_{s,L}^*) &\leq (1-\delta^{T-s})\pi_H(p_H^M) + \left(\frac{\delta - \delta^{T-s+1}}{1-\delta}\right) D_H - \left(\frac{\delta - \delta^{T-s}}{1-\delta}\right) \pi_H(p_L^M), \quad s = t & (61) \end{aligned}$$

**Proof:** See Appendix B ■

Define the following set:

$$A_H(T,t) \equiv \left\{ p : \pi_H(p) = (1-\delta^{T-t})\pi_H(p_H^M) + \left(\frac{\delta - \delta^{T-t+1}}{1-\delta}\right) D_H - \left(\frac{\delta - \delta^{T-t}}{1-\delta}\right) \pi_H(p_L^M) \right\} \quad (62)$$

Note that the coefficients on $\pi_H(p_H^M)$, $D_H$ and $\pi_H(p_L^M)$ in the definition of $A_H(T,t)$ sum to one. It then follows from Assumptions 5 and 6 and the fact that $\pi_H(p_H^M) > \pi_H(p_L^M)$ that the set $A_H(T,t)$ contains at most two points. Let

$$\widehat{p}(T,t) \equiv \min A_H(T,t), \quad \widehat{q}(T,t) \equiv \max A_H(T,t) \quad (63)$$

where $\widehat{p}(T,t) < \infty$ and $\widehat{q}(T,t) \leq \infty$.

To express the incentive compatibility constraints in terms of prices, note that for periods with pooling, the price sequence must satisfy

$$\max\{p_0, \widehat{p}\} \leq p_s^* \leq p_L^M < p_H^M, \quad s = 1, ..., t-1 \quad (64)$$

For the period in which separation is prescribed, the $H$ type's incentive compatibility constraint when condition (129) is satisfied is that

$$p_{t,L}^* \leq \widehat{p} \text{ or } p_{t,L}^* \geq \widehat{q} \quad (65)$$

which is as in the static setting. In this case, only the inequality $p_{t,L}^* \leq \widehat{p}$ is relevant, since $p_{t,L}^* < p_L^M < p_H^M < \widehat{q}$. For the period in which separation is prescribed, the $H$ type's incentive compatibility constraint when condition (129) is violated is that

$$p_{t,L}^* \leq \widehat{p}(T,t) \text{ or } p_{t,L}^* \geq \widehat{q}(T,t) \quad (66)$$

In this case, only the inequality $p_{t,L}^* \leq \widehat{p}(T,t)$ is relevant, since $p_{t,L}^* < p_L^M < p_H^M < \widehat{q}(T,t)$. For later use, note that by definition it is the case that

$$\pi_H(\widehat{p}(T,t)) = (1-\delta^{T-t})\pi_H(p_H^M) + \left(\frac{\delta - \delta^{T-t+1}}{1-\delta}\right) D_H - \left(\frac{\delta - \delta^{T-t}}{1-\delta}\right) \pi_H(p_L^M) = \pi_H(\widehat{q}(T,t)) \quad (67)$$

**Existence of Separating Limit Price Equilibria.** In the static setting, I showed that (28) was a sufficient condition for the set of separating limit pricing equilibria to be non-empty, since it implied that $\widehat{p} > p_0$. I now derive the dynamic counterparts of (28). Note that when (129) is satisfied, the existence of separating limit price equilibria is ensured if $\widehat{p} > p_0(T,t)$ whereas if (129) is violated, then existence is ensured if $\widehat{p}(T,t) > p_0(T,t)$.

The relevant conditions are given as follows:

**Lemma 16.** *(i) Suppose that (129) is satisfied. If*

$$\pi_L(p_L^M) - D_L > \left(\frac{1-\delta}{1-\delta^{T-t}}\right)\left[\pi_H(p_H^M) - D_H\right], \quad t = 1,...,T-1 \tag{68}$$

*then $\widehat{p} > p_0(T,t)$ and the set of limit price equilibria is non-empty.*

*(ii) Suppose that (129) is violated. If*

$$\pi_L(p_L^M) - D_L > \left[\frac{(1-\delta)\delta^{T-t-1}}{1-\delta^{T-t}}\right]\pi_H(p_H^M) - D_H + \left[\frac{\delta-\delta^{T-2}}{1-\delta^{T-t}}\right]\pi_H(p_L^M), \quad t = 1,...,T-1 \tag{69}$$

*then $\widehat{p}(T,t) > p_0(T,t)$ and the set of limit price equilibria is non-empty.*

**Proof.** (i) Solving (57) and (17) for $D_L$ and $D_H$ respectively, substituting into (68) and rearranging yields

$$\pi_L(p_L^M) - \pi_H(p_H^M) > \pi_L(p_0(T,t)) - \pi_H(\widehat{p}) \tag{70}$$

which is equivalent to condition (29). The remainder of the proof follows the same steps as in the static setting.

(ii) Solving (57) and (67) for $D_L$ and $D_H$ respectively, substituting into (69) and rearranging yields

$$\pi_L(p_L^M) - \pi_H(p_H^M) > \pi_L(p_0(T,t)) - \pi_H(\widehat{p}(T,t)) \tag{71}$$

which again is equivalent to condition (29) ■

Note that in both regimes, the relevant sufficient condition for existence becomes easier to satisfy as the horizon recedes.

