Lecture 3: Pulses

The previous lecture covered the theory of waves produced by point sources radiating at a fixed frequency. Of course, this is only an approximation to real life: a fixed frequency implies that the source has been radiating forever, and no experimentalist is prepared to wait forever. In this lecture we'll cover some aspects of the theory of pulses, i.e. waves that last for a finite time and contain a range of frequencies.

1 Pulse propagation in one dimension:

We'll begin our discussion with pulse propagation in one dimension, firstly in a homogeneous medium where the wave speed is independent of frequency: $k_0 = \omega / c$. Besides making the formulae more complicated, there is no fundamental difference between the 1D case illustrated here and the two and three dimensional cases. The integrals may be more difficult to evaluate in e.g.the 2D case, but otherwise the procedure is the same.

1.1 Homogeneous non–dispersive medium:

In lecture 2 we calculated the field due to a point source at $x = x_0$ in a homogeneous 1D medium, radiating waves at a fixed frequency ω.

$$G(x - x_0, t) = \frac{e^{i\frac{\omega}{c}|x - x_0|} e^{-i\omega t}}{2i\frac{\omega}{c}}$$

where I have now included the factor of $e^{-i\omega t}$ that was implicit in all the expressions given in lecture 2. If we sum this field over all frequencies and weight each frequency with a factor $f(\omega)$ then we obtain the field due to a time dependent source

$$\phi(x, t) = \int_{-\infty}^{\infty} d\omega f(\omega) \frac{e^{i\frac{\omega}{c}|x - x_0|}}{2i\frac{\omega}{c}} e^{-i\omega t}. \tag{1}$$

applying the wave operator to (1) we see that

$$\left[\frac{d^2}{dx^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi(x, t) = \int_{-\infty}^{\infty} d\omega f(\omega) \left[\frac{d^2}{dx^2} + k_0^2\right] \frac{e^{i\frac{\omega}{c}|x - x_0|}}{2i\frac{\omega}{c}} e^{-i\omega t} = \delta(x - x_0) \int_{-\infty}^{\infty} d\omega f(\omega) e^{-i\omega t}$$

$$= \delta(x - x_0) f(t)$$

so that the time dependence of the source is given by $f(t) = \int d\omega f(\omega) e^{-i\omega t}$.

Let's consider the following frequency spectrum that is centred around $\omega = \omega_0$

$$f(\omega) = \sqrt{\frac{2}{\pi}} \frac{i}{\Delta} \frac{\omega}{c} e^{-\frac{(\omega - \omega_0)^2}{2\Delta^2}} \tag{2}$$

where the prefactor is chosen to give a convenient amplitude for the pulse. Substituting (2) into (1) we obtain

$$\phi(x, t) = \sqrt{\frac{1}{2\pi}} \frac{1}{\Delta} \int_{-\infty}^{\infty} d\omega e^{-\frac{(\omega - \omega_0)^2}{2\Delta^2}} e^{i\frac{\omega}{c}(|x - x_0| - ct)}$$

which we can evaulate using the Gaussian integral formula[1]

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

find that the field is equal to

$$\boxed{\phi(x,t) = e^{-\frac{\Delta^2}{2c^2}(|x-x_0|-ct)^2} e^{i\frac{\omega_0}{c}(|x-x_0|-ct)}} \tag{3}$$

After a little bit of thought, it is clear that for $t \gg 0$ (3) represents two Gaussian pulses, one on the right of x_0, and one on the left. The envelope of each of these Gaussians moves at velocity c and is multiplied by a travelling wave $\exp\left(\frac{i\omega_0}{c}(\pm x - ct)\right)$ also moving at velocity c with a wave vector $k = \omega_0 / c$ corresponding to the central frequency of the pulse. In this case the dispersion relation $\omega = ck$ is linear, and the pulses move as unchanging entities at velocity $\pm c$.

1.2 Homogeneous dispersive medium:

Let's now consider a slightly more complicated (and realistic) case. Let's imagine that we launch our pulse within a homogeneous medium of refractive index $n(\omega)$ (the refractive index is not a concept specific to optics, but can be used for any wave). The phenomenon where the refractive index depends on frequency is called dispersion. We shall examine this frequency dependence a bit more in the next section.

