Lecture 6:
Scattering from an obstacle

In the previous lecture we examined the theory of boundary conditions for the wave equation, treating the problem of a wave propagating through a small hole in a perfectly reflecting screen. In this lecture we shall continue our discussion of waves interacting with materials, treating the problem of waves scattering from obstacles.

1 A description of the scattering problem:

If we imagine a scalar wave $e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}$ incident onto an obstacle then some of the wave will be scattered into different directions. Imagine we measure the scattered wave far away from the obstacle. The amplitude of the waves falls off with distance in the same way as from a point source located at the position of the scatterer (this rate of fall off is dictated by energy conservation). However, in general the angular dependence of the scattered field is not isotropic like it is from a point source. Therefore, the general form of the waves scattered from an obstacle in 3D (observed at a distance r from the obstacle) is given by

$$\text{far away from scatterer (3D):} \qquad \phi_s(r,\theta,\varphi) = \frac{e^{i\,k_0\,r}}{4\pi k_0 r} S(\theta,\varphi) \tag{1}$$

which is simply the Green function in 3D (lecture 2) divided by k_0, multiplied by an unknown angular dependence $S(\theta,\varphi)$. The factor of k_0 is just a convention to make S dimensionless. Similarly in two dimensions we can write

$$\text{far away from scatterer (2D):} \qquad \phi_s(r,\theta) = \frac{1}{4i} H_0(k_0 r) S(\theta) \sim \frac{1}{4i}\sqrt{\frac{2}{\pi k_0 r}}\, e^{i\,(k_0 r - \pi/4)} S(\theta) \tag{2}$$

which is the Green function in 2D (lecture 2), multiplied by the unknown angular dependent function $S(\theta)$. In the second step I used the form of the Hankel function at large $k_0 r$ (see http://dlmf.nist.gov/10.17). Finally, in one dimension we have

$$\text{far away from scatterer (1D):} \qquad \phi_s(r,\theta) = e^{\pm i\,k_0 x} S_{\pm} \tag{3}$$

which is the one dimensional Green function times $2i k_0$, times the unknown scalar $S_{\pm}$. In the one dimensional case we don't have an unknown angular dependence, but just an unknown scattering amplitude on the far left and far right of the obstacle. In the above formula, the plus sign corresponds to the far right, and the minus sign to the far left. The factor of $2i k_0$ was chosen so that S_- is dimensionless and corresponds to the ordinary reflection coefficient.

In most textbooks, the canonical problem of scattering is to find the unknown angular dependence of the scattered field. In this lecture we shall also give some methods for calculating these unknown functions $S(\theta,\varphi)$, $S(\theta)$ and $S_{\pm}$, but we shall not always be concerned with large distances from the object.

In passing it is worth mentioning that an excellent book that treats many aspects of wave scattering is "*Light Scattering by Small Particles*" by H. C. van de Hulst, Dover (1981) (note that he uses a different convention for the definition of S).

2 Return of the Green function—the Born series:

Consider an object with a refractive index $n(\boldsymbol{x})$ onto which a wave is incident. We look to find how the wave is redirected after it interacts with the object. That is we look to solve the Helmholtz equation with a spatially varying refractive index

$$[\boldsymbol{\nabla}^2 + k_0^2 n^2(\boldsymbol{x})]\phi(\boldsymbol{x}) = 0. \tag{4}$$

We write the refractive index as a background value $n = 1$ plus the part χ that describes the change in the refractive index due to the presence of the object

$$n^2(\boldsymbol{x}) = 1 + \chi(\boldsymbol{x})$$

we can then re–write our Helmholtz equation as the free space equation driven by a source

$$(\boldsymbol{\nabla}^2 + k_0^2)\phi(\boldsymbol{x}) = -k_0^2\chi(\boldsymbol{x})\phi(\boldsymbol{x}) = j(\boldsymbol{x}) \tag{5}$$

But we already know how to solve this equation (see lecture 2). If we know the Green function $G(\boldsymbol{x} - \boldsymbol{x}')$

$$(\boldsymbol{\nabla}^2 + k_0^2)G(\boldsymbol{x} - \boldsymbol{x}') = \delta^{(3)}(x - \boldsymbol{x}')$$

then we can find the field due to any source $j(\boldsymbol{x})$

$$\phi(\boldsymbol{x}) = \int d^3\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}')j(\boldsymbol{x}') = -k_0^2\int d^3\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')\phi(\boldsymbol{x}') \tag{6}$$

