

# Information-Generated Trade in Asset Markets\*

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## Abstract

Excessive trade volume in asset markets is often associated with the presence of noise or liquidity traders and used to justify assumptions in economic models which depart from those of the No Trade theorem. We show that under no trade-type assumptions, it is possible to construct equilibrium outcomes which involve arbitrarily large trade volume of assets when there are more assets than signals. Under this condition, there exist different asset portfolios which allow the same optimal income transfer across signals. These assets are not redundant because they are needed to achieve Pareto efficiency. We show that this reasoning also applies to more general economies where markets are incomplete and where traders' beliefs do not share a restriction.

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# 1 Introduction

Trading volume in asset markets is thought to be a factor in the adjustment of underlying asset prices (O'Hara pp. 160 [13]) and is of interest to analysts since traders may learn from trading volume just as they learn from asset prices. The large trading volume in stock markets<sup>1</sup> is "perhaps the single most embarrassing fact to the standard finance paradigm" (De Bondt and Thaler [3]). This popular assertion stems from analyzing the implications of no-gains from trade theorems (Milgrom and Stokey [11]). If traders have optimal allocations of assets, they should not want to rebalance their asset holdings if new information arrives and they agree on the interpretation of that information. Large trade volume must then stem from the presence of noise or liquidity traders in environments with asymmetric information (Grossman and Stiglitz [7], Glosten and Milgrom [6]), differences in opinion (Varian [14], Harris and Raviv [9]), overconfidence (De Bondt and Thaler [3], Kyle and Wang [10], Glaser and Weber [5]), churning by fund managers (Dow and Gorton [4], Dasgupta and Prat [2]) or any assumption that deviates from the rational paradigm<sup>2</sup>.

In this paper, we show that one need not deviate from the rational paradigm to generate equilibrium outcomes with large trading volume. We consider a setting where asset trading takes place over multiple periods and assets are long-lived. We distinguish between states and (payoff irrelevant) signals. Traders face state-contingent endowment risk that they can hedge by trading assets. They trade them over time as information about the likelihood of states becomes available. The simplest such model is a three period setting where traders trade these assets before and after arrival of a signal. In the last period, traders use the payoff wealth of their assets to choose optimal consumption of goods on spot markets.

In this setting, traders are selecting a period 0 allocation of assets that transfers income

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<sup>1</sup>For example, daily trading volume on the NYSE was between 4 and 6 million shares in January of 2006. See <http://www.nyse.com/attachment/Vol200602.prn>

<sup>2</sup>Morris [12] implies that since overconfidence and differences in opinion are simply different prior beliefs, they fit under the banner of rationality in the sense of Savage.

appropriately across signals. They then use that income wealth to reallocate their asset holdings taking into account new information. This paper makes the point that there are many different period 0 asset allocation that achieve the same income wealth across signals. This is possible because assets are long-lived and because there are more assets than signals. This in turn means that there are equilibria which involve large retrade of assets from period 0 to period 1.

The paper is organized as follows. The above point is made in the following section and is followed by an example. In section (5), we consider the case of an economy with state-contingent complete asset markets and where traders interpret information in the same way. We show the existence of a no-retrade competitive equilibrium which yields an optimal state-contingent allocation of consumption goods. This equilibrium is the competitive equivalent of Milgrom and Stokey's no-trade theorem in a setting with multiple periods and long-lived assets. We then show that if there are more assets than signals, it is possible to generate equilibria which involve arbitrarily large trading volume of assets. These assets are not redundant because they allow traders to achieve an allocation of consumption goods that is Pareto efficient. In section (6), we show that the same kind of reasoning allows to generate arbitrarily large trade volume in a setting where markets are incomplete and where traders do not necessarily share a restriction on their beliefs. When markets are state-contingent incomplete, equilibria typically involve allocations that are not state contingent and that stem from asset holdings that are typically different across signals. These equilibrium outcomes are typically not Pareto efficient. But if there are more assets than signals, the same reasoning can be used to generate equilibria with large trading volume.

In the paper, we make the distinction between the "no-trade" setting and the more general setting to emphasize that while the reasoning applies in the general case, it also applies when the assumptions of the no-trade theorem hold.

## 2 Preliminary Discussion of Results

Consider an economy where traders are subjective expected utility (SEU) maximizers where uncertainty is represented by a space  $\mathbb{S} = \{1, \dots, S\}$  of futures states. The no-trade theorem in a competitive setting assumes that traders have an optimal allocation of the consumption good relative to state contingent claims:

$$\frac{\pi^h(s)u_1^h(\omega^h(s))}{\pi^h(s')u_1^h(\omega^h(s'))} = \frac{\pi^{h'}(s)u_1^{h'}(\omega^h(s))}{\pi^{h'}(s')u_1^{h'}(\omega^h(s'))} \text{ for all } h, h', s, s' \quad (1)$$

The above equation depicts a situation where there is a unique consumption good, and where  $\pi^h(s)$  denotes trader  $h$ 's subjective probability of state  $s$ ,  $u_1^h$  is the marginal utility of consumption of trader  $h$ , and  $\omega^h \in \mathbb{R}^S$  is her allocation of the consumption good across states. Suppose now that a signal is publicly announced from a signal space  $Z = \{\xi_1, \dots, \xi_L\}$ . The initial allocation is Pareto efficient relative to both state and signal contingent claims if:

$$\frac{\pi^h(s, \xi)u_1^h(\omega^h(s))}{\pi^h(s', \xi')u_1^h(\omega^h(s'))} = \frac{\pi^{h'}(s, \xi)u_1^{h'}(\omega^h(s))}{\pi^{h'}(s', \xi')u_1^{h'}(\omega^h(s'))} \text{ for all } h, h', s, s', \xi, \xi' \quad (2)$$

If traders agree on the meaning of the signal, then the above relationship will hold. Their beliefs have to satisfy the following relationship:

$$\frac{\pi^h(\xi|s)}{\pi^h(\xi|s')} = \frac{\pi^{h'}(\xi|s)}{\pi^{h'}(\xi|s')} \text{ for all } h, h', \xi, s, s' \quad (3)$$

Using Bayes' rule, one can show that equations (1, 3) imply equation (2). This is the competitive form of Milgrom and Stokey's No Trade Theorem. One can argue that the optimality of traders' initial allocation can be achieved on a previous round of asset trades: Suppose now that  $\omega = (\omega^1, \dots, \omega^H)$  is not Pareto efficient. Can traders trade assets to achieve an equilibrium

allocation  $x = (x^1, \dots, x^H)$  that is Pareto efficient relative to both state and signal contingent trades? Suppose that they can trade  $S$  state-contingent claims before the public signal arrival. They each face a budget constraint of the type:

$$q(\xi_0) \cdot z^h(\xi_0) = 0$$

Where  $q(\xi_0)$  is an  $S$ -vector of state-contingent claims prices and where  $z^h(\xi_0)$  is trader  $h$ 's holdings of the  $S$  claims. In equilibrium, prices  $q(\xi_0)$  must clear the market for claims. After this round of trade, traders' equilibrium consumption is  $x^h(s) = \omega^h(s) + z_s^h(\xi_0)$  for each state  $s \in \mathbb{S}$ . This allocation is optimal relative to state-contingent claims since markets are (state-contingent) complete. So if traders' beliefs satisfy equation (3), it must be that equation (2) holds at the equilibrium allocations, meaning:

$$\frac{\pi^h(s, \xi) u_1^h(x^h(s))}{\pi^h(s', \xi') u_1^h(x^h(s'))} = \frac{\pi^{h'}(s, \xi) u_1^{h'}(x^h(s))}{\pi^{h'}(s', \xi') u_1^{h'}(x^h(s'))} \text{ for all } h, h', s, s', \xi, \xi'$$

So, upon public signal arrival, traders will not have an incentive to rebalance their asset holdings  $z^h(\xi_0)$ . Formally, if forward-looking traders face a three-period environment in which they trade these state contingent claims both before and after signal arrival, they face the following budget set:

$$\begin{aligned} q(\xi_0) \cdot z^h(\xi_0) &= 0 \\ q(\xi) \cdot [z^h(\xi) - z^h(\xi_0)] &= 0 \text{ for all } \xi \in Z \end{aligned}$$

And, their equilibrium consumption is  $x^h(s) = \omega^h(s) + z_s^h(\xi)$ . Because signal arrival will not cause them to retrade, there exists an equilibrium  $(x, z, q)$  where  $z^h(\xi_0) = z^h(\xi)$  for all  $h, \xi$ . This is the no-trade theorem embedded in a dynamic framework: beginning with an endowment level that

is not necessarily Pareto efficient, traders whose likelihood ratios of beliefs are equated will trade to an equilibrium allocation that is Pareto efficient relative to state-contingent trades. Because they interpret signals in similar ways, signal arrival will not cause them to rebalance their asset holdings and the equilibrium outcome is Pareto efficient relative to state and signal-contingent trades.