For a fixed frequency the 1D Green function we derived in lecture 2 is given by

$$G(x - x') = \frac{e^{ik(\omega)|x-x_0|} e^{-i\omega t}}{2ik(\omega)}$$

where for a non-dispersive medium we have the following relation between the wave–vector and the frequency $k(\omega) = \omega / c$. If the medium has refractive index $n(\omega)$ then this relationship is changed to $k(\omega) = n(\omega)\omega / c$. Therefore the analogue of equation (1) is

$$\phi(x,t) = \int_{-\infty}^{\infty} d\omega f(\omega) \frac{e^{i\frac{n(\omega)\omega}{c}|x-x_0|}}{2i\frac{n(\omega)\omega}{c}} e^{-i\omega t} \tag{4}$$

We consider a similar Gaussian pulse (2) for $f(\omega)$,

$$f(\omega) = \sqrt{\frac{2}{\pi}} \frac{i}{\Delta} \frac{n(\omega)\omega}{c} e^{-\frac{(\omega-\omega_0)^2}{2\Delta^2}}$$

However the resulting integral is now impossible to exactly evaluate, because $n(\omega)$ is an unknown frequency dependent function that can be arbitrarily unpleasant. To make progress we assume that the pulse has a narrow enoughspread in frequency that it samples the refractive index at ω_0 plus a linear change

$$n(\omega) \sim n(\omega_0) + (\omega - \omega_0) n'(\omega_0)$$

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1. To prove this define $I_1 = \int dx e^{-ax^2+bx} = e^{\frac{b^2}{4a^2}} \int dx e^{-a(x-b/2a)^2} = e^{\frac{b^2}{4a}} \int dX e^{-aX^2}$. The problem is now reduced to finding $I_2 = \int dX e^{-aX^2}$. To find this, consider its square $I_2^2 = \int dX \int dY e^{-a(X^2+Y^2)}$. Switch to polar coordinates, $I_2^2 = 2\pi \int_0^{\infty} r dr e^{-ar^2} = -\frac{\pi}{a} \int_0^{\infty} dr \frac{d}{dr} e^{-ar^2} = \frac{\pi}{a}$. We then have the result we were looking for $I_1 = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$.

There is a convention when $n'(\omega_0) > 0$ we say that the material is exhibiting *normal dispersion*, while $n'(\omega_0) < 0$ is said to be *anomalous dispersion*. Applying these assumptions to (4) then gives

$$\begin{aligned}\phi(x,t) &= \sqrt{\frac{1}{2\pi}}\frac{1}{\Delta}\int_{-\infty}^{\infty} d\omega e^{-\frac{(\omega-\omega_0)^2}{2\Delta^2}} e^{i\frac{n(\omega)\omega}{c}\left|x-x_0\right|} e^{-i\omega t} \\ &= \sqrt{\frac{1}{2\pi}}\frac{1}{\Delta} e^{-i\omega_0 t} e^{i\frac{n(\omega_0)\omega_0}{c}|x-x_0|}\int_{-\infty}^{\infty} d\Omega e^{-\left[\frac{1}{2\Delta^2}-\frac{i n'(\omega_0)}{c}|x-x_0|\right]\Omega^2} e^{i\frac{\Omega}{c}\{[n(\omega_0)+n'(\omega_0)\omega_0]|x-x_0|-ct\}}\end{aligned}$$

this is again in the form of a Gaussian integral. After applying the formula for this integral given in the previous section

$$\phi(x,t) = e^{i\frac{n(\omega_0)\omega_0}{c}|x-x_0|} e^{-i\omega_0 t}\sqrt{\frac{1}{1-\frac{2i\Delta^2 n'(\omega_0)}{c}|x-x_0|}}\exp\left[-\frac{\Delta^2}{2c^2}\frac{[(n(\omega_0)+n'(\omega_0)\omega_0)|x-x_0|-ct]^2}{1-\frac{2i\Delta^2 n'(\omega_0)}{c}|x-x_0|}\right] \tag{5}$$

which is now a much different function of position and time than in the non–dispersive case (3). Let's try to understand this a bit. Firstly we have the factor of