However, we shouldn't fool ourselves. We can't find the general solution to the wave equation that easily. As it stands equation (6) is no use because the source of the scattered waves $j(\boldsymbol{x})$ depends on the field ϕ, which is precisely what we don't know. To make progress we can develop the solution as a series in powers of χ

$$\phi(\boldsymbol{x}) = \phi_0 + \phi_1 + \phi_2 + \ldots = \sum_{n=0}^{\infty} \phi_n(\boldsymbol{x}) \tag{7}$$

(just as we do in the perturbation theory you learned in quantum mechanics). In the series (7), ϕ_0 is independent of χ (i.e. it is the incident field), ϕ_1 is linear in χ, ϕ_2 is quadratic in χ, and so on. Substituting the series (7) into (6) and equating quantities that are of the same order of χ we find that the next term in the series (7) is related to the previous one by

$$\phi_{n+1}(\boldsymbol{x}) = -k_0^2\int d^3\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')\phi_n(\boldsymbol{x}')$$

so that,

$$\begin{aligned}
\phi_1 &= -k_0^2\int d^3\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')\phi_0(\boldsymbol{x}') \\
\phi_2 &= -k_0^2\int d^3\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')\phi_1(\boldsymbol{x}') &&= k_0^4\int d^3\boldsymbol{x}' \int d^3\boldsymbol{x}'' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')G(\boldsymbol{x}' - \boldsymbol{x}'')\chi(\boldsymbol{x}'')\phi_0(\boldsymbol{x}'') \\
\phi_3 &= -k_0^2\int d^3\boldsymbol{x}' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')\phi_2(\boldsymbol{x}') &&= -k_0^6\int d^3\boldsymbol{x}' \int d^3\boldsymbol{x}'' \int d^3\boldsymbol{x}''' G(\boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}') \\
& && \quad \times G(\boldsymbol{x}' - \boldsymbol{x}'')\chi(\boldsymbol{x}'')G(\boldsymbol{x}'' - \boldsymbol{x}''')\chi(\boldsymbol{x}''')\phi_0(\boldsymbol{x}''') \\
\ldots & \;\ldots && \ldots
\end{aligned}$$

this series of increasingly complicated integrals is known as the *Born series*. We can understand this series as a physical picture in terms of multiple levels of scattering. In each term of the series we have a chain of Green functions multiplied by χ, and integrated over all possible intermediate positions. We can understand each factor of χ as a scattering event, with G representing the propagation of the wave between these events. Figure 1 shows a schematic of this interpretation.

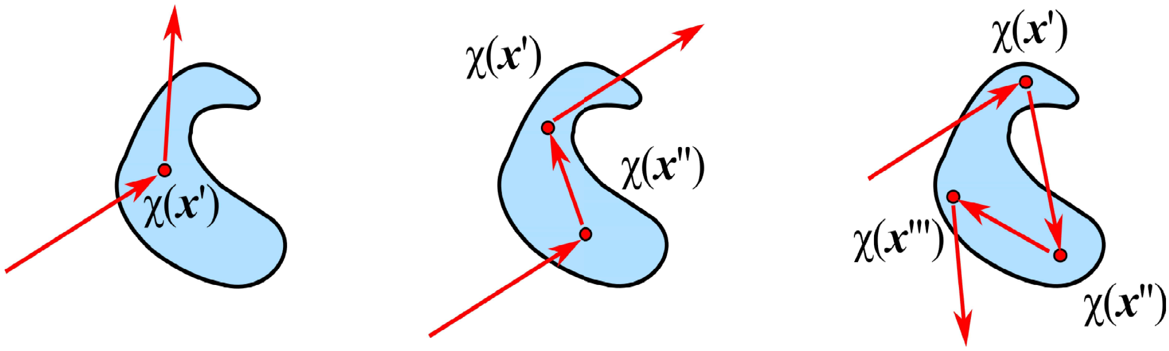

Figure 1. The Born series can be understood as a sum over different kinds of scattering events. The first term ϕ_1 (leftmost figure) can be pictured as a single scattering event at $\boldsymbol{x}'$, where the incident field 'bounces' from the material once. One then integrates over all possible positions of this event. The second term ϕ_2 is equivalent to two such events (at $\boldsymbol{x}'$ and $\boldsymbol{x}''$, see middle panel) and one integrates over all possible positions of these two events. The n^{th} term can then be understood as n such scattering events, and the right hand panel shows the $n=3$ case. This kind of perturbation series has the same form as that which appears in quantum field theory as Feynman diagrams. The main difference here is that the point of interaction is weighted by $\chi(\boldsymbol{x})$, rather than being an interaction with another field (i.e. another particle).