This no-retrade equilibrium outcome involves zero trade volume from the initial node  $\xi_0$  to any node  $\xi \in Z$ . The key feature is that traders are using relative prices of assets  $q(\xi)_{\xi \in Z}$  to transfer income to signals in a way that is optimal. Trader  $h$ 's income in signal  $\xi \in Z$  is:

$$w^h(\xi) = q(\xi) \cdot z^h(\xi_0)$$

Suppose now that in addition to state-completeness of markets, there are at least as many state-contingent claims as there are signals. Under this condition, there are many asset trades  $y^h(\xi_0)$  that yield the income stream  $w^h(\xi)_{\xi \in Z}$ . Indeed, consider the system of equations:

$$q(\xi) \cdot y^h(\xi_0) = w^h(\xi) \text{ for all } \xi \in Z \quad (4)$$

With  $L$  equations and  $S > L$  unknowns  $y^h(\xi_0)$ , there exists a space of solutions of dimension  $(S - L)$  in the space  $\mathbb{R}^S$  of possible asset holdings for trader  $h$ . Given an arbitrary choice of  $y^h(\xi_0)$  for all traders  $h > 1$  that satisfies equation (4), and choosing trader 1's asset holdings  $y^1$  to clear the asset markets, one can show that  $(x, z', q)$  is also an equilibrium where  $(z')^h = (y^h(\xi_0), z^h(\xi)_{\xi \in Z})$ .  $(x, z', q)$  is an equilibrium because all asset and good markets clear, optimality conditions are satisfied and budget constraints hold ( $q(\xi_0) \cdot y^h(\xi_0) = 0$  follows from equation (4) and the fact that prices  $q(\xi)_{\xi \in \xi_0 \cup Z}$  satisfy no-arbitrage).

So there exists an  $(H - 1)(S - L)$  space of equilibria  $(x, z', q)$  which involve arbitrarily large

trade of assets from the initial node  $\xi_0$  to any node  $\xi \in Z$ . All of these equilibria are Pareto efficient because they yield an equilibrium consumption which is the "no-trade" consumption level. The following example illustrates this point.

### 3 An Example

Consider an economy with two traders with log utility over one consumption good. There are three states of the world and two signals denoted  $\{\xi_1, \xi_2\}$ . The common prior on states is  $(1/3, 1/3, 1/3)$ . The signals have equal probability and beliefs conditioning on signals are  $(1/6, 1/3, 1/2)$  for signal  $\xi_1$  and  $(1/2, 1/3, 1/6)$  for signal  $\xi_2$ . State contingent endowments are  $(1, 1/2, 0)$  for trader one and  $(0, 1/2, 1)$  for trader two.

The complete markets equilibrium consumption for each trader is state-contingent and is equal to  $(1/2, 1/2, 1/2)$  and the price of state-contingent assets is  $(1/3, 1/3, 1/3)$  in period 0, and  $(1/3, 2/3, 1)$  given signal  $\xi_1$  and  $(1, 2/3, 1/3)$  given signal  $\xi_2$ . Because markets are state-contingent complete, traders' equilibrium allocations of assets  $z^h$  in period 0 remain equilibrium allocations at each signal. It is straightforward to compute that:

$$\begin{aligned} z^1(\xi_0) &= z^1(\xi) = (-1/2, 0, 1/2) \text{ for each } \xi \in Z \\ z^2(\xi_0) &= z^2(\xi) = (1/2, 0, -1/2) \text{ for each } \xi \in Z \end{aligned}$$

If  $(x, q)$  denote equilibrium consumption and prices of assets, respectively then  $(x, z, q)$  is a competitive equilibrium involving no retrade of assets from period 0 to period 1. Consider however

the following alternative allocation of assets:

$$\begin{aligned} y^1(\xi_0) &= (\eta, -1 - 2\eta, 1 + \eta) \\ y^2(\xi_0) &= (-\eta, 1 + 2\eta, -1 - \eta) \end{aligned}$$

Let  $(z')^h = (y^h(\xi_0), z^h(\xi)_{\xi \in Z})$  for  $h = 1, 2$ . One can check that  $(x, z', q)$  is also a competitive equilibrium for any  $\eta \in \mathbb{R}$ . Since  $\eta$  is arbitrary, it is possible to generate a competitive equilibrium that involves arbitrarily large trading volume for any of the state-contingent assets. This example demonstrates that what traders are doing at time 0 is selecting asset holdings that moves wealth appropriately across signals. Because there are more states than signals this does not pin down time 0 asset trades.

We elaborate on this result in section (5) and generalize the result to economies where traders do not share a belief restriction and where markets are not state-contingent complete. The reasoning above relies on the fact that the equilibrium matrix  $q(\xi)_{\xi \in Z}$  has full rank  $L$ . We show the genericity of the full rank condition in section (6) and also explore implications for existence of a fully revealing rational expectations equilibrium.

## 4 Description of the Economy

There are three time periods. We denote period zero by the node  $\xi_0$ . In period one, a signal is revealed to traders. The signal space is denoted  $Z = \{\xi_1, \dots, \xi_L\}$ . A typical element of that space is  $\xi \in Z$ . In period two, a state of the world is revealed. We denote the state space by  $\mathbb{S} = \{1, \dots, S\}$ . A typical element of the state space is  $s \in \mathbb{S}$ . There are  $H$  agents with preferences for period 2

consumption (there are  $K$  consumption goods). These preferences have an expected utility form.

$$U^h(x^h) = \sum_{s,\xi} \pi^h(s,\xi) u^h(x^h(s,\xi)) = \mathbb{E}^h(u^h) \quad (5)$$

$\pi^h$  is a probability distribution on the joint space  $Z \times \mathbb{S}$  and represents trader  $h$ 's beliefs:  $\pi^h(s,\xi)$  is trader  $h$ 's belief that the state/signal  $(s,\xi)$  will occur.  $x^h(s,\xi) \in \mathbb{R}^K$  is trader  $h$ 's consumption if signal  $\xi$  and state  $s$  occur. Traders have an endowment of the consumption goods which is independent of any signal. So  $\omega^h(s) \in \mathbb{R}_{++}^K$  is trader  $h$ 's endowment of the goods in state  $s \in \mathbb{S}$  for all signals  $\xi \in Z$ . Let  $\omega^h \in \mathbb{R}_{++}^{SK}$  denote trader  $h$ 's overall endowment. Traders can hedge against their endowment risk by trading  $J$  different assets. They are issued in period 0 and pay off in period 2. The payoffs of these assets are independent of any signal. So, let  $V_k^j(s)$  be the payoff of asset  $j \in J$  in state  $s \in \mathbb{S}$  in terms of consumption good  $k$  for all  $\xi \in Z$ . Finally, let  $\pi = (\pi^h)_{h \in H}, \omega = (\omega^h)_{h \in H}, u = (u^h)_{h \in H}$ .

**Definition 1** *The economy  $\mathcal{E}(u, \pi, \omega, V)$  consists of the description of preferences, endowments and the payoff matrix  $V$ .*

**Assumption - U0** (i)  $U^h : \mathbb{R}^{SLK} \rightarrow \mathbb{R}$  is continuous on  $\mathbb{R}_+^{SLK}$  and  $\mathcal{C}^\infty$  on  $\mathbb{R}_{++}^{SLK}$  (ii)  $U^h(x) = \{x' \in \mathbb{R}_+^{SLK} : U^h(x') \geq U^h(x)\} \subset \mathbb{R}_{++}^{SLK}$  for all  $x \in \mathbb{R}_{++}^{SLK}$  (iii)  $\frac{\partial U^h(x)}{\partial x_{slk}} > 0$  for all  $x \in \mathbb{R}_{++}^{SLK}$  (iv)  $\sum_{j=1}^{SLK} \sum_{k=1}^{SLK} h_j h_k \frac{\partial^2 U^h(x)}{\partial x_j \partial x_k} < 0$  for all  $h \in \mathbb{R}^{SLK}, h \neq 0$  such that  $\sum_{j=1}^{SLK} h_j \frac{\partial U^h(x)}{\partial x_{slk}} = 0$ .