**Comparative Statics.** It is immediately clear that the constraints in periods characterized by pooling in the separating equilibria are unaffected by the length of the horizon. In periods where separation is prescribed however, the constraints do explicitly depend on the horizon. First consider the $L$ types' incentive compatibility constraint $p_{t,L}^* \geq p_0(T,t)$. The cutoff $p_0(T,t)$ is decreasing in $T$ since $p_0(T,t) \leq p_L^M$ and the right-hand side of the equality defining the set $A_L(T,t)$ is decreasing in $T$. In the limit $T \to \infty$, $p_0(T,t)$ is implicitly given by

$$\lim_{T\to\infty} \pi_L(p_0(T,t)) = \left(1 - \frac{\delta}{1-\delta}\right)\pi_L(p_L^M) + \left(\frac{\delta}{1-\delta}\right)D_L \tag{72}$$

This means that as the horizon recedes, the $L$ type's incentive compatibility constraint becomes more easy to satisfy. Now turn to the $H$ type. Under condition (129), the $H$ type's incentive compatibility constraints are unaffected by changes in $T$. When (129) is violated however, the appropriate constraint is $p_{t,L}^* \leq \widehat{p}(T,t)$. The cutoff $\widehat{p}(T,t)$ is decreasing in $T$ since $\widehat{p}(T,t) \leq p_H^M$ and the right-hand side of the equality defining the set $A_H(T,t)$ is decreasing in $T$. In the limit $T \to \infty$, $\widehat{p}(T,t)$ is implicitly given by

$$\lim_{T\to\infty} \pi_H(\widehat{p}(T,t)) = \pi_H(p_H^M) + \left(\frac{\delta}{1-\delta}\right)\left(D_H - \pi_H(p_L^M)\right) \tag{73}$$

This means that as the horizon recedes, the $H$ type's incentive compatibility constraint becomes more difficult to satisfy.

Recall that (28) implied that $\widehat{p} > p_0$. In fact, in the static setting (28) turned out also to imply that $p_0 > 0$, i.e. that prices are positive in any separating limit price equilibrium. Interestingly, this is no longer necessarily the case in the dynamic version of the game. In particular, I have:

**Proposition 17.** *If (129) is violated, there exists a unique* $\delta^* < 1$ *such that* $\lim_{T\to\infty} \pi_H(\widehat{p}(T,t)) \leq 0$ *for* $\delta \geq \delta^*$.

**Proof:** For $\lim_{T\to\infty} \pi_H(\widehat{p}(T,t)) \leq 0$, it must be that

$$\pi_H(p_L^M) - D_H \geq \frac{1-\delta}{\delta}\pi_H(p_H^M) \tag{74}$$

The left-hand side is strictly positive because (129) is violated, while the right-hand side is strictly decreasing in $\delta$, tending to zero. It follows from the intermediate value theorem that there exists a unique $\delta^*$ such that $\lim_{T\to\infty} \widehat{p}(T,t) = 0$ when evaluated at $\delta = \delta^*$. It is given by

$$\delta^* = \frac{\pi_H(p_H^M)}{\pi_H(p_H^M) - D_H + \pi_H(p_L^M)} < 1 \tag{75}$$

■

**Proposition 18.** *If (129) is violated, then* $lim_{\delta\to 1} \lim_{T\to\infty} \pi_H(\widehat{p}(T,t)) = -\infty$

**Proof:** The result follows from taking the limit $\delta \to 1$ of (73) since when (129) is violated, $\pi_H(p_L^M) > D_H$ ■

Returning to the two regimes discussed earlier, I now show the following:

**Lemma 19.** *Condition (129) holds if and only if* $\widehat{p}(T,t) \geq p_L^M$.

**Proof:** From (67) it follows that

$$\delta D_H = \left(\frac{1-\delta}{1-\delta^{T-t}}\right)\left[\pi_H(\widehat{p}(T,t)) - (1-\delta^{T-t})\pi_H(p_H^M) + \left(\frac{\delta-\delta^{T-t}}{1-\delta}\right)\pi_H(p_L^M)\right] \tag{76}$$

Substituting this in (129) yields

$$\pi_H(\widehat{p}(T,t)) \leq \pi_H(p_L^M) \tag{77}$$

Since $\widehat{p}(T,t) < p_H^M$ and $p_L^M < p_H^M$, the result follows from the inequality and Assumption 5 ■

In other words, in the monopoly price regime, the set of separating equilibria expands with the length of the horizon.

**Equilibrium Selection.** I now determine which of the equilibria in the dynamic game can be deemed reasonable in that they satisfy the forward induction criterion. I do this explicitly for the case where (129) is violated. The case where (129) is satisfied is similar, with $\widehat{p}(T,t)$ replaced by $\widehat{p}$.

**Lemma 20.** *(i) Suppose that* $p_0(T,t) < \widehat{p}(T,t) \leq p_L^M$. *Then only* $p_{t,L}^* = \widehat{p}(T,t)$ *satisfies the forward induction criterion. (ii) Suppose that* $p_0(T,t) < p_L^M \leq \widehat{p}(T,t)$. *Then only* $p_{t,L}^* = p_L^M$ *satisfies the forward induction criterion.*

**Proof:** (i) Suppose that $p_0(T,t) < \widehat{p}(T,t) \leq p_L^M$ and let $p'$ satisfy $p_L^* < p' < \widehat{p}(T,t)$. Whichever strategy $E$ picks, it is a strictly dominated strategy for $H$ to choose $p'$. If $s_E(p') = 1$, then since $p' < \widehat{p}(T,t) \leq p_L^M \leq p_H^M$, Assumption 5 implies that the $H$ type can benefit from switching to $p_H^M$, thereby earning $\pi_H(p_H^M) - \pi_H(p') > 0$. Next, suppose that $s_E(p') = 0$. In equilibrium, the $H$ type should set the price $p_H^M$ and can never earn more out of equilibrium than by playing his optimal off equilibrium strategy. But the first element of this strategy is precisely given by $\widehat{p}(T,t)$. It follows that the $H$ type is better off by switching from $p'$ to $\widehat{p}(T,t)$. After removing the price $p'$ from the $H$ type's strategy set, $E$ must set $s_E(p') = 0$, since $p'$ could only have been set by the $L$ type. But since $p' < \widehat{p}(T,t) \leq p_L^M$, it follows from Assumption 5 that the $L$ type is better off by increasing his price to $\widehat{p}(T,t)$. The proof of (ii) follows similar steps as that of (i) ■

**Lemma 21.** *In the monopoly price regime, essentially all sensible equilibria are characterized by separation in the first period.*

**Proof:** It is known from the static analysis that in the monopoly price regime, essentially no sensible pooling equilibria exist. The result then follows immediately from observing that the incentive compatibility constraint of the $H$ type in periods of pre-separation pooling are identical to the incentive compatibility constraint in the static setting ■

**3.2. Pooling Limit Price Equilibria.** In this setting, a pooling equilibrium consists of a sequence $\sigma = \{p_t^*\}_{t=1}^{T-1}$. This means that in every period, the entrant cannot distinguish the two types. For pooling to be feasible in period $t = 1.,,,T-1$, I must impose the conditions

$$\mu D_E(H) + (1-\mu) D_E(L) < \left(\frac{1-\delta}{1-\delta^{T-t+1}}\right) K, \quad t = 2, ..., T \tag{78}$$

Note that these constraints are easier to satisfy the farther away the final period is. In order to avoid time varying necessary conditions, I instead impose the following condition that ensures that pooling is feasible in any period $t = 1, ..., T-1$:

**Assumption** $\mu D_E(H) + (1-\mu) D_E(L) < (1-\delta) K$

Note that as in the static setup, the best alternative for either type to setting the pooling equilibrium price is to set his monopoly price.

