Problem set 3: Pulses

(a) Show that the monochromatic 1D Green function for a source in front (i.e. $x' < 0$) of a mirror at $x = 0$ is given by

$$G(x - x', \omega) = \frac{\mathrm{e}^{\mathrm{i}k_0|x-x'|} - \mathrm{e}^{-\mathrm{i}k_0(x+x')}}{2\mathrm{i}k_0}$$

(for this problem a mirror means the field is zero at $x = 0$).

(b) Using the result of the previous problem find the general solution to the wave equation for a time dependent point source at the position $x_0 < 0$ in front of a mirror, i.e. the solution to

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \delta(x - x_0)s(t)$$

(c) For the particular case where the source is given by

$$j(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega \mathrm{e}^{-\frac{(\omega-\omega_0)^2}{2\Delta^2}} \tilde{j}(\omega) \mathrm{e}^{-\mathrm{i}\omega t}$$

calculate the field φ from the solution you found in the previous section. Plot the result in jupyter. Having found this solution for a pulse incident onto a mirror, can you see a much simpler way you could have obtained the same answer?

(d) In a general dispersive medium, the field due to a time dependent source of waves in one dimension can be written as

$$\varphi(x, t) = \int_{-\infty}^{\infty} d\omega s(\omega) \frac{\mathrm{e}^{\mathrm{i}n(\omega)k_0|x-x_0|-\omega t}}{2\mathrm{i}n(\omega)k_0} \tag{1}$$

where the frequency dependent refractive index is $n(\omega)$. Assuming that $s(\omega)$ has the following dependence on frequency

$$s(\omega) = 2\mathrm{i}n(\omega)k_0 \mathrm{e}^{-\frac{(\omega-\omega_0)^2}{2\Delta^2}}$$

expand the refractive index to second order around the carrier frequency ω_0, so that the effects of the curvature of the dispersion relation are included

$$n(\omega) \sim n(\omega_0) + (\omega - \omega_0)n'(\omega_0) + \frac{1}{2}(\omega - \omega_0)^2 n''(\omega_0)$$

and carry out the integral (1) (you may need to use the integral representation of the Airy function http://dlmf.nist.gov/9). What effect does the curvature of the dispersion relation have on the pulse propagation?