**Assumption - U1**  $U^h$  as the form (5),  $u^h$  is strictly increasing, continuous on  $\mathbb{R}_+$ , infinitely differentiable on  $\mathbb{R}_{++}$ . Also,  $u^{h''} < 0$ ,  $\lim_{x \rightarrow 0} u^h(x) = +\infty$  and  $\pi^h(s,\xi) > 0$  for all  $s,\xi$ .

If preferences have the form (5), and satisfy assumption U1 then  $U^h$  satisfies the assumption U0 of smooth preferences of Debreu ([1]). These assumptions on preferences will be useful in proving results in appendix 2. In the following definition,  $q(\xi)$  is the  $J$ -vector of asset prices at

nodes  $\xi \in \{\xi_0\} \cup Z$  and  $z^h(\xi)$  is the  $J$ -vector of asset holdings of trader  $h$  at nodes  $\xi \in \{\xi_0\} \cup Z$ . The spot market prices at node  $(s, \xi)$  is the  $K$ -vector  $p(s, \xi)$ .

**Definition 2** *Given an economy  $\mathcal{E}(u, \pi, \omega, V)$ , an equilibrium  $(x, z, p, q)$  is a collection of equilibrium consumption, net trades of assets, spot prices and asset prices, respectively such that:*

1. *Given prices  $(p, q)$ , traders choose  $(x^h, z^h)$  to maximize  $U^h(x^h)$  subject to the budget constraints:*

$$q(\xi_0) \cdot z^h(\xi_0) = 0$$

$$q(\xi) \cdot [z^h(\xi) - z^h(\xi_0)] = 0 \text{ for all } \xi \in Z$$

$$p(s, \xi) \cdot (x^h(s, \xi) - \omega_s^h) = p(s, \xi) \cdot V(s) \cdot z^h(\xi) \text{ for all } s \in \mathbb{S} \text{ and } \xi \in Z$$

2. *Markets clear:*

$$\sum_{h \in H} z^h(\xi) = 0 \text{ for all } \xi \in \{\xi_0\} \cup Z$$

$$\sum_{h \in H} x_l^h(s, \xi) - \omega_l^h(s) = 0 \text{ for all } s \in S, \xi \in Z \text{ and } l \in \{1, \dots, K\}$$

## 5 Inessential Trade under No-Trade Assumptions

All proofs for this section are in appendix 1.

**State-Completeness (SC)** There are  $J = S$  assets where the payoff of asset  $s$  is one unit of the first consumption good in state  $s$  and zero otherwise.

**Signal-Completeness (SigC)** There are  $J \geq L$  assets.

**Belief Restriction 1 (BR1)** Agents beliefs satisfy:

$$\frac{\pi^h(\xi|s)}{\pi^h(\xi|s')} = \frac{\pi^{h'}(\xi|s)}{\pi^{h'}(\xi|s')} \text{ for all } h, h', \xi, s, s'$$

**Definition 3** *Given an economy  $\mathcal{E}(u, \pi, \omega, V)$ , an equilibrium  $(x, z, p, q)$  involves no re-trade if  $z(\xi_0) = z(\xi)$  for all  $\xi \in Z$ .*

Under assumptions (SC, BR1), one can show that there exists an equilibrium which involves no-retrade and that this equilibrium is Pareto efficient (see lemma (14) in appendix 1), but one cannot rule out the existence of other equilibria which involve retrade of assets. This is not the case in Milgrom and Stokey's original framework. In a competitive setting, they show that if traders begin with an allocation of consumption goods that is Pareto efficient relative to state-contingent trades then public information arrival will not cause them to trade any state-contingent assets, as long as their beliefs satisfy (BR1). So state-contingent Pareto efficiency implies Pareto efficiency in the usual sense (in this case, relative to both state and signal contingent trade) if traders likelihood ratios are equal (condition BR1). The only possible equilibrium is the no-trade equilibrium.

The multiperiod equilibrium concept poses problems not found in Milgrom and Stokey's original paper. Our model allows traders to achieve an equilibrium allocation of assets across signals whose payoff wealth is used on a second round of asset trades which is contingent on the public signal. The payoff wealth of this final allocation of assets is then used to achieve a Pareto efficient allocation of equilibrium consumption on spot markets once the true state is revealed. Given an equilibrium  $(x, z, p, q)$ , traders' payoff wealth across signals is:

$$q(\xi) \cdot z^h(\xi_0) \text{ for each signal } \xi \in Z$$

This wealth is used to rebalance traders' asset holdings conditioning on the information obtained from the signal  $\xi \in Z$ . Once the true state  $s \in \mathbb{S}$  of the world is revealed, their payoff wealth is  $p(s, \xi) \cdot V(s) \cdot z^h(\xi)$  for each signal  $\xi \in Z$ . They use this wealth, along with their endowment wealth, to achieve optimum consumption of goods on the spot market. The difference with the Milgrom and Stokey setup is that there are different ways to achieve the same payoff wealth across signals. Also, once the final allocation of assets is achieved for a given signal, its value is affected by spot prices. These issues do not arise in the Milgrom and Stokey setup.

In their setup, the no-trade equilibrium exists, regardless of the market structure but here the market structure matters. If markets are not (SC) then equilibrium outcomes will typically not achieve efficiency. Also, markets may be (SC) in different ways. For example, we assume that assets pay off only in terms of the first consumption good. Allocative efficiency is achieved on spot markets once the state is revealed. But many other configurations of asset payoffs achieve state-completeness of markets, and this may be an endogenous condition, just as completeness in the asset market is endogenous in a Radner equilibrium when assets pay off in units of many consumption goods. To each state-complete asset structure corresponds a different no retrade equilibrium (the final consumption is the same but the way it is achieved varies with the asset structure). The existence of equilibria involving retrade of assets when markets are (SC) depends on whether markets are signal contingent complete (SigC) and whether the matrix of equilibrium prices  $q(\xi)_{\xi \in Z}$  has full rank  $L$ . Under this additional condition, an outcome  $(x, z, p, q)$  must be Pareto efficient. This is because traders can use intermediate equilibrium prices to transfer income across nodes without restriction.

**Belief Restriction 2 (BR2)** Agents beliefs satisfy:

$$\pi^h(\xi|s) = \pi^{h'}(\xi|s) \text{ for all } h, h', \xi, s$$

If outcomes are Pareto efficient and if traders' beliefs satisfy (BR2) then the averaged-out state-contingent allocation  $y$  is feasible and is strictly preferred to  $x$  because of traders' risk aversion:

$$y^h(s) = \sum_{\xi} \pi^h(\xi|s)x^h(s, \xi) = \mathbb{E}^h(x^h(s, \cdot)|s) \text{ for all } h, s$$

But since  $x$  is Pareto efficient, this must be a contradiction. Hence  $x$  is state-contingent. Since traders are not transferring income across signals in a way that affects their consumption, we say that allocation  $x$  was achieved without essential trade of assets.

**Definition 4** *Given an economy  $\mathcal{E}(u, \pi, \omega, V)$ , an equilibrium  $(x, z, p, q)$  involves essential asset trades if there exists a trader  $h$  such that  $U^h(x^h) \neq U^h(y^h)$  for some  $h$ .*

Notice that in the one-good case, an inessential trade equilibrium is necessarily a no-trade equilibrium. This is because for all traders  $h : x^h(s, \xi) = \omega_s^h + z_s^h(\xi)$ . But since  $x^h$  is state-contingent, we get  $x^h(s, \xi) = x^h(s, \xi')$  for all  $\xi, \xi'$  and so:  $z_s^h(\xi) = z_s^h(\xi')$  for all  $\xi, \xi'$ . In the case of multiple goods, it may be that  $z_s^h(\xi) \neq z_s^h(\xi')$  but the wealth payoff  $p(s, \xi) \cdot V(s) \cdot z^h(\xi)$  at node  $\xi$  is equal to that in node  $\xi'$  for all  $s$ . In this case, optimal consumption may still be state-contingent even though the asset holdings across signals are different: this is because spot prices across different signal/states may counteract the effects of different asset holdings across signals and a trader would obtain the same wealth payoff of assets.