As in the static setting, I have:

**Lemma 22.** $\overline{p}_t = p_t^*$ *and* $p_t^* \leq p_L^M, \quad t = 1, ...T-1.$

**Proof:** See proof in the static analysis ■

**The Incentive Compatibility Constraints.** As was the case in the static setting, the best alternative for each type to setting the pooling price is to set the monopoly price and inviting entry. With this in mind, the following can be shown to hold:

**Lemma 23.** *For the price sequence* $\{p_t^*\}_{t=1}^{T-1}$ *to constitute a pooling limit price equilibrium, it must satisfy*

$$\pi_L(p_t^*) \geq (1-\delta)\pi(p_L^M) + \delta D_L, \quad p_t^* < p_L^M, \quad t = 1, ..., T-1 \tag{79}$$

$$\pi_H(p_t^*) \geq (1-\delta)\pi(p_H^M) + \delta D_H, \quad t = 1, ..., T-1 \tag{80}$$

**Proof:** See Appendix C ■

In terms of prices, I thus have that

$$\max\{p_0, \widehat{p}\} \leq p_t^* \leq p_L^M < p_H^M, \quad t = 1, ..., T-1 \tag{81}$$

**Equilibrium Selection.** Note that the incentive compatibility constraints in the dynamic pooling equilibrium are in fact equivalent to their static counterparts (42) and (41). It then follows from the same arguments as in the static analysis that only $p_t^* = p_L^M$ is a sensible pooling equilibrium in that it satisfies the forward induction criterion.

**Comparative Statics.** As is the case in periods where pooling is prescribed in the separating equilibria, the constraints in the pooling equilibria do not depend explicitly on the remaining number of periods. It follows that the pooling equilibria are in fact not affected by the dynamic extension of the model.

**3.3. Existence of Sensible Limit Price Equilibria.** As was the case in the static analysis, one may characterize two distinct regimes, namely a monopoly price regime and a limit price regime. In the monopoly price regime, the only sensible outcome is separation on monopoly prices in the first period, whereas in the limit price regime, both sensible pooling and separating equilibria coexist. In the sensible pooling equilibrium, both types of incumbent set the efficient type's monopoly price and thus the equilibrium involves limit pricing. One may see this as a repetition of the static outcome. In the sensible separating limit price equilibrium however, since the benefits from entry deterrence increase with the horizon and the patience of the players, credibly signaling to be of the efficient type may involve incurring arbitrarily large losses in periods where separation is prescribed. Depending on the model specification and mode of competition in the market game, this may actually involve setting negative prices.[9]

## 4. The Perturbed Dynamic Model

As discussed in the previous section, the main problem with the dynamic game is that it is not clear how to deal with post separation beliefs. If the incumbent plays pure strategies and the outcome of his strategy is perfectly observed by the entrant, then the incumbent's posterior will be degenerate and Bayes' rule be inapplicable in subsequent periods. There are three natural ways of circumventing this difficulty. The first is to consider noisy signals, such as in Saloner (1984). With noisy signals, the entrant can never exclude any type from the incumbent's type support. Secondly, one may instead analyze trembling hand perfect equilibria in the sense of Selten (1975) which exclude reasoning based on pure strategies. Last, one may perturb the game in question such that even if the incumbent's strategy is perfectly informative of his type, this information does not spill over and inform the entrant of all future relevant information. This is most easily done by letting the incumbent's type evolve randomly over time (see Mester, 1992 for a similar insight). This is the approach that will be studied next.

Consider the setting in which the incumbent's type evolves randomly over time. For simplicity, suppose that in moving from one period to the next, the incumbent's type switches with some small probability $\varepsilon > 0$. This construct avoids the aforementioned problems of determining what happens to beliefs after separation has occurred. In order to serve as a small perturbation of the original game, the limiting case where $\varepsilon \to 0$ will be the main object of interest. As was the case in the previously studied scenarios, the entrant perfectly observes the incumbent's type upon entry. Furthermore, the entrant is assumed to perfectly observe any future switches in the incumbent's type.

In what follows, attention is restricted to stationary Markov strategies for both the incumbent and the entrant. For the incumbent, this means that the mapping from his type

[9] This will be the case, e.g., in a model with constant marginal costs and linear demand as that considered by Tirole (1988). In fact, the efficient incumbent would have to give its customers infinitely large subsidies to credibly convey his identity.

space to his price is independent of past play (as long as entry has not occurred yet). In particular, this means that both types of incumbent set the same price in each period in the pooling equilibrium. In the separating equilibrium, each type sets a different price from the other, but these prices do no vary across time.