$$\sigma(x) = \sqrt{1-\frac{2i\Delta^2 n'(\omega_0)}{c}|x-x_0|}$$

which indicates a change in both the width and the phase of the Gaussian. In the case where n' is imaginary and positive then $\sigma(x)$ is purely real and positive. The pulse thus grows wider as $|x-x_0|$ increases and its amplitude diminishes. For real n' the pulse both widens and accumulates an additional position dependent phase. The peak of the Gaussian is at

$$(n(\omega_0)+n'(\omega_0)\omega_0)|x-x_0| = ct$$

which can be written in terms of the group velocity,

$$v_g = \left[\frac{dk(\omega_0)}{d\omega_0}\right]^{-1} = \frac{c}{n(\omega_0)+n'(\omega_0)\omega_0}$$

with the above notation, the field (5) can be written as

$$\boxed{\phi(x,t) = e^{i\omega_0\left[\frac{|x-x_0|}{v_p}-t\right]}\frac{1}{\sigma(x)}\exp\left\{-\frac{\Delta^2}{2\sigma^2(x)}\left[\frac{|x-x_0|}{v_g}-t\right]^2\right\}} \tag{6}$$

where $v_p = c/n(\omega_0)$ is the phase velocity. We see that the centre of the two Gaussians now moves with the group velocity, while the phase within the envelope moves at the phase velocity. The quantity $\sigma(x)$ serves to reshape the pulses as they propagate. Notice that when the dispersion disappears $n'(\omega_0) = 0$ then $v_g = v_p$ and (6) reduces to the same form as (3).

Having added in the dispersion we can see that it is more difficult to answer the question "How fast is the pulse moving?" because it depends what you decide to track. Follow the centre of the pulse and it moves at the group velocity, follow one of the oscillations within the pulse and it will be the phase velocity. There is clearly some arbitrariness in this game. The important thing is how fast energy can be transported by the pulse between two points (how fast can a source communicate with a detector?). This is why we shouldn't be too worried when people tell us things like, "I can do an experiment where the phase and group velocity of light are both greater than c, Einstein was wrong!". Yes, both phase and group velocity can be greater than c. But your favourite pie shop can also be moved from Bolton to New Zealand in an arbitrarily short time by simply changing your mind (light takes about 0.06s to travel from Bolton to New Zealand, and I don't know how fast you can change your mind). We need to be careful that the variables we use are using are physically meaningful.

You can see animations and play with the parameters of both the dispersive and non–dispersive pulse in the corresponding Jupyter notebook for this lecture.

2 Why does the refractive index depend on frequency?

Before moving on to talk a bit more about the speed at which signals travel through a dispersive material we should discuss the phenomenon of dispersion in a bit more detail. First we should think about why the refractive index exhibits a frequency dependence.

When a wave travels through a material, it pushes the constituent parts of the material about a bit. The response of the material to these pushes then leads to the emission of additional waves, and the sum of all the waves can then appear (for example) as an overall slowing of the total wave.

Let us consider a material where we decompose the wave speed as c_∞ + something, with the 'something' being the response of the material to the wave. The quantity c_∞ is the speed of the wave at infinite frequency, where the response of the material becomes irrelevant[2]. We can write the wave equation as

$$\nabla^2\phi - \frac{1}{c_\infty^2}\frac{\partial^2\phi}{\partial t^2} = j \tag{7}$$

where j is the additional source of the waves emitted by the material as the wave propagates. Because j is a function of the wave amplitude we can Taylor expand j in powers of ϕ. To leading order in ϕ we can write

$$j(x,t) = \frac{1}{c_\infty^2}\int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx' \chi_{\rm nl}(x-x', t-t')\frac{\partial^2\phi(x'.t')}{\partial t'^2} \tag{8}$$

i.e. the source of waves at the point x is generally due to some sum of all the past behaviour of ϕ for $t' < t$, over all points x' in the medium. The quantity $\chi_{\rm nl}$ is a *susceptibility* and is specific to the material supporting the wave, being a description of the response of the material to the wave. The second time derivative of ϕ is present because I've assumed the material responds to *changes* in ϕ, and that ϕ responds to changes in the material. The factor of $1/c_\infty^2$ is just present for convenience. Typical materials have a quite localised spatial response so that we approximate $\chi_{\rm nl}$ as