In general the Born series may not converge, so that taking more and more terms does not necessarily get one closer to the exact answer (in the same way that $1/(1-x)=1+x+x^2+x^3+\ldots$ is not a good series expansion when $|x|>1$). However, there are ways of modifying the series to make it converge. Nevertheless, for small scatterers (relative to the wavelength $\lambda=2\pi/k_0$) and for low index contrasts ($\chi<1$), one can get good answers from the first few terms of the series. Taking just the first term is known as the *Born approximation* and is valid for such weakly scattering objects

$$\phi\sim\phi_0+\phi_1$$

For example, if we have an incident plane wave the zeroth order term in the Born series is $\phi_0=e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}$, and within the Born approximation the total wave is given by

$$\phi(\boldsymbol{x})\sim e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}-k_0^2\int d^3\boldsymbol{x}'G(\boldsymbol{x}-\boldsymbol{x}')\chi(\boldsymbol{x}')e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}'} \tag{8}$$

So that we can identify the scattered field as

$$\phi_s(\boldsymbol{x})=-k_0^2\int d^3\boldsymbol{x}'G(\boldsymbol{x}-\boldsymbol{x}')\chi(\boldsymbol{x}')e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}'}$$

Now consider this for large distances from the scatterer in the case of three dimensional scattering where the Green function is $G(x-x')=-\exp(i\,k_0|\boldsymbol{x}-\boldsymbol{x}'|)/4\pi|\boldsymbol{x}-\boldsymbol{x}'|$

$$\begin{aligned}\phi_s(\boldsymbol{x})\ &=k_0^2\int d^3\boldsymbol{x}'\frac{e^{i\,k_0|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\chi(\boldsymbol{x}')e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}'}\\ &\sim\frac{k_0^2e^{i\,k_0|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}\int d^3\boldsymbol{x}'\chi(\boldsymbol{x}')e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}'}\\ &=\frac{e^{i\,k_0|\boldsymbol{x}|}}{4\pi k_0|\boldsymbol{x}|}[k_0^3\tilde{\chi}(-\boldsymbol{k})]\end{aligned} \tag{9}$$

By comparison with (1) we can see that the scattering amplitude $S(\theta,\varphi)$ is angle independent and equal to $k_0^3\tilde{\chi}(-\boldsymbol{k})$, i.e. proportional to the Fourier amplitude of the refractive index profile of the scatterer at the wave–vector $-\boldsymbol{k}$.

Despite the fact that (8) is far from exact, it can give useful insights into how a wave scatters from an object. The advantage of having an analytic expression is that one can immediately see how the scattering depends on the parameters of the system, In the accompanying Jupyter notebook we compare results computed using the Born approximation, and results computed using COMSOL.

2.1 Example 1—1D refractive index profiles:

Let's consider the simplest application of the Born approximation: to 1D wave propagation. In this case the Green function is given by (see lecture 2)

$$G(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{(k_0+i\eta)^2-k^2} = \frac{e^{ik_0|x-x'|}}{2ik_0} \tag{10}$$

Inserting the integral representation (10) into (8) we find

$$\phi(x) \sim e^{ik_0x} - k_0^2 \int dx' \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{(k_0+i\eta)^2-k^2} \chi(x') e^{ik_0x'}$$

Performing the integral over x' we can re–write thie scattered field in terms of the Fourier transform of χ

$$\phi(x) \sim e^{ik_0x} - k_0^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta)^2-k^2} \tilde{\chi}(k-k_0) \tag{11}$$

where

$$\tilde{\chi}(k) = \int_{-\infty}^{\infty} dx \chi(x) e^{-ikx}.$$

Although in general we cannot make any progress with the integral in (11), we can evaluate it with Cauchy's theorem at the limits $x \to \pm\infty$. For instance when $x \to -\infty$

