Problem set 2: Source of waves

(a) Use the Einstein and de–Broglie relations, $E = \hbar\omega$, and $\boldsymbol{p} = \hbar\boldsymbol{k}$ to show that if a relativistic particle of rest mass m_0 were represented by a plane wave $\psi = \exp(\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t))$ then the relativistic energy formula

$$\frac{E^2}{c^2} - \boldsymbol{p}^2 = m_0^2 c^2$$

implies ψ obeys the following wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \boldsymbol{\nabla}^2\psi + \frac{m_0^2c^2}{\hbar^2}\psi = 0$$

This is known as the *Klein–Gordon* equation, and is often used as a simple starting point for quantum field theory (but it is just the wave equation with an extra term).

(b) Find the Fourier representation of the Green function for the Klein–Gordon equation in one spatial dimension plus time, i.e. the solution to

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{m_0^2c^2}{\hbar^2}\right] G(x-x', t-t') = \delta(t-t')\delta(x-x')$$

List the possible choices of positions for the poles of the Green function in the Fourier domain. This Green function occurs all over the place in high energy physics textbooks.

(c) Use Cauchy's integral formula to evaluate the Fourier integrals from part (b). What is the physical meaning for the different choices of poles? You may find it useful to refer to the derivation of the Green function for the two dimensional Helmholtz equation.

(d) Use the 1D Green function for the Helmholtz equation to solve the 1D Helmholtz equation including two delta function scatterers spaced by a distance a

$$\frac{d^2\varphi}{dx^2} + k_0^2\left[1 + \alpha_1\delta(x) + \alpha_2\delta(x-a)\right]\varphi = 0$$

for a wave incident from the left. Under what circumstances is the reflection from the scatterers zero?