$$\chi_{\rm nl}(x-x', t-t') = \delta(x-x')\chi(t-t') \tag{9}$$

Substituting (8) and (9) into (7) we obtain

$$\nabla^2\phi - \frac{1}{c_\infty^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{c_\infty^2}\int_{-\infty}^{t} dt' \chi(t-t')\frac{\partial^2\phi(x.t')}{\partial t'^2} = 0 \tag{10}$$

which is a wave equation that is non-local in time (i.e. it depends on all previous times, not just 'now'). This equation simplifies a lot if we write it in the Fourier domain

$$\phi(x,t) = \int_{-\infty}^{\infty} d\omega \tilde{\phi}(x,\omega) e^{-i\omega t} \tag{11}$$

Substituting (11) into (10) we find

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\left[\nabla^2\tilde{\phi}(x,\omega) + \frac{\omega^2}{c_\infty^2}\left(1 + \int_{-\infty}^{t} dt' \chi(t-t') e^{-i\omega(t'-t)}\right)\tilde{\phi}(x,\omega)\right] = 0$$

2. In electromagnetism c_∞ is easy to make sense of, it is the speed of light in vacuum. In acoustics we need to be more pragmatic, and c_∞ can be considered as some large frequency (much larger than the frequency range of interest) where the wave speed is approximately constant. See e.g. M. O'Donnell, E. T. Jaynes and J. G. Miller *J. Acoust. Soc. Am.* **69** 696 (1981).

which implies the usual Helmholtz equation in a dispersive medium of refractive index $n(\omega)$

$$\nabla^2\tilde{\phi}(x,\omega)+\frac{\omega^2 n^2(\omega)}{c_\infty^2}\tilde{\phi}(x,\omega)=0$$

where the square of the refractive index is given by (changing the integration variable from t' to τ)

$$n^2(\omega)=1+\int_0^\infty d\tau\chi(\tau)e^{i\omega\tau}=1+\chi(\omega) \tag{12}$$

where $\chi(\omega)$ is the frequency dependent susceptibility. We thus see that the phenomenon of dispersion is due to the fact that a material takes a finite time to respond to the wave as it moves through the medium. If we assume an instantaneous response to the wave then $\chi(\tau)=\text{const.}\times\delta(\tau)$, and the frequency dependence disappears from the refractive index (12).

There is something very important to notice about (12). Because the integral extends only over positive τ, we can see that $\chi(\omega)$ is finite and decaying as we move further up into the upper half complex ω plane ($\text{Im}[\omega]>0$)[3]. In short: *because the material only responds to the past behaviour of the wave, $\chi(\omega)$ is an analytic function in the upper half complex frequency plane.*

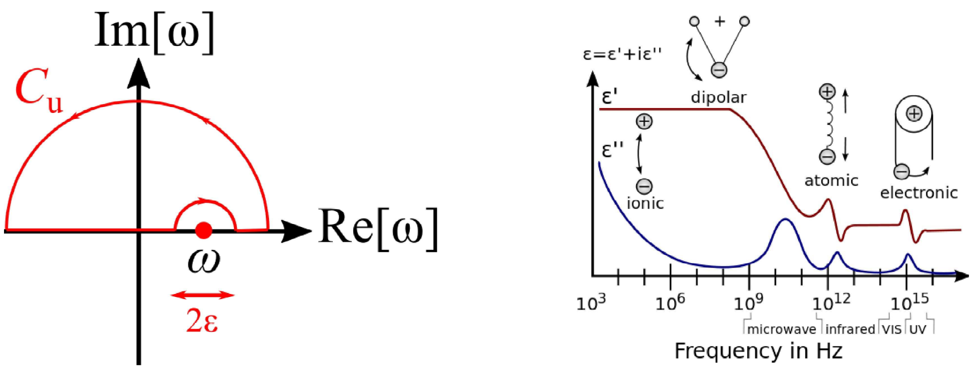

Figure 1. (Left) The frequency dependent susceptibility $\chi(\omega)$ is analytic in the upper half frequency plane because it is built from factors of $e^{i\omega\tau}$ with positive τ, all of which smoothly decay for increasing $\text{Im}[\omega]$. Therefore if we apply the Cauchy integral formula to χ with the evaluation point ω outside of the contour C_u it will yield zero. This zero valued integral can be split up into a contribution from integrating along the real line $\text{P}\int ....$, and integrating along a small semicircle (the integrand is zero along the infinite semicircle). Equating the sum of the two to zero gives the Kramers–Kronig relations. (Right, taken from wikipedia) Due to the Kramers–Kronig relations the real and imaginary parts of the electric permittivity $\varepsilon(\omega)=1+\chi(\omega)$ are correlated in frequency. This figure shows a typical frequency dependence of a dielectric material.