Denote by $v(i,1)$ the value to an incumbent of type $i = H, L$ of the remainder of the game when entry has occurred at some previous stage. Analogously, denote by $v(i,0)$ the equivalent value to an incumbent of type $i = H, L$ when no entry has occurred at previous stages. Assuming that the players play the infinite repetition of the stage game Nash equilibrium post entry, the values are given by

$$v(L,1) = D_L + \delta\left[(1-\varepsilon)v(L,1) + \varepsilon v(H,1)\right] \tag{82}$$

$$v(H,1) = D_H + \delta\left[(1-\varepsilon)v(H,1) + \varepsilon v(L,1)\right] \tag{83}$$

Solving this system of equations yields

$$v(L,1) = \frac{D_L\left[1-\delta(1-\varepsilon)\right] + \delta\varepsilon D_H}{(1-\delta)(1-\delta+2\delta\varepsilon)} \tag{84}$$

$$v(H,1) = \frac{D_H\left[1-\delta(1-\varepsilon)\right] + \delta\varepsilon D_L}{(1-\delta)(1-\delta+2\delta\varepsilon)} \tag{85}$$

Next, suppose that both types of incumbent pool on price $p^*$ and that both strategies and equilibrium selection is stationary. The values are then given by

$$v^p(L,0) = \pi_L(p^*) + \delta\left[(1-\varepsilon)v^p(L,0) + \varepsilon v^p(H,0)\right] \tag{86}$$

$$v^p(H,0) = \pi_H(p^*) + \delta\left[(1-\varepsilon)v^p(H,0) + \varepsilon v^p(L,0)\right] \tag{87}$$

By solving, these values reduce to

$$v^p(L,0) = \frac{\pi_L(p^*)\left[1-\delta(1-\varepsilon)\right] + \delta\varepsilon\pi_H(p^*)}{(1-\delta)(1-\delta+2\delta\varepsilon)} \tag{88}$$

$$v^p(H,0) = \frac{\pi_H(p^*)\left[1-\delta(1-\varepsilon)\right] + \delta\varepsilon\pi_L(p^*)}{(1-\delta)(1-\delta+2\delta\varepsilon)} \tag{89}$$

Turning to the situation where the types separate in each period, note that

$$v^s(L,0) = \pi_L(p_L^*) + \delta\left[(1-\varepsilon)v^s(L,0) + \varepsilon v^s(H,0)\right] \tag{90}$$

$$v^s(H,0) = \pi_H(p_H^M) + \delta\left[(1-\varepsilon)v(H,1) + \varepsilon v(L,1)\right] \tag{91}$$

Solving these equations yields

$$v^s(L,0) = \frac{\pi_L(p_L^*) + \delta\varepsilon v^s(H,0)}{1-\delta(1-\varepsilon)} \tag{92}$$

$$v^s(H,0) = \pi_H(p_H^M) + \delta\frac{D_H\left[1-\delta-\varepsilon+2\delta\varepsilon\right] + D_L\varepsilon}{(1-\delta)(1-\delta+2\delta\varepsilon)} \tag{93}$$

For later use, note that all the above value functions are bounded for finite $\varepsilon > 0$.

In what follows, I restrict attention to the following monotone decision rule for the entrant, for $t = 1, ..., T-1$, if entry has not occurred by time $t$:

$$s_E(p_t) = \begin{cases} 1 \text{ if } & p_t > \overline{p}_t \\ 0 \text{ if } & p_t \leq \overline{p}_t \end{cases} \tag{94}$$

for a sequence $\{\overline{p}_t\}_{1=1}^{\infty}$ to be determined by $E$.

**4.1. Separating Equilibrium.** A necessary condition for a separating equilibrium to exist is that

**Assumptions**

**3**** $\frac{D_E(L)[1-\delta(1-\varepsilon)]+\delta\varepsilon D_E(H)}{(1-\delta)(1-\delta+2\delta\varepsilon)} < K, \quad \forall\varepsilon$

**4**** $\frac{D_E(H)[1-\delta(1-\varepsilon)]+\delta\varepsilon D_E(L)}{(1-\delta)(1-\delta+2\delta\varepsilon)} > K, \quad \forall\varepsilon$

**Characterization.** The following result is immediately noted:

**Lemma 24.** *$\overline{p}_t = p_L^*$ and $\overline{p}_t < p_L^M$, $\quad t = 1, 2, ....$*

**Proof:** See proof in static analysis ■

As was previously the case, the best alternative strategy to the equilibrium (separating) strategy is the monopoly price for the $L$ type and the $L$ type's equilibrium strategy for the $H$ type. The relevant incentive compatibility constraints are thus as follows:

$$\pi_L(p_L^*) + \delta\left[(1-\varepsilon)v^s(L,0) + \varepsilon v^s(H,0)\right] \geq \pi_L(p_L^M) + \delta\left[(1-\varepsilon)v(L,1) + \varepsilon v(H,1)\right] \quad (95)$$
$$\pi_H(p_H^M) + \delta\left[(1-\varepsilon)v(H,1) + \varepsilon v(L,1)\right] \geq \pi_H(p_L^*) + \delta\left[(1-\varepsilon)v^s(H,0) + \varepsilon v^s(L,0)\right] \quad (96)$$

Define the following sets:

$$A_H^s(\varepsilon) \equiv \left\{p : \pi_H(p) = \pi_H(p_H^M) + \delta\left[(1-\varepsilon)v(H,1) + \varepsilon v(L,1) - (1-\varepsilon)v^s(H,0) - \varepsilon v^s(L,0)\right]\right\} \quad (97)$$
$$A_L^s(\varepsilon) \equiv \left\{p : \pi_L(p) = \pi_L(p_L^M) + \delta\left[(1-\varepsilon)v(L,1) + \varepsilon v(H,1) - (1-\varepsilon)v^s(L,0) - \varepsilon v^s(H,0)\right]\right\} \quad (98)$$

These sets have at most two points. Let

$$\widehat{p}^s(\varepsilon) \equiv \min A_H^s(\varepsilon), \quad \widehat{q}^s(\varepsilon) \equiv \max A_H^s(\varepsilon), \quad p_0^s(\varepsilon) \equiv \min A_L^s(\varepsilon), \quad q_0^s(\varepsilon) \equiv \max A_L^s(\varepsilon) \quad (99)$$

For finite $\varepsilon > 0$, it can be verified that $\widehat{p}^s(\varepsilon) < \infty$, $\widehat{q}^s(\varepsilon) \leq \infty$, $p_0^s(\varepsilon) < \infty$ and $q_0^s(\varepsilon) \leq \infty$.