$$\begin{aligned} \phi(x) \sim e^{ik_0x} - k_0^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta)^2-k^2} \tilde{\chi}(k-k_0) &= e^{ik_0x} - k_0^2 \int_{C_-} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta)^2-k^2} \tilde{\chi}(k-k_0) \\ &= e^{ik_0x} - k_0^2 \int_{C_-} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta-k)(k_0+i\eta+k)} \tilde{\chi}(k-k_0) \\ &= e^{ik_0x} + \frac{ik_0}{2} \tilde{\chi}(-2k_0) e^{-ik_0x} \qquad x \to -\infty \end{aligned} \tag{12}$$

where the contour could be closed in the lower half k plane because the factor of $\exp(ikx)$ damps the rest of the integrand out in the lower half plane (whatever the behaviour of χ, so long as $\chi(x) \to 0$ as $|x| \to \infty$) when x is very large and negative (to see this consider that $\exp(ikx) = \exp(ik_rx - k_ix)$, is a very small number when $k_i \ll 0$ and $x \ll 0$). Similarly, when $x \to +\infty$

$$\begin{aligned} \phi(x) \sim e^{ik_0x} - k_0^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta)^2-k^2} \tilde{\chi}(k-k_0) &= e^{ik_0x} - k_0^2 \int_{C_+} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta)^2-k^2} \tilde{\chi}(k-k_0) \\ &= e^{ik_0x} - k_0^2 \int_{C_+} \frac{dk}{2\pi} \frac{e^{ikx}}{(k_0+i\eta-k)(k_0+i\eta+k)} \tilde{\chi}(k-k_0) \\ &= \left(1 + \frac{ik_0}{2} \tilde{\chi}(0)\right) e^{ik_0x} \qquad x \to +\infty \end{aligned} \tag{13}$$

From the form of the fields (12) and (13) we can identify the general expressions for the reflection and transmission coefficients predicted by the Born approximation:

$$\begin{aligned} r(k_0) &= \frac{ik_0}{2}\tilde{\chi}(-2k_0) \\ \overline{r}(k_0) &= \frac{ik_0}{2}\tilde{\chi}(2k_0) \\ t(k_0) &= 1 + \frac{ik_0}{2}\tilde{\chi}(0) \end{aligned} \tag{14}$$

where I included the reflection coefficients for waves incident from the left r and right $\overline{r}$ (because finding the second of these just requires the change of the argument of $\tilde{\chi}$ from $-2k_0$ to $2k_0$). These expressions have a very clear physical interpretation. In 1D, when a wave reflects it needs to change its wave–vector from $+k_0$ to $-k_0$, i.e. a change of $-2k_0$ (using the quantum mechanical formula for the momentum of a wave $p = \hbar k$, you can think of this as a change of momentum of $-2\hbar k_0$). This change in wave–vector is obtained from the spatial dependence of the refractive index, and hence (for a single scattering event) the reflection coefficient is proportional to $\tilde{\chi}(-2k_0)$. Meanwhile, for a weak refractive index contrast we expect the transmission coefficient to be the phase change due to propagation through the refractive index

$$\begin{aligned} e^{ik_0\int_{-\infty}^{x} n(x')dx'} = e^{ik_0\int_{-\infty}^{x}\sqrt{1+\chi(x')}dx'} \sim e^{ik_0\int_{-\infty}^{x}\left[1+\frac{1}{2}\chi(x')\right]dx'} &\sim e^{ik_0 x}\left[1 + \frac{ik_0}{2}\int_{-\infty}^{x}\chi(x')dx'\right] \\ &\to e^{ik_0 x}\left[1 + \frac{ik_0}{2}\int_{-\infty}^{\infty}\chi(x')dx'\right] \quad x\to\infty \\ &= e^{ik_0 x}\left[1 + \frac{ik_0}{2}\tilde{\chi}(0)\right] \quad x\to\infty \end{aligned}$$

where in the third step I dropped an infinite phase factor (which is related to our choice of the origin of the coordinate system). The point to make here is that the expected first order change in the phase of the wave due to the presence of the spatially varying refractive index is equivalent to the transmission coefficient obtained from the Born approximation (14).

We can also use the results from the Born approximation get some general insight into the scattering from inhomogeneous media. For example, if the material is lossless then the refractive index is equal to a real number. This means that $\chi(x)$ is a real function of position

$$\chi(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\tilde{\chi}(k)e^{ikx} = \chi^*(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\tilde{\chi}^*(k)e^{-ikx} \tag{15}$$

which implies that

$$\tilde{\chi}^*(-k) = \tilde{\chi}(k)$$

meaning that the reflection coefficients (14) for incidence from the left and the right of a 1D lossless inhomogeneous medium are only different by a phase. In fact this is true in general, irrespective of the Born approximation. When the scattering medium is lossy then (15) no longer holds, and r and $\overline{r}$ can be arbitrarily different (e.g. one could be unity and the other zero).