**Theorem 5** *Under assumptions (SC, SigC, BR2), any equilibrium  $(x, z, p, q)$  with the property that the  $L \times J$  matrix  $q(\xi)_{\xi \in Z}$  has rank  $L$  cannot involve essential trade of assets.*

For a given outcome  $(x, z, p, q)$ , the fact that  $q(\xi)_{\xi \in Z}$  has full rank  $L$  is an equilibrium condition. Since the space of  $L \times J$  matrices that has rank less than  $L$  is a lower-dimensional set, one expects that the rank condition on prices holds for a large set of parameters. If markets are both state and

signal contingent complete, and traders beliefs satisfy (BR1), then the no-retrade equilibrium's asset prices  $q(\xi)_{\xi \in Z}$  satisfies the rank condition as long as beliefs satisfy the following minimal assumption on beliefs (lemma (15) in appendix 1):

**Rank Condition on Beliefs (RC)** For all  $h$ , the  $L \times S$  matrix of beliefs  $\pi^h$  has full rank  $L$ .

Given an equilibrium  $(x, z, p, q)$  such that the matrix  $q(\xi)_{\xi \in Z}$  has rank  $L$ , then one can construct many other equilibria  $(x, z', p, q)$  where  $z' \neq z$ . For each trader  $h$ , there is an  $(J - L)$  dimensional subset of  $(z')^h(\xi_0)$  such that:

$$q(\xi) \cdot \left[ z^h(\xi) - (z^h)'(\xi_0) \right] = 0 \text{ for all } \xi \in Z$$

This can be done for all  $h > 1$ . Set  $(z')^1(\xi_0)$  so that all markets clear. Finally, through no-arbitrage, one can show that  $q(\xi_0) \cdot z^h(\xi_0) = 0$  for all  $h$ . This leads us to an  $(H - 1)(J - 1)$  dimensional space of equilibria involving possibly large trade volume. As a special case, under assumptions which imply the existence of an equilibrium with no-retrade of assets, we have the following

**Theorem 6** *Under assumptions (SC, SigC, RC, BR1), there exists an  $(H - 1)(J - L)$  dimensional space of distinct inessential equilibria.*

In general, if assumption (SigC) holds and there exists an equilibrium  $(x, z, p, q)$  where  $q(\xi)_{\xi \in Z}$  has rank  $L$ , then one can construct a  $(H - 1)(J - L)$  dimensional space of equilibria with the property that period 0 asset holdings are different from  $z^h(\xi_0)$  but achieve the same wealth  $q(\xi)z^h(\xi_0)$  in node  $\xi \in Z$ . There are different ways of achieving the same optimal level of wealth across nodes when the number of assets is no less than the number of signals (lemma (16) in appendix 1).

**Definition 7** Given asset trades  $z \in \mathbb{R}^{HJ(L+1)}$ , trade volume for asset  $j$  from node  $\xi_0$  to node  $\xi$  is a scalar:

$$V_j(z, \xi) = \frac{1}{2H} \sum_h |z_j^h(\xi_0) - z_j^h(\xi)|$$

Under the assumptions that imply the existence of a no-retrade equilibrium and for any level of volume of trade  $N$ , one can construct an equilibrium from the  $(H - 1)(J - L)$  space of inessential equilibria where the amount of retrade exceeds  $N$ .

**Corollary 8** to theorem(6). Under assumptions (SC,SigC,RC,BR1), for any  $N > 0$ , there exists an inessential equilibrium such that  $V_j(z, \xi) > N$  for all  $j, \xi$ .

The construction exploits the fact that some assets are redundant relative to the number of signals: there are many different asset portfolios that achieve some desired wealth level  $q(\xi) \cdot z^h(\xi_0)$  across nodes  $\xi \in Z$ , however all assets are necessary to achieve Pareto efficiency. For example, if markets are (SC,SigC) and the rank condition on asset prices holds, then outcomes are Pareto efficient although there are more assets than necessary to hedge against signal risk.

## 6 Trade and Rational Expectations

In the previous section, we have shown that the no-retrade equilibrium has the desired rank property for prices. This equilibrium can be used to construct other equilibria that achieve the same level of consumption but with different portfolios of assets. Corollary (8) to theorem (6) in fact shows that one can find equilibria which involve arbitrarily large retrade of assets. Because of the particular structure of prices in the no-retrade equilibrium, it is easy to link the rank of the matrix  $q(\xi)_{\xi \in Z}$  directly to traders' beliefs. The only property however that is needed to construct equilibria with large trade volume is assumption (SigC) along with the rank condition on prices. This section explores this more general question. We show genericity of the rank condition in

the space of beliefs. For simplicity, we look at the case of one consumption good ( $K = 1$ ). Let  $\Psi = \mathbb{R}_{++}^{(SL-1)H}$  denote the space of traders' beliefs.

**Theorem 9** *Given an economy  $\mathcal{E}(u, \cdot, \omega, V)$  under assumption (SigC), there exists an open set with null complement  $\Psi_0 \subset \Psi$  of beliefs such that the economy  $\mathcal{E}(u, \pi, \omega, V)$  admits an equilibrium  $(x, z, q)$  whenever  $\pi \in \Psi_0$ . The matrix  $q(\xi)_{\xi \in Z}$  has full rank  $L$ .*

**Proof.** Appendix 2. ■

**Corollary 10** *to theorem (9). Under assumption (SigC), given  $N > 0$  and for each  $\pi \in \Psi_0$ , the economy  $\mathcal{E}(u, \pi, \omega, V)$  admits an equilibrium  $(x, z, q)$  such that  $V_j(z, \xi) > N$  for all  $j, \xi$ .*

**Proof.** Follows from theorem (9) and the proof of corollary (8) to theorem (6). ■

Notice that if the prices  $q(\xi)$  are a multiple of prices  $q(\xi')$  for some nodes  $\xi$  and  $\xi'$  in  $Z$ , then necessarily the matrix  $q(\xi)_{\xi \in Z}$  cannot have full rank  $L$  when  $J \geq L$ . So theorem (9) also implies genericity of fully revealing asset prices when information about the signal space is non-public or asymmetric. Suppose that traders' original information partition of the signal space is some set  $Z^h$ . Traders have rational expectations if they can invert the price function and recover any information about the underlying signal. If  $(x, z, q)$  is an equilibrium for a given economy, let  $I$  denote the mapping from signals to prices:

$$I : Z \rightarrow \mathbb{R}^{SJ}$$

$$I(\xi) = q(\xi) \text{ for } \xi \in Z$$

Define the information partition revealed by prices as  $Z(q)$  where the equivalence class of signal  $\xi$  is  $Z(q)(\xi) = I^{-1}(q(\xi))$ . We say that traders have rational expectations if they know the mapping  $I$ . Trader  $h$ 's equilibrium information partition is the set  $Z^h(q) = Z^h \cap Z(q)$ , the equilibrium

information partition which incorporates all information available in  $Z^h$  and in equilibrium prices  $q$ . A rational expectations equilibrium is an equilibrium where traders' decisions are measurable with respect to their equilibrium information partition.

**Definition 11** *Given an economy  $\mathcal{E}(u, \pi, \omega, V)$ , an rational expectations equilibrium  $(x, z, p, q)$  is a collection of equilibrium consumption, net trades of assets and asset prices, respectively such that  $(x, z, p, q)$  is an equilibrium where trades  $z^h(\xi)$  are measurable with respect to the equilibrium information partition  $Z^h(q)$  for all  $h \in H$ .*

An equilibrium is fully revealing if  $Z^h(q)[\xi] = \{\xi\}$  for all  $\xi \in Z$  and all  $h \in H$ .

**Corollary 12** *to theorem (9). Given an economy  $\mathcal{E}(u, \cdot, \omega, V)$  under assumption (SigC), there exists an open set with null complement  $\Psi_0 \subset \Psi$  of beliefs such that all equilibria of the economy  $\mathcal{E}(u, \pi, \omega, V)$  are fully revealing rational expectations equilibria whenever  $\pi \in \Psi_0$ .*

## 7 Conclusion

In this paper, we show that one can exploit the structure of economies with multiple periods and long-lived assets to generate equilibria which involve large trading volume of assets.

We first show this in a setting where the assumptions of the no-trade theorem apply. The result relies on asset redundancy relative to the signal space: there are many asset trades that achieve the same optimal wealth level across signals. These assets however are not redundant relative to the state space and hence are needed to achieve an equilibrium allocation that is Pareto efficient. The construction of these equilibria with large trade volume requires that the matrix of asset prices across signals has full rank. In the "no-trade" setting, this condition holds as long as traders' beliefs satisfy a rank condition.

We also consider a more general case where markets are state-contingent incomplete and where traders beliefs do not necessarily share a restriction. In these economies, outcomes are typically Pareto inefficient and typically involve equilibrium consumption levels that are not state-contingent. In this more general setting, we show that one can also generate equilibria that involve arbitrarily large trading volume of assets as long as there are more assets than signals and the full rank condition on asset prices is satisfied. In this more general setting, it is difficult to directly link equilibrium asset prices to traders' beliefs. In the one-good case, we show the genericity of this full rank condition in that space. The genericity result can be used to show generic existence of a fully revealing rational expectations equilibrium.