In terms of prices, the incentive compatibility constraint for the $L$ type is

$$p_0^s(\varepsilon) \leq p_L^* \leq q_0^s(\varepsilon) \quad (100)$$

In turn, the incentive compatibility constraint for the $H$ type is given by

$$p_L^* \leq \widehat{p}^s(\varepsilon) \text{ or } p_L^* \geq \widehat{q}^s(\varepsilon) \quad (101)$$

The relevant inequality is that $p_L^* \leq \widehat{p}^s(\varepsilon)$, since $p_L^* < p_L^M < p_H^M < \widehat{q}^s(\varepsilon)$. The equilibrium separating price sequence must therefore satisfy

$$p_0^s(\varepsilon) \leq p_L^* \leq \widehat{p}^s(\varepsilon) \quad (102)$$

**Proposition 25.** *$lim_{\varepsilon\to 0} p_0^s(\varepsilon) = p_0$, $lim_{\varepsilon\to 0}\widehat{p}^s(\varepsilon) = \widehat{p}$.*

**Proof:** As $\varepsilon \to 0$, the constraints (95)-(96) converge to

$$\pi_L(p_L^*) \geq (1-\delta)\pi_L(p_L^M) + \delta D_L \quad (103)$$
$$\pi_H(p_L^*) \leq (1-\delta)\pi_H(p_H^M) + \delta D_H \quad (104)$$

Note that these constraints coincide with the constraints in the static setting (13) and (22) and the result follows ■

**Existence and Equilibrium Selection.** A sufficient condition for a separating equilibrium to exist is that $\widehat{p}^s(\varepsilon) > p_0^s(\varepsilon)$. But since $lim_{\varepsilon\to 0}\widehat{p}^s(\varepsilon) = \widehat{p}$ and $lim_{\varepsilon\to 0}p_0^s(\varepsilon) = p_0$, existence in the limit, which is the scenario of interest, is ensured under condition (28), i.e. the same condition that ensured equilibrium existence in the static setting. Turning to equilibrium selection, once the restriction to stationary Markov strategies is imposed, similar arguments to those in the static setting yield that the only reasonable separating limit price equilibrium is the perpetual repetition of the least-cost separating equilibrium where the incumbent of type $H$ always sets the monopoly price $p_H^* = p_H^M$ (and invites entry) and the incumbent of type $L$ deters entry by setting price $p_L^* = \min\left\{\widehat{p}, p_L^M\right\}$. It is worth emphasizing that the equilibrium selection is itself stationary and Markovian in the sense that when a particular price is ruled out today, it is done so with the understanding that it will be ruled out in all future periods. In other words, since a strategy is a mapping from the type space to the price space, equilibrium selection is performed by restricting the price space of this mapping, in effect reducing the prices that can be chosen in any given period.

**4.2. Pooling Equilibrium.** A necessary condition for a pooling equilibrium to exist is that

$$\mu\left[\frac{D_E(H)\left[1-\delta(1-\varepsilon)\right]+\delta\varepsilon D_E(L)}{(1-\delta)(1-\delta+2\delta\varepsilon)}\right]+(1-\mu)\left[\frac{D_E(L)\left[1-\delta(1-\varepsilon)\right]+\delta\varepsilon D_E(H)}{(1-\delta)(1-\delta+2\delta\varepsilon)}\right]-K<0,\quad\forall\varepsilon \tag{105}$$

**Characterization.** The following result is immediately noted:

**Lemma 26.** *$\overline{p}_t = p^*$ and $p^* \leq p_L^M$, $\quad t = 1, 2, ....$*

**Proof:** See proof in the static analysis ■

As in the preceding analysis of pooling equilibria, it is the case for each type that the best alternative strategy to the (pooling) equilibrium strategy is to set the monopoly price and inviting entry.

The relevant incentive compatibility constrains are as follows:

$$\pi_L(p^*)+\delta\left[(1-\varepsilon)v^p(L,0)+\varepsilon v^p(H,0)\right] \;\geq\; \pi_L(p_L^M)+\delta\left[(1-\varepsilon)v(L,1)+\varepsilon v(H,1)\right],\quad p^*<p_L^M \tag{106}$$

$$\pi_H(p^*)+\delta\left[(1-\varepsilon)v^p(H,0)+\varepsilon v^p(L,0)\right] \;\geq\; \pi_H(p_H^M)+\delta\left[(1-\varepsilon)v(H,1)+\varepsilon v(L,1)\right] \tag{107}$$

Define the following sets:

$$A_H^p(\varepsilon) \equiv \left\{p:\pi_H(p)=\pi_H(p_H^M)+\delta\left[(1-\varepsilon)v(H,1)+\varepsilon v(L,1)-(1-\varepsilon)v^p(H,0)-\varepsilon v^p(L,0)\right]\right\} \tag{108}$$

$$A_L^p(\varepsilon) \equiv \left\{p:\pi_L(p)=\pi_L(p_L^M)+\delta\left[(1-\varepsilon)v(L,1)+\varepsilon v(H,1)-(1-\varepsilon)v^p(L,0)-\varepsilon v^p(H,0)\right]\right\} \tag{109}$$

These sets have at most two points. Let

$$\widehat{p}^p(\varepsilon)\equiv\min A_H^p(\varepsilon),\quad \widehat{q}^p(\varepsilon)\equiv\max A_H^p(\varepsilon),\quad p_0^p(\varepsilon)\equiv\min A_L^p(\varepsilon),\quad q_0^p(\varepsilon)\equiv\max A_L^p(\varepsilon) \tag{110}$$

For finite $\varepsilon > 0$, it can be verified that $\widehat{p}^p(\varepsilon) < \infty$, $\widehat{q}^p(\varepsilon) \leq \infty$, $p_0^p(\varepsilon) < \infty$ and $q_0^p(\varepsilon) \leq \infty$.

In terms of prices, the incentive compatibility constraints are as follows:

$$\max\left\{p_0^p(\varepsilon),\widehat{p}^p(\varepsilon)\right\}\leq p^*\leq p_L^M<p_H^M \tag{111}$$

**Proposition 27.** *$lim_{\varepsilon\to 0}p_0^p(\varepsilon) = p_0$, $lim_{\varepsilon\to 0}\widehat{p}^p(\varepsilon) = \widehat{p}$.*

**Proof:** As $\varepsilon \to 0$, the constraints (106)-(107) converge to

$$\pi_L(p^*) \;\geq\; (1-\delta)\,\pi_L(p_L^M)+\delta D_L,\quad p^*<p_L^M \tag{112}$$

$$\pi_H(p^*) \;\geq\; (1-\delta)\,\pi_H(p_H^M)+\delta D_H \tag{113}$$

Note that these constraints coincide with the equivalent constraints in the static setting, namely (41) and (42) and the result follows ■

**Equilibrium Selection.** As was the case in the separating equilibrium of the perturbed game, arguments analogous to those in the static setting show that the only reasonable pooling equilibrium in stationary Markov strategies is that both firms perpetually set prices $p^* = p_L^M$, i.e. pool on the $L$ type incumbent's monopoly price.