2.2 Example 2—Diffraction from a periodic structure:

Another simple and useful application of the Born approximation is to understand the interaction of waves with periodic structures. Suppose we have a wave propagating through some weak, but infinitely extended and periodic variation of the refractive index $n^2(\boldsymbol{x}) = 1 + \chi(\boldsymbol{x})$. We assume cubic symmetry. Because of the periodicity, the susceptibility of the material can be represented as a Fourier sum

$$\chi(\boldsymbol{x}) = \sum_{n,m}\chi_{nm}e^{\frac{2\pi i}{L}(nx+my)}$$

We can then insert this expression for $\chi(\boldsymbol{x})$ into (8) and work out the scattered field, within the Born approximation

$$\phi(\boldsymbol{x}) \sim e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} - k_0^2 \sum_{n,m} \chi_{nm} \int d^3\boldsymbol{x}' G(\boldsymbol{x}-\boldsymbol{x}') e^{\frac{2\pi i}{L}(nx'+my')} e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}'}$$

defining

$$\boldsymbol{k}_{n,m} = \left(k_x + \frac{2\pi n}{L}\right)\hat{\boldsymbol{x}} + \left(k_y + \frac{2\pi m}{L}\right)\hat{\boldsymbol{y}}$$

we can change integration variables to $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{x}'$

$$\begin{aligned}\phi(\boldsymbol{x}) &= e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + k_0^2 \sum_{n,m} \chi_{nm} e^{i\,\boldsymbol{k}_{n,m}\cdot\boldsymbol{x}} \int d^3\boldsymbol{X} G(\boldsymbol{X}) e^{-i\,\boldsymbol{k}_{n,m}\cdot\boldsymbol{X}} \\ &= e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + k_0^2 \sum_{n,m} \chi_{nm} e^{i\,\boldsymbol{k}_{n,m}\cdot\boldsymbol{x}} \tilde{G}(\boldsymbol{k}_{n,m})\end{aligned}$$

where the Fourier transform of the Green function is given by (see lecture 2)

$$\tilde{G}(\boldsymbol{k}_{n,m}) = \frac{1}{(k_0 + i\eta)^2 - \boldsymbol{k}_{n,m}^2}. \tag{16}$$

Now let's look at the field scattered by the medium

$$\phi_s(\boldsymbol{x}) = k_0^2 \sum_{n,m} \chi_{nm} e^{i\,\boldsymbol{k}_{n,m}\cdot\boldsymbol{x}} \tilde{G}(\boldsymbol{k}_{n,m}) \tag{17}$$

this consists of a collection of waves all propagating in the discrete directions corresponding to the values of $\boldsymbol{k}_{n,m}$ (these are known as the *diffracted orders*). We can understand the origin of the scattered field (17) as follows: the incident wave with wave–vector $\boldsymbol{k}$ scatters from the medium and picks up an integer multiple of the wave–vector $\Delta\boldsymbol{k}_{n,m} = 2\pi(n\,\hat{\boldsymbol{x}} + m\,\hat{\boldsymbol{y}})/L$ corresponding to the periodic variation of the medium. This scattering is weighted by $\chi_{n,m}$ (the degree to which the refractive index exhibits a variation with wave–vector $\Delta\boldsymbol{k}_{n,m}$), and $\tilde{G}(\boldsymbol{k}_{n,m})$ (how far the wave–vector is from fulfilling the free–space dispersion relation).

In the next lecture we shall treat the problem of waves travelling through periodic media in more detail. It is useful to explore what the Born approximation gets right and what it misses out from the general case.

- What it gets right is that it correctly predicts the wave–vectors $\boldsymbol{k}_{n,m}$ that are allowed to propagate through the medium. It also correctly predicts when the medium will strongly scatter the incident waves. For example when $\boldsymbol{k} = \pi\hat{\boldsymbol{x}}/L = k_0\hat{\boldsymbol{x}}$ (a point known as the *Brillouin zone boundary*), the amplitude of the $\boldsymbol{k}_{-1,0} = -\pi\hat{\boldsymbol{x}}/L = -k_0\hat{\boldsymbol{x}}$ wave will be large because the denominator of $\tilde{G}$ (16) will be very close to zero.

