Problem set 8: Wave propagation through inhomogeneous media

1) Consider the Helmholtz equation for a wave confined to the surface of a sphere of radius R

$$\left(\boldsymbol{\nabla}^2 + k_0^2\right)\phi = \left[\frac{1}{R^2\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{R^2\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2} + k_0^2\right]\phi(\theta,\phi) = 0$$

Transform this equation to a new coordinate system (this is known as the stereographic projection, mapping this sphere onto a plane with plane polar coordinates r and φ

$$r(\theta) = \frac{R\sin(\theta)}{1\cos(\theta)}$$

$$\varphi = \varphi$$

where $r \in [0,\infty]$ and $\varphi \in [0, 2\pi)$. Show that in this new coordinate system, the Helmholtz equation takes the form

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + k_0^2\left(\frac{2}{1+r^2/R^2}\right)^2\right]\phi = 0$$

The refractive index profile

$$n(r) = \frac{2}{1+(r/R)^2}$$

is known as the *Maxwell Fish Eye* profile. It was invented by Maxwell as a profile where all of the light rays travel in circles (because propagation in this profile is equivalent to propagation on the surface of a sphere). He apparently invented it over breakfast while considering the eye of a kipper (no idea how this worked).

2) Find the motion of the rays in the refractive index profile (1) (either numerically or analytically).

3) Use the ray tracing program provided with this lecture (or write your own) to find the motion of rays incident onto the following refractive index profile

$$n(r) = \begin{cases} \left(Q - \frac{1}{3Q}\right)^2 & r \leq a \\ 1 & r > a \end{cases}$$

where

$$Q(r) = \left[-\frac{a}{r} + \sqrt{\frac{a^2}{r^2} + \frac{1}{27}}\right]^{1/3}$$

What do you see?