## 5. Discussion

In this paper I analyzed a dynamic model of limit pricing and compared it with the outcome of a static model. I showed that there are two regimes of interest. In one, the *monopoly price regime*, the only sensible equilibrium involves separation in the first period on monopoly prices. In the other, the *limit price regime*, sensible pooling limit price equilibria and sensible separating limit price equilibria coexist. While the former is essentially a repetition of the static outcome, with both types of incumbent pooling on the efficient type's monopoly price, the latter may differ radically from the separating limit price equilibrium in the static setting. While the basic forces at work are similar, the fact that the game is repeated may make the benefits from entry deterrence arbitrarily large. In turn, this means that the efficient incumbent must, in order to credibly convey his identity, make arbitrarily large (possibly infinite) losses in initial periods. While this outcome is fully consistent from a theoretical perspective, it is not clear to which extent this result is of empirical relevance. In particular, such equilibria would require firm behavior that is rarely (if ever) observed in practice. Last, I find that equilibrium existence is easier to obtain in the dynamic setting than in a static setting. Last, I analyze a perturbed version of the dynamic model and find that the equilibria of the perturbed game, separating as well as pooling, correspond to perpetual repetitions of the static equilibrium.

There are several directions in which the current analysis can be extended, the first of which is to study generalized probationary strategies in which the incumbent is expected to engage in repeated costly signaling. More interestingly though, would be to relax the assumption that the entrant learns the incumbent's type upon entry. Relaxing this assumption would be a substantial departure from the current setting, as it would add features of predatory pricing to the model.

## Appendix

### A. Proof of Lemma 14

**Proof:** I first derive the condition for the separating equilibrium price. The incentive compatibility constraints for the $L$ type are given by

$$\pi_L(p_{1,L}^*) + \sum_{i=2}^{T} \delta^{i-1} \pi_L(p_L^M) \geq \pi_L(p_L^M) + \sum_{i=2}^{T} \delta^{i-1} D_L \tag{114}$$

$$\sum_{i=1}^{K+1} \delta^{i-1} \pi_L(p_i^*) + \delta^{K+1} \pi_L(p_{K+2,L}^*) + \sum_{i=K+3}^{T} \delta^{i-1} \pi_L(p_L^M) \tag{115}$$
$$\geq \sum_{i=1}^{M+1} \delta^{i-1} \pi_L(p_i^*) + \delta^{M+1} \pi_L(p_L^M) + \sum_{i=M+3}^{T} \delta^{i-1} D_L$$

for $0 \leq M \leq K = 0, 1, ..., T-3$. The first constraint (114) reduces to

$$\pi_L(p_{1,L}^*) \geq \left(1 - \frac{\delta - \delta^T}{1-\delta}\right) \pi_L(p_L^M) + \left(\frac{\delta - \delta^T}{1-\delta}\right) D_L \tag{116}$$

Next, evaluate (115) at $M = K$ and rearrange to get

$$\pi_L(p^*_{K+2,L}) \geq \left(1 - \frac{\delta - \delta^{T-K-1}}{1-\delta}\right)\pi_L(p^M_L) + \left(\frac{\delta - \delta^{T-K-1}}{1-\delta}\right)D_L \tag{117}$$

which determines the separating prices. Next, evaluate (115) at two arbitrary consecutive periods $M = K - j$ and $M = K - j - 1$ respectively, with $j = 1, ..., K - 1$. These yield

$$\delta^{K-j+1}\pi_L(p^*_{K-j+2}) - \left(\delta^{K-j+1} - \sum_{i=K+3}^{T}\delta^{i-1}\right)\pi_L(p^M_L) \geq \tag{118}$$
$$\sum_{i=K-j+3}^{T}\delta^{i-1}D_L - \delta^{K+1}\pi_L(p^*_{K+2,L}) - \sum_{i=K-j+3}^{K+1}\delta^{i-1}\pi_L(p^*_i)$$
$$\delta^{K-j}\pi_L(p^*_{K-j+1}) - \left(\delta^{K-j} - \sum_{i=K+3}^{T}\delta^{i-1}\right)\pi_L(p^M_L) \geq \tag{119}$$
$$\sum_{i=K-j+2}^{T}\delta^{i-1}D_L - \delta^{K+1}\pi_L(p^*_{K+2,L}) - \sum_{i=K-j+3}^{K+1}\delta^{i-1}\pi_L(p^*_i) - \delta^{K-j+1}\pi_L(p^*_{K-ji+2})$$

Substituting (118) in (119), rearranging and reducing yields

$$\pi_L(p^*_{K-j+1}) \geq (1-\delta)\pi_L(p^M_L) + \delta D_L \tag{120}$$

Last, if the equilibrium requires pooling in only the first period, then it must be that

$$\pi_L(p^*_1) \geq \left(1 - \sum_{i=3}^{T}\delta^{i-1}\right)\pi_L(p^M_L) + \sum_{i=2}^{T}\delta^{i-1}D_L - \delta\pi_L(p^*_{K+2,L}) \tag{121}$$