- However, while the Born approximation is correct that there is a large amount of diffraction at the Brillouin zone boundary, it is so strong that the approximation of weak scattering is invalid. In the general case a so–called *band gap* opens at this value of $\boldsymbol{k}$, and the scattering is so strong that is suppresses any propagation throught the medium. This is not predicted by the Born approximation. It also doesn't predict that the relationship between k_0 and $\boldsymbol{k}$ will be modified by the presence of the medium, which in general it is. We shall explore all of this in the next lecture.

3 Rayleigh scattering:

Rayleigh scattering is the theory of waves scattering from particles with a refractive index such that the wavelength inside the particle is much larger than the particle size. This regime is often found in nature when visible light (wavelength of hundreds of nanometers) scatters from molecules (size of a few nanometers). It is the mechanism that explains why the sky is blue, and why the sunset is red.

We have developed a scalar theory of scattering, ignoring the polarization of light. Despite this, the scalar Born approximation is enough to explain why the sky is blue (the sky would be blue even if light didn't have polarization).

Figure 2. The sky is blue despite being illuminated with white(ish) sunlight. This is due to the phenomenon of Rayleigh scattering, which is the generic phenomenon where small particles scatter high frequencies more than low frequencies. The intensity of the scattering has a k_0^4 dependence, which can be derived from the Born approximation, ignoring the effect of polarization.

Suppose we are a large distance r from a small spherical particle of radius R and we use the far–field formula derived from the Born approximation (9). The scattered field equals

$$\begin{aligned}\phi_s(\boldsymbol{x}) &\sim \frac{k_0^2\phi_0 e^{i k_0 r}}{4\pi r}\int d^3\boldsymbol{x}'\chi(\boldsymbol{x}')e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}'}\\ &= \frac{k_0^2\phi_0 e^{i k_0 r}}{4\pi r}(n^2-1)\int d^3\boldsymbol{x}'\\ &= \frac{k_0^2\phi_0 e^{i k_0 r}}{r}(n^2-1)\frac{R^3}{3}\end{aligned} \tag{18}$$

where ϕ_0 is the incident field amplitude and in the second step I replaced $\chi(\boldsymbol{x}')$ with n^2-1 which is uniform within the sphere, and zero outside, and I assumed $k_0 R \ll 1$, such that $\exp(i\,\boldsymbol{k}\cdot\boldsymbol{x}') \sim 1$ within the sphere. The field (18) corresponds to a scattered intensity

$$I_s = |\phi_s|^2 = I_0 \frac{k_0^4}{r^2}\left(\frac{n^2-1}{3}\right)^2 R^6. \tag{19}$$

Meanwhile, from the full vector theory of Rayleigh scattering (where θ is the angle relative to the direction of incidence) the expression for the scattered intensity is

$$I_s = I_0\left(\frac{1+\cos^2(\theta)}{2}\right)\frac{k_0^4}{r^2}\left(\frac{n^2-1}{n^2+2}\right)^2 R^6 \sim I_0\left(\frac{1+\cos^2(\theta)}{2}\right)\frac{k_0^4}{r^2}\left(\frac{n^2-1}{3}\right)^2 R^6 \tag{20}$$

where in the second step I took the limit of $n^2-1=\chi\ll 1$, which is where the Born approximation is valid. Clearly all that the scalar theory leaves out is the factor of $[1+\cos^2(\theta)]/2$, which is due the directional dependence of dipole radiation.

As far as the colour of the sky is concerned, the important feature of the scattered intensity (19) is that it scales with k_0^4. This means that the higher frequencies (the blue and violet part of the visible spectrum) are scattered much more than the lower frequencies (the red). Therefore sunlight illuminates the sky with a colour that is skewed towards the blue end of the spectrum. When viewing the sun through a thick wodge of atmosphere (e.g. at sunset) we see only what is left over from lots of scattering, so that the color of the sun is skewed towards the red (which has scattered least). In the Jupyter notebook accompanying this lecture the solar spectrum is combined with the Rayleigh scattering formula to show the dominant colours scattered by the particles in the sky.