So, if $\chi(\omega)$ is an analytic function in the upper half frequency plane then we can play games with Cauchy's integral theorem as we did in lecture 2. For instance

$$0=\oint_{C_u}\frac{\chi(\omega')}{\omega'-\omega}d\omega' \tag{13}$$

where ω is any frequency on the real line, and C_u is a contour in the upper half frequency plane. Formula (13) becomes useful if we choose a C_u that passes along the real axis, skirts above ω in a very small clockwise semi–circle, continues along the remainder of the real axis and then closes in a semi–circle at infinity (see figure 1). We can write this contour integral as

$$0=\text{P}\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega'+i\chi(\omega)\int_\pi^0\frac{1}{\epsilon e^{i\theta}}\epsilon e^{i\theta}d\theta=\text{P}\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega'-i\pi\chi(\omega) \tag{14}$$

3. If you're struggling to see this, just think about the function $\int_0^\infty\chi(\tau)e^{i\omega\tau}$. Suppose this is a finite valued function on the real ω axis. Now add an imaginary part to ω, $\omega\to\omega+i\delta$. The integral becomes $\int_0^\infty\chi(\tau)e^{i\omega\tau}e^{-\delta\tau}$, which now has the large values of τ damped out from the integral. The resulting function oscillates less and is reduced in amplitude compared to the behaviour on the real axis.

where 'P' means the integral along the real line, excluding the problematic region of width 2ϵ where $\omega' \sim \omega$, and the small semicircular integral is performed in polar coordinates $\omega' - \omega = \epsilon e^{i\theta}$. The infinite semicircle does not contribute because $\chi(\omega) = 0$ at infinity in the upper half ω plane. Taking the real and imaginary parts of (14) separately we obtain

$$\boxed{\begin{aligned} \mathrm{Re}[\chi(\omega)] &= \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}[\chi(\omega')]}{\omega'-\omega}d\omega' \\ \mathrm{Im}[\chi(\omega)] &= -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}[\chi(\omega')]}{\omega'-\omega}d\omega' \end{aligned}} \tag{15}$$

where the limit of $\epsilon \to 0$ has been take so that the 'P' has an unambiguous meaning. These relations between the real and imaginary parts of the frequency dependent susceptibility are known as *the Kramers–Kronig relations*. They are fundamental to physics, being an expression of the one-way nature of time: the fact that the behaviour of the material 'now' depends only on what happened in the past, and doesn't know anything about the future. Were a genuine clairvoyant to exist, their response function (perhaps their response to an acoustic stimulus) would violate the Kramers–Kronig relations.

The Kramers–Kronig relations always govern the relationship between the real and imaginary parts of the susceptibility. One important message is that you cannot assume any dispersion you like, you are always constrained by (15).

A Jupyter notebook is provided with this lecture so you can play around with numerically evaulating the Kramers–Kronig relations to see what kind of real/imaginary parts arise from their partner imaginary/real parts.

3 Application–The Kramers–Kronig relations and relativity:

Now we have spent some time thinking a bit about how pulses move in dispersive materials, and the the origin of material dispersion, let's put this knowledge to use. As stated above, both the phase and group velocity of electromagnetic waves can be greater than the speed of light in vacuum, c (there are several papers on this topic, see for example L. J. Wang, A. Kuzmich & A. Dogariu *Nature* **406** 277 (2000)). *So how can we be sure that we can't use some funny kind of material to transmit messages faster than c*? Let's make ourselves sure, using our newfound knowledge to prove that it is impossible. In this section we still deal with scalar waves (polarization again plays no role here), but c always refers to the speed of light, and $n(\omega)$ is the optical refractive index.