Substituting for the value of $\pi_L(p^*_{K+2,L})$ given by (117) and rearranging yields

$$\pi_L(p^*_1) \geq (1-\delta)\pi_L(p^M_L) + \delta D_L$$

■

## B. Proof of Lemma 15

**Proof:** The constraints are as follows:

$$\sum_{i=1}^{K+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_H^M)+\sum_{i=K+3}^{T}\delta^{i-1}D_H\geq$$

$$\sum_{i=1}^{M+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{M+1}\pi_H(p_H^M)+\sum_{i=M+3}^{T}\delta^{i-1}D_H,\quad 0\leq M<K=0,...,T-3 \tag{122}$$

$$\pi_H(p_H^M)+\sum_{i=2}^{T}\delta^{i-1}D_H\geq\pi_H(p_{1,L}^*)+\delta\pi_H(p_H^M)+\sum_{i=3}^{T}\delta^{i-1}D_H \tag{123}$$

$$\sum_{i=1}^{K+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_H^M)+\sum_{i=K+3}^{T}\delta^{i-1}D_H\geq$$

$$\sum_{i=1}^{K+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_{K+2,L}^*)+\delta^{K+2}\pi_H(p_H^M)+\sum_{i=K+4}^{T}\delta^{i-1}D_H,\quad K=0,...,T-4 \tag{124}$$

$$\pi_H(p_H^M)+\sum_{i=K+2}^{T}\delta^{i-1}D_H\geq\pi_H(p_{1,L}^*)+\sum_{i=2}^{K+2}\delta^{i-1}\pi_H(p_L^M)+\delta^{K+2}\pi_H(p_H^M)+\sum_{i=K+4}^{T}\delta^{i-1}D_H,$$
$$K=0,...,T-4 \tag{125}$$

$$\sum_{i=1}^{K+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_H^M)+\sum_{i=K+3}^{T}\delta^{i-1}D_H\geq$$

$$\sum_{i=1}^{K+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_{K+2,L}^*)+\sum_{i=K+3}^{M+2}\delta^{i-1}\pi_H(p_L^M)+\delta^{M+2}\pi_H(p_H^M)+\sum_{i=M+4}^{T}\delta^{i-1}D_H,$$
$$0\leq K<M=0,...,T-4 \tag{126}$$

$$\pi_H(p_H^M)+\sum_{i=2}^{T}\delta^{i-1}D_H\geq\pi_H(p_{1,L}^*)+\sum_{i=2}^{T-1}\delta^{i-1}\pi_H(p_L^M)+\delta^{T-1}\pi_H(p_H^M) \tag{127}$$

$$\sum_{i=1}^{K+1}\delta^{j-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_H^M)+\sum_{i=K+3}^{T}\delta^{i-1}D_H\geq$$

$$\sum_{i=1}^{K+1}\delta^{i-1}\pi_H(p_i^*)+\delta^{K+1}\pi_H(p_{K+2,L}^*)+\sum_{i=K+3}^{T-1}\delta^{i-1}\pi_H(p_L^M)+\delta^{T-1}\pi_H(p_H^M),$$
$$K=0,...,T-4 \tag{128}$$

These sets of constraints will be explained in turn. Roughly, the $H$ type's off equilibrium behavior can be described by the sequence *pool-mimic-reveal*. That is, first $H$ pools whenever the $L$ type pools, then the $H$ type mimics $L$'s behavior for some number of periods and then he reveal his type, subsequently earning duopoly profits following entry by $E$. The first set (122) considers the possibility of the $H$ type revealing his type by setting the monopoly price earlier than the period in which the $L$ type separates. These constraints will determine the pooling constraints for the $H$ type. Next, the constraints (123) and (124) consider the $H$ type mimicking the $L$ type for a single period, in the cases of no prior pooling and an arbitrary number of prior periods with pooling respectively. Constraints (125) and (126) consider the $H$ type mimicking the $L$ type for a number of periods, in the cases of no prior pooling and an arbitrary number of prior periods with pooling respectively. Last, constraints (127) and (128) consider the possibility of the $H$ type perpetually mimicking the $L$ type, again in the cases of no prior pooling and an arbitrary number of prior periods with pooling respectively.

The first equations in parts $i$ and $ii$ of the Lemma follow from the constraints (122) and the same steps as those for the $L$ type.

The next step is to order the magnitudes of the right-hand sides of constraints (123)-(128). Straightforward comparison shows that the order depends on whether or not

$$(1-\delta)\pi_H(p_H^M)+\delta D_H \geq \pi_H(p_L^M) \tag{129}$$

If (129) is satisfied, then (123)-(124) imply (125)-(128), whereas if (129) is violated, then (123)-(126) are implied by (127)-(128).

The incentive compatibility constraints if the condition is satisfied are thus (124), which reduce to

$$\pi_H(p^*_{K+2,L}) \leq (1-\delta)\pi_H(p_H^M)+\delta D_H \tag{130}$$

for $K=0,...,T-4$, while the equivalent constraint for the first period follows from (123). If the condition is violated, then the relevant incentive compatibility constraints are (128), which reduce to

$$\pi_H(p^*_{K+2,L}) \leq \left(1-\delta^{T-K-2}\right)\pi_H(p_H^M)+\left(\frac{\delta-\delta^{T-K-1}}{1-\delta}\right)D_H-\left(\frac{\delta-\delta^{T-K-2}}{1-\delta}\right)\pi_H(p_L^M) \tag{131}$$

for $K=0,...,T-4$, while the equivalent constraint for the first period follows from (127) ■

## C. Proof of Lemma 23

**Proof:** The incentive compatibility constraints for the $L$ type are given by[10]

$$\sum_{i=1}^{T-1}\delta^{i-1}\pi_L(p_i^*)+\delta^{T-1}\pi_L(p_L^M) \geq \pi_L(p_L^M)+\sum_{i=2}^{T}\delta^{i-1}D_L \tag{132}$$

$$\sum_{i=1}^{T-1}\delta^{i-1}\pi_L(p_i^*)+\delta^{T-1}\pi_L(p_L^M) \geq \sum_{i=1}^{K+1}\delta^{i-1}\pi_L(p_i^*)+\delta^{K+1}\pi_L(p_L^M)+\sum_{i=K+3}^{T}\delta^{i-1}D_L \tag{133}$$

for $K=0,1,...,T-3$. The set of constraints (133), (one for each $K$) compares the equilibrium strategy with a strategy that pools until (and including) period $K+1$ and deviates in period $K+2$. Solving (132) for $\pi_L(p_1^*)$ yields

$$\pi_L(p_1^*) \geq (1-\delta^{T-1})\pi_L(p_L^M)+\sum_{i=2}^{T}\delta^{i-1}D_L-\sum_{i=3}^{T-1}\delta^{i-1}\pi_L(p_i^*)-\delta\pi_L(p_2^*) \tag{134}$$

Evaluating (133) at $K=0$ and rearranging yields

$$\delta\pi_L(p_2^*) \geq \sum_{i=K+3}^{T}\delta^{i-1}D_L+\delta\pi_L(p_L^M)-\sum_{i=3}^{T-1}\delta^{i-1}\pi_L(p_i^*) \tag{135}$$

Substituting this in (134) and rearranging yields

$$\pi_L(p_1^*) \geq (1-\delta)\pi_L(p_L^M)+\delta D_L$$

[10] It is without loss of generality to consider a deviation in period 1, since if there is pooling in periods $s=1,...,t-1$, then the period $t$ problem is essentially the same as that faced in period 1.