4 Mie scattering:

As in the previous section, there are many situations where we are interested in scattering from a geometrically simple object. For example, for three dimensional problems we are often faced with the scattering of a wave from a sphere. Similarly in two dimensional problems the simplest object is the cylinder (a circular scatterer). In both the cases of scattering from a sphere and a cylinder there is an exact solution (both for scalar and vector waves). These exact solutions are known as *Mie theory* (strangely as exact solutions are not usually given the title of a theory. We shall now derive the exact scattering of a scalar wave from a cylinder (the simplest case of Mie theory). There won't be time to discuss this in the lecture—I'm including this for the interested.

Start by dividing our wave ϕ up into an incident part ϕ_i plus a scattered part ϕ_s

$$\phi(\boldsymbol{x}) = \phi_i(\boldsymbol{x}) + \phi_s(\boldsymbol{x})$$

We assume that the incident wave is propagating along the $x-$axis with wave–vector $\boldsymbol{k}=k_0\hat{\boldsymbol{x}}$. We take the cylinder to be centred at $x=y=0$. Due to the cylindrical symmetry around this point, we decompose the field into a series of Bessel functions (using the generating function that was discussed in the material at the end of lecture 1)

$$\phi_i(\boldsymbol{x}) = e^{ik_0x} = e^{ik_0r\cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(k_0r)e^{il\theta}$$

where the cylindrical polar coordinates are r,θ. Each term in this sum (with angular dependence $\exp(il\theta)$) is known as a *partial wave*. We treat each of these partial waves separately, applying the continuity of ϕ and $\partial\phi/\partial r$ (see lecture 5) at the boundary between the background medium and the cylinder. Inside the cylinder (radius R) the wavenumber k_0 in the argument of the Bessel functions is changed to nk_0 (where n is the refractive index of the cylinder). Therefore the field inside the cylinder is

$$\phi(r<R) = A_l\, J_l(nk_0r)$$

where A_l is an unknown amplitude. To match up the field inside and outside the cylinder we need the scattered field ϕ_s in addition to the incident field ϕ_i. The scattered field outside the cylinder must be some combination of the two Bessel function J_l and Y_l, such that the wave is entirely outgoing. We already know from our treatment of Green functions in lecture 2 that the Hankel function of the first kind is a purely outgoing wave. Therefore,

$$\phi_s(r>R) = B_l\, H_l^{(1)}(k_0r)$$

where B_l is the unknown amplitude, determining how much the cylinder scatters the l^{th} incident partial wave. Applying the continuity of ϕ at the radius $r = R$ we find the value of the unknown A_l

$$i^l J_l(k_0 R) + B_l H_l^{(1)}(k_0 R) = A_l J_l(n k_0 R) \rightarrow \frac{i^l J_l(k_0 R) + B_l H_l^{(1)}(k_0 R)}{J_l(n k_0 R)} = A_n$$

Following this with the continuity of $\partial\phi/\partial r$ at $r = R$ we can also find B_l

$$B_l = -i^l \left[\frac{J_l'(k_0 R) - J_l(k_0 R) n \frac{J_l'(n k_0 R)}{J_l(n k_0 R)}}{H_l^{(1)\prime}(k_0 R) - H_l^{(1)}(k_0 R)\, n \frac{J_l'(n k_0 R)}{J_l(n k_0 R)}} \right]$$

Having determined A_l and $B_,$ we have thus completely solved the problem, and the field outside the cylinder is given by

$$\begin{aligned} \phi(r > R) &= e^{i k_0 x} + \sum_{l=-\infty}^{\infty} B_l H_l^{(1)}(k_0 r) \\ &= e^{i k_0 x} - \sum_{l=-\infty}^{\infty} i^l \left[\frac{J_l(n k_0 R) J_l'(k_0 R) - n J_l(k_0 R) J_l'(n k_0 R)}{J_l(n k_0 R) H_l^{(1)\prime}(k_0 R) - n H_l^{(1)}(k_0 R) J_l'(n k_0 R)} \right] H_l^{(1)}(k_0 r) e^{i l \theta} \end{aligned}$$

This is the exact solution for a plane wave scattering from a cylinder of refractive index n. Such exact series are rather difficult to use to extract simple physics, but form the basis of many scattering calculations. For instance, a more sophisticated theory of the rainbow (that explains features due to diffraction that go beyond simple ray tracing) is based on such scattering series (see e.g. J. Adam "*The mathematical physics of rainbows and glories*" *Phys. Rep.* **356** 229 (2002)).