## 8 Appendix 1

**Definition 13** Given an economy  $\mathcal{E}(u, \pi, \omega, V)$ , a Radner equilibrium  $(x, z, p, q)$  is a collection of equilibrium consumption, net trades of assets and asset prices, respectively such that:

1. Given prices  $(p, q)$ , traders choose  $(x^h, z^h)$  to maximize  $U^h(x^h)$  subject to the budget constraints:

$$q(\xi_0) \cdot z^h(\xi_0) = 0$$

$$p(s) \cdot (x^h(s) - \omega^h(s)) = p(s) \cdot V(s) \cdot z^h(\xi_0) \text{ for all } s \in \mathbb{S}$$

2. Markets clear:

$$\begin{aligned} \sum_{h \in H} z^h(\xi_0) &= 0 \\ \sum_{h \in H} x_l^h(s) - \omega_l^h(s) &= 0 \text{ for all } s \in S, l \in \{1, \dots, L\} \text{ and } \xi \in Z \end{aligned}$$

**Lemma 14** Under assumptions (SC, BR1), there exists an equilibrium  $(x, z, p, q)$  which involves no retrade in asset markets and the equilibrium allocation  $x$  is Pareto efficient.

**Proof.** Let  $(x, z, p, q)$  be a Radner equilibrium. We wish to construct a (three-period) equilibrium  $(x', z', p', q')$ . Set:

$$x'(s, \xi) = x(s) \text{ for all } s, \xi \in Z$$

$$z'(\xi) = z(\xi) \text{ for all } \xi \in Z_0$$

Finally, set  $q'(\xi_0) = q(\xi_0)$  and:

$$\frac{q'_s(\xi)}{q'_{s'}(\xi)} = \frac{q_s(\xi_0)}{q_{s'}(\xi_0)} r(\xi, s, s') \text{ for all } \xi, s, s'$$

Using optimality conditions, one can check that  $(x', z', p', q')$  is an equilibrium which involves no-retrade. ■

**Lemma 15** *Under assumptions (SC, SigC, RC, BR1), there exists an equilibrium  $(x, z, p, q)$  which involves no retrade in asset markets and the  $L \times J$  matrix  $q(\xi)_{\xi \in Z}$  has rank  $L$ .*

**Proof.** Lemma (15). The no-retrade equilibrium  $(x, z, p, q)$  has the following asset prices:

$$\frac{q_s(\xi)}{q_{s'}(\xi)} = \frac{q_s(\xi_0)}{q_{s'}(\xi_0)} r(\xi, s, s') \text{ for all } \xi, s, s'$$

Normalize prices  $q_1(\xi) = 0$  for all  $\xi \in Z_0$ . So the  $L \times S$  matrix  $q(\xi)_{\xi \in Z}$  is:

$$\begin{bmatrix} q_1(\xi_0)r(\xi_1, 1, 1) & \cdots & q_S(\xi_0)r(\xi_1, S, 1) \\ \vdots & \ddots & \vdots \\ q_1(\xi_0)r(\xi_L, 1, 1) & \cdots & q_S(\xi_0)r(\xi_L, S, 1) \end{bmatrix}$$

This matrix has the same rank as:

$$\begin{bmatrix} r(\xi_1, 1, 1) & \cdots & r(\xi_1, S, 1) \\ \vdots & \ddots & \vdots \\ r(\xi_L, 1, 1) & \cdots & r(\xi_L, S, 1) \end{bmatrix}$$

This matrix has the same rank as the  $S \times L$  matrix of beliefs  $\pi^h$  which has full rank  $L$  by assumption (RC). ■

**Lemma 16** *Under assumptions (SigC), if an equilibrium  $(x, z, p, q)$  exists such that the  $L \times J$  matrix  $q(\xi)_{\xi \in Z}$  has rank  $L$  then there exists an  $(H - 1)(J - L)$  dimensional space of equilibria  $(x, y, p, q)$  where  $y = (z'(\xi_0), z(\xi)_{\xi \in Z})$  and where  $z'(\xi_0) \neq z(\xi_0)$ .*

**Proof.** Consider the following set:

$$Y^h = \{y^h(\xi_0) \in \mathbb{R}^J : q(\xi) \cdot y^h(\xi_0) = q(\xi) \cdot z^h(\xi) \text{ for all } \xi \in Z\}$$

Let  $Y^{-1} = Y^2 \times \dots \times Y^H$ . Choose  $(y_2(\xi_0), \dots, y_H(\xi_0)) \in Y^{-1}$  and let:

$$y^1(\xi_0) = - \sum_{h>1} y_h(\xi_0)$$

Let  $z' = (y(\xi_0), z(\xi))$ . Note that  $(x, z', p, q)$  is an equilibrium. Markets clear and all optimality conditions are satisfied. The only thing to check is that  $q(\xi_0) \cdot y^h(\xi_0) = 0$ . By no-arbitrage, there is a vector  $\lambda \in \mathbb{R}_{++}^L$  such that:

$$q_j(\xi_0) = \sum_{\xi} \lambda(\xi) q_j(\xi) \text{ for all } j$$

We know that:

$$q(\xi) \cdot y^h(\xi_0) = q(\xi) \cdot z^h(\xi)$$

So:

$$\sum_{\xi \in Z} \lambda(\xi) q(\xi) \cdot y^h(\xi_0) = \sum_{\xi \in Z} \lambda(\xi) q(\xi) \cdot z^h(\xi)$$

So:

$$q(\xi_0) \cdot y^h(\xi_0) = \sum_{\xi \in Z} \lambda(\xi) q(\xi) \cdot z^h(\xi)$$

But we also have:

$$q(\xi_0) \cdot z^h(\xi_0) = \sum_{\xi \in Z} \lambda(\xi) q(\xi) \cdot z^h(\xi)$$

So:

$$q(\xi_0) \cdot y^h(\xi_0) = q(\xi_0) \cdot z^h(\xi_0) = 0$$

Finally, note that the space  $Y^h$  has dimension  $J - L$  (since  $q(\xi)_{\xi \in Z}$  has rank  $L$ ). So  $Y^{-1}$  has dimension  $(H - 1)(J - L)$ . The lemma follows. ■

**Proof.** Theorem (6). This follows from lemmas (15, 16). ■

**Proof.** Corollary (8). Under assumptions (SC,SigC,RC,BR1), for any  $N > 0$ , there exists an inessential equilibrium such that  $V_j(z, \xi) > N$  for all  $j, \xi$ .

$$V_j(z, \xi) = \frac{1}{2H} \sum_h |z_j^h(\xi_0) - z_j^h(\xi)|$$

Let  $(x, z, p, q)$  be the no-retrade equilibrium. Choose any  $t^h(N', \xi_0) \in Null [q(\xi)_{\xi \in Z}] \cap \{y^h \in \mathbb{R}^S : |y_s^h| > N'\}$ . This exists for all  $N'$  because the  $Null [q(\xi)_{\xi \in Z}]$  is a subspace of  $\mathbb{R}^S$  which doesn't interest the axes (this is because all prices are positive). Set  $y^h(N', \xi_0) = t^h(N', \xi_0) + z^h(\xi_0)$  for all  $h > 1$  and set  $y^1(N', \xi_0) = -\sum_{h>1} y_h(N', \xi_0)$ . Let  $z'(N') = (y(N', \xi_0), z(\xi))$ . Finally, note that  $(x, z'(N'), p, q)$  is an equilibrium and that for all  $N > 0$ , there is an  $N' > 0$  such that  $V_j(z'(N'), \xi) > N$  for all  $\xi, j$ . ■

**Lemma 17** *Under assumptions (SC,SigC), any equilibrium  $(x, z, p, q)$  with the property that the  $L \times J$  matrix  $q(\xi)_{\xi \in Z}$  has rank  $L$  has a Pareto optimal allocation.*

**Proof.** In equilibrium, there exist a vector of multipliers  $\lambda^h \in \mathbb{R}_{++}^{L+1}$  associated with the  $(L + 1)$  budget constraints of each trader  $h$  such that the following NA relationship holds:

$$q_s(\xi_0) = \sum_{\xi} \frac{\lambda^h(\xi)}{\lambda^h(\xi_0)} q_s(\xi)$$

By assumption (SigC) and since  $q(\xi)_{\xi \in Z}$  has full rank, the solution to the above system of

equations is unique:

$$\frac{\lambda^h(\xi)}{\lambda^h(\xi_0)} = \frac{\lambda^{h'}(\xi)}{\lambda^{h'}(\xi_0)} \text{ for all } h, h' \quad (6)$$

