Problem set 7: Scattering from an infinite number of obstacles (Part 2)

1) Consider a hexagonal lattice with real space lattice vectors

$$\boldsymbol{a}_1 = a\hat{\boldsymbol{x}}$$
$$\boldsymbol{a}_2 = a\left[\frac{1}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}\right] \tag{1}$$

and suppose that we are faced with the problem of solving the two dimensional Helmholtz equation (assuming the Bloch condition $\varphi(\boldsymbol{x}) = \exp(\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{x})u_{\boldsymbol{K}}(\boldsymbol{x})$ for a pair of scatterers at points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ in the unit cell: i.e. the equation

$$\left[\boldsymbol{\nabla}^2 + k_0^2\chi(\boldsymbol{x})\right]\varphi(\boldsymbol{x}) = 0$$

where

$$\chi(\boldsymbol{x}) = \alpha\sum_{n,m}\left[\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}_0-n\boldsymbol{a}_1-m\boldsymbol{a}_2)+\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}_1-n\boldsymbol{a}_1-m\boldsymbol{a}_2)\right]$$

For a value $\boldsymbol{K} = \boldsymbol{K}_0$ at the corner of the Brillouin zone

$$\boldsymbol{K}_0 = \frac{4\pi}{3a}\left[\frac{1}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}\right]$$

use degenerate perturbation theory (with χ as a perturbation) to find the frequency splitting of the first three eigenstates

$$|0\rangle = \frac{1}{\sqrt{A}}$$
$$|1\rangle = \frac{1}{\sqrt{A}}\mathrm{e}^{-\mathrm{i}\boldsymbol{b}_2\cdot\boldsymbol{x}}$$
$$|2\rangle = \frac{1}{\sqrt{A}}\mathrm{e}^{-\mathrm{i}(\boldsymbol{b}_1+\boldsymbol{b}_2)\cdot\boldsymbol{x}} \tag{2}$$

find the dependence of the frequency shift on the relative position $\boldsymbol{x}_1 - \boldsymbol{x}_0$ of the scatterers in the unit cell. For what separation do two out of three of the frequencies stay degenerate?