For arbitrary $K$, (133) reduces to

$$\sum_{i=K+2}^{T-1} \delta^{i-1}\pi_L(p_i^*) + \delta^{T-1}\pi_L(p_L^M) \geq \delta^{K+1}\pi_L(p_L^M) + \sum_{i=K+3}^{T} \delta^{i-t} D_L \tag{136}$$

Straightforward manipulation yields that this inequality can be rewritten as

$$\sum_{i=0}^{T-K-3} \delta^{i}\pi_L(p_{i+K+2}^*) + \delta^{T-K-2}\pi_L(p_L^M) \geq \pi_L(p_L^M) + \sum_{i=0}^{T-K-3} \delta^{i+1} D_L \tag{137}$$

In particular, this implies that

$$\pi_L(p_{K+2}^*) \geq \left(1-\delta^{T-K-2}\right)\pi_L(p_L^M) + \sum_{i=0}^{T-K-3} \delta^{i+1} D_L - \sum_{i=2}^{T-K-3} \delta^{i}\pi_L(p_{i+K+2}^*) - \delta\pi_L(p_{K+3}^*) \tag{138}$$

But the constraint on $\pi_L(p_{K+3}^*)$ is in turn given by

$$\delta\pi_L(p_{K+3}^*) \geq \left(\delta-\delta^{T-K-2}\right)\pi_L(p_L^M) + \sum_{i=1}^{T-K-3} \delta^{i+1} D_L - \sum_{i=2}^{T-K-3} \delta^{i}\pi_L(p_{i+K+2}^*) \tag{139}$$

Substituting this back in (138) and rearranging, yields the following constraints:

$$\pi_L(p_{K+2}^*) \geq (1-\delta)\pi_L(p_L^M) + \delta D_L \tag{140}$$

for $K = 0, 1, ..., T-3$. Similar steps yield the equivalent constraints for the $H$ type ■

REFERENCES


[1] BAIN, J. S. (1949): A Note on Pricing in Monopoly and Oligopoly, *American Economic Review*, 39(2), 448-464.

[2] BARON, D. P. (1973): Limit Pricing, Potential Entry, and Barriers to Entry, *American Economic Review*, 63(4), 666-674.

[3] BERCK, P. AND J. M. PERLOFF (1988): The Dynamic Annihilation of a Rational Competitive Fringe by a Low-Cost Dominant Firm, *Journal of Economic Dynamics and Control*, 12(4), 659-678.

[4] CARLTON, D. W. AND J. M. PERLOFF (2004): Modern Industrial Organization, *Adison-Wesley.*

[5] DE BONDT, R. R. (1976): Limit Pricing, Uncertain Entry, and the Entry Lag, *Econometrica*, 44(5), 939-946.

[6] FLAHERTY, M. T. (1980): Dynamic Limit Pricing, Barriers to Entry, and Rational Firms, *Journal of Economic Theory*, 23(2), 160-182.

[7] FRIEDMAN, J. W. (1979): On Entry Preventing Behavior and Limit Price Models of Entry, Applied Game Theory, edited by S. J. Brams, A. Schotter and Schwodiauer, Physica Verlag, Wurzburg-Wien, 236-253.

[8] GASKINS, D. W., JR. (1971): Dynamic Limit Pricing: Optimal Pricing Under Threat of Entry, *Journal of Economic Theory*, 3(3), 306-622.

[9] HARRINGTON, J. E. (1986): Limit Pricing when the Potential Entrant is Uncertain of its Cost Function, *Econometrica*, 54(2), 429-437.

[10] JUDD, K. L. AND B. C. PETERSEN (1986): Dynamic Limit Pricing and Internal Finance, *Journal of Economic Theory*, 39(2), 368-399.

[11] JUN, B. H. AND I.-U. PARK (2005): Anti-Limit Pricing, *mimeo.*

[12] KAMIEN, M. I. AND N. L. SCHWARTZ (1971): Limit Pricing and Uncertain Entry, *Econometrica*, 39(3), 441-454.

[13] KAYA, A. (2005): Repeated Signaling Games, *mimeo.*

[14] LEBLANC, G. (1992): Signalling Strength: Limit Pricing and Predatory Pricing, *RAND Journal of Economics*, 23(4), 493-506.

[15] LIPPMAN, S. A. (1980): Optimal Pricing to Retard Entry, *Review of Economic Studies*, 47(4), 723-731.

[16] MADRIGAL, V., T. C. C. TAN AND S. R. DA C. WERLANG (1987): Support Restrictions and Sequential Equilibria, *Journal of Economic Theory*, 43, 329-334.

[17] MATTHEWS, S. A. AND L. J. MIRMAN (1983): Equilibrium Limit Pricing: The Effects of Private Information and Stochastic Demand, *Econometrica*, 51(4), 981-996.

[18] MESTER, L. J. (1992): Perpetual Signalling with Imperfectly Correlated Costs, *Rand Journal of Economics*, 23(4), 548-563.

[19] Milgrom, P. and J. Roberts (1982): Limit Pricing and Entry under Incomplete Information: An Equilibrium Analysis, *Econometrica*, 50(2), 443-459.

[20] Noldeke, G. and E. van Damme (1990): Switching Away from Probability One Beliefs, *mimeo.*

[21] Saloner, G. (1984): Dynamic Equilibrium Limit-Pricing in an Uncertain Environment, *MIT Working Paper* #342.

[22] Selten, R. (1975): A Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games, *International Journal of Game Theory*, 4(1), 25–55.

[23] Tirole, J. (1988): The Theory of Industrial Organization, *MIT Press.*