Set:

$$\mu(\xi_0) = \frac{\lambda^h(\xi)}{\lambda^{h'}(\xi)} = \frac{\lambda^h(\xi_0)}{\lambda^{h'}(\xi_0)}$$

Also, assumption (SC) implies that for all  $\xi \in Z$  :

$$\lambda^h(\xi)q_s(\xi) = \pi^h(s, \xi)u_l^h(x^h(s, \xi))\frac{p_1(s, \xi)}{p_l(s, \xi)}$$

This equation implies that:

$$\mu(\xi_0) = \frac{\pi^h(s, \xi)u_l^h(x^h(s, \xi))}{\pi^{h'}(s, \xi)u_l^{h'}(x^{h'}(s, \xi))} \text{ for all } h, h', s, \xi$$

So:

$$\frac{\pi^h(s, \xi)u_l^h(x^h(s, \xi))}{\pi^{h'}(s, \xi)u_l^{h'}(x^{h'}(s, \xi))} = \frac{\pi^h(s', \xi')u_l^h(x^h(s', \xi'))}{\pi^{h'}(s', \xi')u_l^{h'}(x^{h'}(s', \xi'))}$$

Or:

$$\frac{\pi^h(s, \xi)u_l^h(x^h(s, \xi))}{\pi^h(s', \xi')u_l^h(x^h(s', \xi'))} = \frac{\pi^{h'}(s, \xi)u_l^{h'}(x^{h'}(s, \xi))}{\pi^{h'}(s', \xi')u_l^{h'}(x^{h'}(s', \xi'))}$$

This equation implies that  $x$  is Pareto optimal. ■

**Proof.** Theorem (5). Consider an equilibrium  $(x, z, p, q)$  which involves essential retrade, meaning, there exists a trader  $h$  such that  $U^h(x^h) \neq U^h(y^h)$  where  $y$  is the following state contingent trade:

$$y^h(s) = \sum_{\xi} \pi(\xi|s)x^h(s, \xi)$$

Here,  $\pi^h(\xi|s) = \pi(\xi|s)$  for all  $h$  by assumption (BR2). Notice that  $y$  is feasible. Also,

$$\begin{aligned}
U^h(y^h) &= \sum_{s,\xi} \pi^h(s, \xi) u^h(y^h(s)) \\
&= \sum_s \pi^h(s) u^h(y^h(s)) \\
&= \sum_s \pi^h(s) u^h \left( \sum_{\xi} \pi(\xi|s) x^h(s, \xi) \right) \\
&\geq \sum_s \pi^h(s) \left[ \sum_{\xi} \pi(\xi|s) u^h(x^h(s, \xi)) \right] \\
&= \sum_{s,\xi} \pi^h(s, \xi) u^h(x^h(s, \xi))
\end{aligned}$$

The inequality must be strict for some trader. Since retrade is essential, we know that the above holds with a strict inequality for some trader  $h$ . So the allocation  $y$  Pareto dominates  $x$ . But  $x$  is Pareto optimal, by lemma (17), a contradiction. ■

## 9 Appendix 2

To prove theorem (9), we need some preliminary results. In this section, we assume that  $K = 1$  (one good economy).

### 9.1 Budget Sets

Given an equilibrium  $(x, z, q)$ , define the budget set:

$$B(q, \omega, V) = \left\{ x \in \mathbb{R}^{SL} \left| \begin{array}{l} q(\xi_0)z(\xi_0) = 0 \\ q(\xi)z(\xi) = q(\xi)z(\xi_0) \text{ for all } \xi \in Z \\ x(\xi) - \omega(\xi) = Vz(\xi) \end{array} \right. \right\}$$

Define the contingent market budget set:

$$B(\pi, \omega) = \{x \in \mathbb{R}^{SL} | \pi(x - \omega) = 0\}$$

Define the no arbitrage budget set:

$$\mathcal{B}(\pi, \omega, V) = \left\{ x \in \mathbb{R}^{SL} \left| \begin{array}{l} \pi(x - \omega) = 0 \\ x(\xi) - \omega \in \langle V \rangle \\ \pi \circ (x - \omega) \in \langle W(\pi) \rangle \end{array} \right. \right\}$$

Where  $\pi(\xi)$  is a  $1 \times S$  vector, and  $\pi \circ (x - \omega)$  is a  $L \times 1$  vector:

$$\pi \circ (x - \omega) = \begin{bmatrix} \pi(\xi_1)(x(\xi_1) - \omega) \\ \vdots \\ \pi(\xi_L)(x(\xi_L) - \omega) \end{bmatrix}$$

And where  $W(\pi)$  is a  $L \times J$  matrix:

$$W(\pi) = \begin{bmatrix} \pi(\xi_1)V \\ \vdots \\ \pi(\xi_L)V \end{bmatrix}$$

Define the pseudo no-arbitrage budget set:

$$\mathcal{B}_\psi(\pi, \omega, V) = \left\{ x \in \mathbb{R}^{SL} \left| \begin{array}{l} \pi(x - \omega) = 0 \\ x(\xi) - \omega \in \langle V \rangle \end{array} \right. \right\}$$

## 9.2 Equilibrium Concepts

**Definition 18** Given an economy  $\mathcal{E}(u, \psi, \omega, V)$ , a no-arbitrage equilibrium is a collection  $(x, \pi)$  such that

- (i)  $x^1 \in \arg \max U^1$  subject to  $x^1 \in B(\pi, \omega)$
- (ii)  $x^h \in \arg \max U^h$  subject to  $x^h \in \mathcal{B}(\pi, \omega, V)$
- (iii)  $\sum_{h=1}^I x^h - \omega^h = 0$

**Definition 19** Given an economy  $\mathcal{E}(u, \psi, \omega, V)$ , a pseudo no-arbitrage ( $\psi$ -NA) equilibrium is a collection  $(x, \pi)$  such that

- (i)  $x^1 \in \arg \max U^1$  subject to  $x^1 \in B(\pi, \omega)$
- (ii)  $x^h \in \arg \max U^h$  subject to  $x^h \in \mathcal{B}_\psi(\pi, \omega, V)$
- (iii)  $\sum_{h=1}^H x^h - \omega^h = 0$

## 9.3 Preliminary Results

**Lemma 20** Given an economy  $\mathcal{E}(u, \psi, \omega, V)$ ,

(i) If  $(x, z, q)$  is an equilibrium and if  $\pi_s^1(\xi) = \psi_s^1(\xi)u^1(x_s^1(\xi))$  for all  $s$  and  $\xi$  then  $(x, \pi^1)$  is a no arbitrage equilibrium.

(ii) If  $(x, \pi)$  is a no arbitrage equilibrium then there exist portfolios  $z$  and security prices  $q$  so that  $(x, z, q)$  is an equilibrium.

**Proof.** Straightforward. ■

**Lemma 21** Given an economy  $\mathcal{E}(u, \psi, \omega, V)$  where  $J \geq L$ , suppose that  $(x, \pi)$  is a  $\psi$ -NA equilibrium where  $W(\pi)$  has full rank  $L$  then  $(x, \pi)$  is a NA equilibrium.

**Proof.** Clearly, given state prices  $\pi$ :

$$\mathcal{B}(\pi, \omega, V) \subset \mathcal{B}_\psi(\pi, \omega, V)$$

Since  $J \geq L$  and  $W(\pi)$  has full rank  $L$  then it must be that:

$$\pi \circ (x^h - \omega^h) \in \langle W(\pi) \rangle$$

So:

$$\mathcal{B}_\psi(\pi, \omega, V) \subset \mathcal{B}(\pi, \omega, V)$$

So  $x^h$  is optimal given state prices  $\pi$ .  $x$  is feasible. ■

**Lemma 22** *Given an economy  $\mathcal{E}(u, \psi, \omega, V)$  where  $J \geq L$ , the set  $\Pi_0 \subset \mathbb{R}_{++}^{SL-1}$  of state prices  $\pi$  such that  $W(\pi)$  has rank less than  $L$  is a closed set of null measure in  $\mathbb{R}_{++}^{SL-1}$ .*

**Proof.** Let  $Z_r$  be the closed set of measure 0 of matrices of rank  $r$  in the space  $\mathbb{R}^{LJ}$ . It is well known that this set is closed and is a smooth submanifold of  $\mathbb{R}^{LJ}$  of codimension  $(L-r) \times (J-r)$ . The mapping  $W(\cdot)$  is a linear submersion (since  $S \geq J$  it follows that  $LS \geq LJ$ ) transversal to  $Z_r$  (proof below) so the set of state prices  $\pi \in \mathbb{R}^{LS}$  such that its image is the set  $Z_r$  is a set of codimension  $(L-r) \times (J-r)$  in the space  $\mathbb{R}^{LS}$ . This means that it's a set of dimension:

$$\dim(W^{-1}(Z_r)) = LS - (L-r) \times (J-r)$$

This is a set of measure zero in the space  $\mathbb{R}^{LS-1}$ . Taking the union over all such set  $Z_r$  gives the following result:

$$\dim \left( \bigcup_{r < L} W^{-1}(Z_r) \right) = \dim \left( W^{-1} \left( \bigcup_{r < L} Z_r \right) \right) < LS$$

Finally, make a transformation from the space of state prices in  $\mathbb{R}^{LS}$  to the space of normalized state prices where  $\pi_{LS} = 1$  in the space  $\mathbb{R}^{LS-1}$ . So, the set  $\Pi_0$  is a set of measure 0 in the space  $\mathbb{R}^{LS-1}$ . That it is closed follows from the continuity of  $W(\cdot)$  and of the normalization transformation.