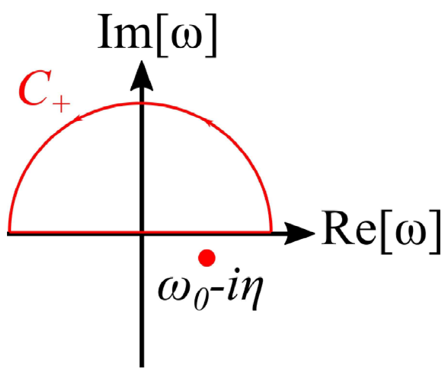

Figure 2. The frequency integral for a monochromatic source that is suddenly turned on at $t = 0$ can be turned into the closed contour C_+ when $|x|/c > t$. Due to the Kramers–Kronig relations $n(\omega)$ and hence the whole integrand is analytic within C_+. This implies that the integral is zero: thus no disturbance can travel faster than the speed of light in vacuum.

As in section 1.2, let's start from the general expression for a pulse emitted in a dispersive medium.

$$\phi(x,t)=\int_{-\infty}^{\infty}d\omega f(\omega)\frac{e^{i\left[\frac{n(\omega)\omega}{c}|x-x_0|-\omega t\right]}}{2i\frac{n(\omega)\omega}{c}} \tag{16}$$

To unambiguously find how fast changes in the field are propagated through the medium, we'll consider the case where a source at $x=0$ is abruptly switched on at the time $t=0$. Then, if we sit and observe the field over time at any point x, we *should* see nothing before $t=|x|/c$.

Suppose the source is turned on at $t=0$ and then oscillates at a fixed frequency ω_0 such that the field at $x=0$ takes the form

$$\phi(0,t)=\begin{cases}0 & t<0\\ e^{-i\omega_0 t} & t\geq 0\end{cases}$$

The corresponding spectrum $\phi(0,\omega)$ is

$$\phi(0,\omega)=\int_{-\infty}^{\infty}\phi(0,t)e^{i\omega t}dt=\int_{0}^{\infty}e^{-i(\omega_0-i\eta)t}e^{i\omega t}dt=-\frac{1}{i(\omega-\omega_0+i\eta)} \tag{17}$$

where I inserted the factor of η to make the integral converge. We'll take the limit of $\eta\to 0$ at the end (as we did for the Green functions in lecture 2). Equation (17) can be fulfilled if we choose the source such that

$$f(\omega)=-\frac{2i\frac{n(\omega)\omega}{c}}{i(\omega-\omega_0+i\eta)} \tag{18}$$

and our wave field becomes

$$\phi(x,t)=\int_{-\infty}^{\infty}d\omega\frac{e^{i\left[\frac{n(\omega)\omega}{c}|x|-\omega t\right]}}{(\omega-\omega_0+i\eta)}. \tag{19}$$

We can use Cauchy's theorem, together with the Kramers–Kronig relations to make some progress with this integral. From the previous section we know that $n(\omega)=1+\chi(\omega)$ tends to a value of one as we go further up into the upper half frequency plane. In the upper half plane the exponent in (19) thus takes the same form as it does in vacuum $\exp\left[i\omega\left(\frac{|x|}{c}-t\right)\right]$, tending to zero with increasing $\mathrm{Im}[\omega]$ when $|x|/c>t$. As a consequence of this decaying exponential, when $|x|/c>t$ the integral in (19) can be replaced with a closed semi–circular contour C_+ in the upper half plane (see figure 2)

$$\phi(x,t)=\int_{C_+}d\omega\frac{e^{i\left[\frac{n(\omega)\omega}{c}|x|-\omega t\right]}}{(\omega-\omega_0+i\eta)}\qquad |x|/c>t \tag{20}$$

given that—due to the Kramers–Kronig relations—the refractive index $n(\omega)$ is an analytic function in the upper half frequency plane, and given that $\omega_0-i\eta$ lies outside of the contour we can use Cauchy's theorem to deduce that

$$\boxed{\phi(x,t)=0\qquad |x|/c>t} \tag{21}$$

This means that if you suddenly turn on an electromagnetic source in *any* material, no wave can reach you more quickly than it could have done in vacuum. It is worth dwelling on how we got this result: through looking at the behaviour of the refractive index and the wave at complex frequencies we deduced that no signal can be propagated faster than light. In wave physics the theory of analytic functions is deeply tied to the way in which cause follows effect.