Problem set 4: Beams

As discussed in the lectures, the propagation of a beam through free space is governed by the following propagator

$$K(\boldsymbol{x}, z; \boldsymbol{x}', z') = \left(\frac{k_0}{2\pi \mathrm{i}(z - z')}\right) \exp\left(\frac{\mathrm{i}k_0(\boldsymbol{x} - \boldsymbol{x}')^2}{2(z - z')}\right)$$

which can be used to propagate any beam cross–section $\psi_0(\boldsymbol{x})$ (defined for a fixed z) along the z–axis, using the formula

$$\psi(\boldsymbol{x}, z) = \int d^2\boldsymbol{x}' K(\boldsymbol{x}, z; \boldsymbol{x}', z')\psi_0(\boldsymbol{x}')$$

(i) Suppose the initial beam cross–section is a delta function $\psi_0 = \delta^{(2)}(\boldsymbol{x})$. Find the cross–section of the beam at other values of z, and plot it. Describe what you see. What do you think has gone wrong?

(ii) Use the propagator to find the diffraction pattern from a uniformly illuminated slit of width d (along x) and infinitely extended along the y–axis. Plot the wave field you find.

Hint: You may need to use the integral representation of one or more of the following functions: the error function, the complementary error function or the Fresnel integral (all of which are available in the *scipy* and *mpmath* libraries).

(iii) The same as in (i), but for the inverse situation, where we have a wave incident onto an infinitely long opaque rectangle of width d.

(iv) Plot the sum of the results of (i) and (ii). Prove that if we sum the field ψ_A behind any uniformly illuminated aperture and that behind a uniformly illuminated opaque object $\bar{\psi}_A$, of the same size and shape as the aperture, then the result is unity

$$\psi_A + \bar{\psi}_A = 1$$

(where I assumed that the incident field has unit amplitude in both cases). This result is known as *Babinet's principle*, and implies that the diffraction from an aperture is very closely related to the diffraction from an opaque object of the same size and shape: $\psi_A = 1 - \bar{\psi}_A$.