Proof that  $W \pitchfork Z_r$ :

Define the following  $S \times 1$  vector.

$$V^s = [V_1^s, \dots, V_J^s]^T \text{ for all } s \in \Theta$$

Also, define the  $LJ \times L$  matrix:

$$M(V^s) = \text{diag}[V^S, \dots, V^S]$$

Consider the vector  $\pi \in \mathbb{R}^{LS}$  such that:

$$\begin{aligned} \pi &= (\pi_1, \dots, \pi_S) \\ &= (\pi_1(\xi_1), \dots, \pi_1(\xi_L), \dots, \pi_S(\xi_1), \dots, \pi_S(\xi_L)) \end{aligned}$$

Then we have the following  $LJ \times L$  derivative matrix:

$$D_{\pi_s} W = M(V^s)$$

And finally the  $LJ \times LS$  derivative of  $W(\cdot)$  :

$$D_{\pi} W = [M(V^1), \dots, M(V^S)]$$

The derivative mapping is surjective on tangent spaces if for every  $dy \in \mathbb{R}^{LJ}$ , there exists a  $d\pi \in \mathbb{R}^{LS}$  such that  $D_\pi W d\pi = dy$ . Equivalently (local control), for every base element of the type  $dy = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{LJ}$ , there exists a  $d\pi \in \mathbb{R}^{LS}$  such that  $D_\pi W d\pi = dy$ . Given  $(\xi, j) \in Z \times J$ , does there exist a vector  $d\pi \in \mathbb{R}^{LS}$  such that:

$$\begin{aligned} \sum_{s \in \Theta} V_j^s d\pi^s(\xi) &= dy(\xi, j) = 1 \\ \sum_{s \in \Theta} V_j^s d\pi^s(\xi') &= 0 \text{ for all } \xi' \neq \xi \in Z \end{aligned}$$

Choose  $\tilde{s}_j$  such that  $V_j^{\tilde{s}_j} > 0$  and:

$$\begin{aligned} d\pi^{\tilde{s}_j}(\xi) &= \frac{1}{V_j^{\tilde{s}_j}} \\ d\pi^{s'}(\xi') &= 0 \text{ for all others} \end{aligned}$$

This shows that the derivative mapping is surjective on tangent spaces and so necessarily,  $W \pitchfork Z_r$ . ■

## 9.4 Proof of Theorem (9)

**Proof.** We think of a trader's beliefs as lying in the space  $\mathbb{R}_{++}^{SL-1}$ . Beliefs are indexed from 1 to  $LS$ . A typical trader's beliefs are  $\psi = (\psi_1, \dots, \psi_{LS})$ . A typical state price is  $\pi = (\pi_1, \dots, \pi_{LS})$ .

$\psi^1$  is trader 1's beliefs with  $\psi_{LS}^1 = 1$ . Prices are normalized so that  $\pi_{LS} = 1$ .  $\omega$  and  $V$  are given. Define trader 1's demand function.

$$G^1 : \mathbb{R}^{SL-1} \times \mathbb{R}^{SL-1} \rightarrow \mathbb{R}_{++}^{SL}$$

$$G^1(\pi, \psi) = \arg \max [U^h \text{ st. } \pi(x^1 - \omega^1) = 0]$$

For  $h \geq 2$  :

$$F^h : \mathbb{R}^{SL-1} \times \mathbb{R}^{SL-1} \rightarrow \mathbb{R}_{++}^{SL}$$

$$F^h(\pi, \psi) = \arg \max [U^h \text{ st. } \pi(x^h - \omega^h) = 0 \text{ and } x^h(\xi) - \omega^h \in \langle V \rangle]$$

Define the aggregate excess demand function for all but the last commodity  $LS$ .

$$Z : \mathbb{R}_{++}^{SL-1} \times \mathbb{R}^{H(SL-1)} \rightarrow \mathbb{R}^{SL-1}$$

$$Z(\pi, \psi) = G^1(\pi, \psi^1) + \sum_{h>1} F^h(\pi, \psi^h) - \sum_h \omega^h$$

This function is  $C^\infty$  (proof is at the end of the argument). Walras law implies that if all but the last market clear, so does the last market. The equilibrium manifold is:

$$\mathcal{E} = \{(\pi, \psi) \in \mathbb{R}^{SL-1} \times \mathbb{R}^{H(SL-1)} : Z(\pi, \psi) = 0\}$$

Let's prove that 0 is a regular value of  $Z$ . Note that the FOCs for the 1<sup>st</sup> trader are the following:

$$\psi_i^1 u^1(x_i^1) = \pi_i \lambda \text{ for } i = 1, \dots, LS - 1$$

$$u^1(x_{LS}^1) = \lambda$$

$$\pi x^1 = \pi \omega^1$$

Or, eliminating the last demand function:

$$\begin{aligned} \psi_i^1 u^1(x_i^1) - \pi_i \lambda &= 0 \text{ for } i = 1, \dots, LS - 1 \\ u^1 \left( - \sum_{i < LS} \pi_i (x_i^1 - \omega_i^1) \right) - \lambda &= 0 \end{aligned}$$

Consider  $(x^1, \lambda) \in \mathbb{R}^{SL}$  as a function of  $\psi^1 \in \mathbb{R}^{SL-1}$ , defined implicitly by the above equations.

We use the implicit function theorem to obtain  $D_\psi Z$ . First, define the following  $LS \times LS$  matrix.

$$A_x = \begin{bmatrix} \psi_1^1 u^{1''}(x_1^1) & 0 & \cdots & 0 & -\pi_1 \\ 0 & \psi_2^1 u^{1''}(x_2^1) & 0 & \cdots & -\pi_2 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & 0 & \psi_{LS-1}^1 u^{1''}(x_{LS-1}^1) & -\pi_{LS-1} \\ -\pi_1 u^{1''}(x_{LS}^1) & -\pi_2 u^{1''}(x_{LS}^1) & \cdots & -\pi_{LS-1} u^{1''}(x_{LS}^1) & -1 \end{bmatrix}$$

Evidently, this matrix has the same rank as:

$$\tilde{A}_x = \begin{bmatrix} \psi_1^1 u^{1''}(x_1^1) & 0 & \cdots & 0 & -\pi_1 \\ 0 & \psi_2^1 u^{1''}(x_2^1) & 0 & \cdots & -\pi_2 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & 0 & \psi_{LS-1}^1 u^{1''}(x_{LS-1}^1) & -\pi_{LS-1} \\ -\pi_1 & -\pi_2 & \cdots & -\pi_{LS-1} & -\frac{1}{u^{1''}(x_{LS}^1)} \end{bmatrix}$$

Partition this matrix into the following way:

$$\tilde{A}_x = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Where  $D = -\frac{1}{u^{1''}(x_{LS}^1)}$ . Then using theorem 2.2 in Zhang [15] implies that:

$$\det \tilde{A}_x = \det A \det(D - CA^{-1}B)$$

Note that  $A$  is invertible and that all terms in  $(D - CA^{-1}B)$  are strictly negative so  $\det \tilde{A}_x \neq 0$ .

So  $\text{Rank}(A_x^{-1}) = LS$ . Also, define the following  $LS \times (LS - 1)$  matrix.

$$A_y = \begin{bmatrix} u^{1'}(x_1^1) & 0 & \cdots & 0 \\ 0 & u^{1'}(x_2^1) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & u^{1'}(x_{LS-1}^1) \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$D_{\psi^1}Z = A_x^{-1} \cdot A_y$$

Recall that:

$$\text{Rank}(A_x^{-1}) = LS$$

Also:

$$\text{Rank}(A_y) = LS - 1$$

A theorem by Sylvester (see theorem 2.6 of Zhang [15]) implies that:

$$\text{Rank}(A_x^{-1}) + \text{Rank}(A_y) - LS \leq \text{Rank}(A_x^{-1} \cdot A_y) \leq \min(\text{Rank}(A_x^{-1}), \text{Rank}(A_y))$$

So:

$$\text{Rank}(D_{\psi^1}Z) = LS - 1$$

Since 0 is a regular value of  $Z$ , we can use the pre-image theorem and claim that  $\mathcal{E}$  is a smooth submanifold of  $\mathbb{R}_{++}^{SL-1} \times \mathbb{R}^{H(SL-1)}$  of dimension:

$$\begin{aligned} \dim \mathcal{E} &= SL - 1 + H(SL - 1) - (SL - 1) \\ &= H(SL - 1) \end{aligned}$$

The projection map is defined as:

$$\begin{aligned} \Pi &: \mathcal{E} \rightarrow \mathbb{R}_{++}^{H(SL-1)} \\ \Pi(\pi, \psi, \omega) &= \psi \end{aligned}$$

$\Pi$  is proper. Indeed, consider a compact set  $X \subset \mathbb{R}_{++}^{H(SL-1)}$  and a sequence  $\{\psi_n\}_n \in X$ . Let's show there exists a converging subsequence of  $\{\pi_n, \psi_n\}_n \in \Pi^{-1}(X)$ . The sequence  $\{\psi_n\}_n \in X$  has a converging subsequence which we call  $\{\psi_{nm}\}_{nm} \in X$ . It converges to  $\psi \in X$ . Note that  $\pi_{nm}$  must be bounded otherwise  $\|Z(\pi_{nm}, \psi_{nm})\| \rightarrow \infty$  (the price of the good  $SL$  would go to zero). But this can't be true since  $Z(\pi_{nm}, \psi_{nm}) = 0$ . For the same reason,  $\pi_{nm}$  cannot converge to  $\partial\mathbb{R}_{++}^{SL-1}$ . Any bounded sequence has a converging subsequence. Denote it by  $\{\pi_{nml}, \psi_{nml}\}_{nml} \in \Pi^{-1}(X)$ . It converges to  $\{\pi, \psi\} \in \mathbb{R}_{++}^{SL-1} \times \mathbb{R}_{++}^{H(SL-1)}$ . By continuity of  $Z$ , it must be that  $Z(\pi, \psi) = 0$ . So  $\Pi$  is proper.

Let  $\Psi_R$  denote the set of regular values of  $\Pi$ . Sard implies that  $\Psi_R^C$  is null.  $\Pi$  is proper implies that  $\Psi_R$  is open. The Stack of Records theorem states that: For every  $\psi \in \Psi_R$  there exists a neighborhood  $U$  of  $\psi$  in  $\Psi_R$  (it exists because  $\Psi_R$  is open) and smooth functions  $\phi^i : U \rightarrow \mathbb{R}_{++}^{SL}$  for  $i = 1, \dots, T$  where  $T$  is the number of pseudo equilibria for the economy  $\psi \in \mathbb{R}^{H(SL-1)}$  such that:

$$\Pi^{-1}(\psi) = \{(\phi^i(\psi), \psi)_{i=1}^T\} \text{ for all } \psi \in U$$

$\phi^i$  is a submersion. Indeed,  $Z(\phi^i(\psi), \psi) = 0$  implies that:

$$D_\pi Z(\phi^i(\psi))D_\psi \phi^i(\psi) + D_\psi Z(\phi^i(\psi), \psi) = 0$$

So that (exploiting our previous result):

$$\text{Rank} (D_\pi Z(\phi^i(\psi))D_\psi \phi^i(\psi)) = LS - 1$$

Sylvester's theorem (theorem 2.6 in Zhang [15]) implies that:

$$\text{Rank}(D_\psi \phi^i(\psi)) = LS - 1 + \dim \{ \text{Im} (D_\pi Z(\phi^i(\psi))) \cap \text{Ker} (D_\psi \phi^i(\psi)) \}$$

So, necessarily:

$$\text{Rank}(D_\psi \phi^i(\psi)) = LS - 1$$

So  $\phi^i$  is a submersion so  $(\phi^i)^{-1}(\mathbb{R}^{SL-1} - \Pi_0)$  is open in  $U$  with null complement (use lemma (20)). Let

$$V = \bigcap_i (\phi^i)^{-1}(\mathbb{R}^{SL-1} - \Pi_0)$$

$V$  is open in  $U$  with null complement. So for every  $\psi \in V$ , the resulting pseudo equilibrium is in fact an equilibrium which is signal-contingent complete (use lemma (21)). Making a local to global argument, we get a set  $\Psi_0 \subset \Psi_R$  such that  $\Psi_0$  is open and has null complement in  $\Psi_R$  and hence in  $\mathbb{R}^{SL-1}$  and for every  $\psi \in \Psi_0$ , an pseudo equilibrium is an equilibrium.

Proof that  $Z$  is  $\mathcal{C}^\infty$  :

We will show that  $F^h$  is  $\mathcal{C}^\infty$ . Recall the mapping:

$$\begin{aligned} F^h & : \mathbb{R}^{SL-1} \times \mathbb{R}^{SL-1} \rightarrow \mathbb{R}_{++}^{SL} \\ F^h(\pi, \psi) & = \arg \max [U^h \text{ st. } \pi(x^h - \omega^h) = 0 \text{ and } x^h(\xi) - \omega^h \in \langle V \rangle] \end{aligned}$$

Define  $z^h$  so that:

$$x^h(\xi) - \omega^h = Vz^h(\xi) \tag{7}$$

Define the Lagrangian:

$$\Xi^h(x^h) = \sum_{s,\xi} \psi_s^h(\xi) u^h(\omega_s^h + V^s z^h(\xi)) - \lambda \sum_{\xi} \pi(\xi) \cdot Vz^h(\xi)$$

The first order conditions with respect to  $z_j^h(\xi)$  are:

$$\sum_s \psi_s^h(\xi) u^{h'}(x_s^h(\xi)) V_j^s = \lambda \pi(\xi) \cdot V_j \text{ for all } j, \xi$$

Rewritten:

$$\sum_s \left[ \psi_s^h(\xi) u^{h'}(\omega_s^h + V^s z^h(\xi)) - \lambda \pi_s(\xi) \right] V_j^s = 0 \text{ for all } j, \xi$$

Define the  $1 \times S$  gradient.

$$\nabla_{\xi} \Xi^h(x^h) = \left[ \psi_s^h(\xi) u^{h'}(\omega_s^h + V^s z^h(\xi)) - \lambda \pi_s(\xi) \right]_{s \in S}$$

Define the  $S \times 1$  payoff of security  $j \in J$ :

$$V_j = [V_j^1, \dots, V_j^S]^T$$

Then we can rewrite FOCs thus:

$$\nabla_{\xi} \Xi^h(x^h) \cdot V_j = 0 \text{ for all } j, \xi$$

Define the  $1 \times LS$  gradient vector:

$$\nabla \Xi^h(x^h) = [\nabla_{\xi} \Xi^h(x^h)]_{\xi \in Z}$$

Define the  $LS \times LJ$  matrix:

$$\tilde{V} = \text{diag}[V, \dots, V]$$

Then all FOCs are:

$$\nabla \Xi^h(x^h) \cdot \tilde{V} = 0$$

The functions  $z_j^h(\xi)$  are defined implicitly in the above equations. By the implicit function theorem, these functions are smooth if the matrix

$$D_{x^h} \left( \Xi^h(x^h) \cdot \tilde{V} \right) = \tilde{V}^T D^2 \Xi^h(x^h) \tilde{V}$$

is invertible when  $\nabla \Xi^h(x^h) \cdot \tilde{V} = 0$ . If it were not, there would exist a  $LJ \times 1$  vector  $\xi \neq 0$  such that:

$$\xi^T \tilde{V}^T D^2 \Xi^h(x^h) \tilde{V} \xi = 0 \text{ and } \nabla \Xi^h(x^h) \cdot \tilde{V} \xi = 0$$

The function  $\Xi^h$  inherits all properties (*U.i*, ..., *U.iv*) from  $U^h$ . In particular, the existence of such a vector would contradict property (*U.iv*) and hence the  $z_j^h(\xi)$  are smooth. Smoothness of  $x_s^h(\xi)$  and hence  $F^h$  follows from equation (7). Smoothness of  $G^1$  can be obtained using a similar argument. ■

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