Lecture 9:
Propagation through inhomogeneous media

In the previous lectures we treated the problem of wave propagation in free space, scattering from simple obstacles, and propagation through periodic media. We only really satisfactorily solved the problem of propagation through inhomogeneous media (where the refractive index varies as a continuous function of position) when we dealt with periodic media. The periodicity was an advantage, allowing us to decompose the wave into a discrete set of modes. The problem is much harder in general. If we want to solve

$$[\boldsymbol{\nabla}^2 + k_0^2 n^2(\boldsymbol{x})]\phi(\boldsymbol{x}) = 0$$

for $\phi(\boldsymbol{x})$ then we are faced with a very difficult problem. The possible forms of $n(\boldsymbol{x})$ are endless (we are dealing with the space of all possible functions of position!), and so any hope of a general solution $\phi(\boldsymbol{x})$ is foundless. There are however some very useful approximations that we can make that let us get some information about $\phi(\boldsymbol{x})$. In this lecture we'll deal with some exact and approximate techniques that allow us to find $\phi(\boldsymbol{x})$ for a wide class of functions $n(\boldsymbol{x})$.

1 Geometrical optics and the WKB approximation:

From an early age we are encouraged to think of light as a bundle of rays (we have sun–screen put on us "to ward off the harmful rays of the sun"). For most of our education this concept of light rays is continued, but left quite vague: the school physics teacher may draw some lines passing through a lens., but at the same time tell us that light is a wave (what the hell are these lines supposed to be?). Even an undergraduate physics education (at least mine!) can cover the design of lenses without clearly explaining where the wave ends and the ray begins. We should be clear, light is a wave—so what's this ray stuff all about?

The theory of rays is not specific to optics, but is a limit one can take of any wave equation. This limit is called the limit of geometrical optics. It involves assuming a large separation of scales. Say the distance a is that over which the refractive index $n(\boldsymbol{x})$ changes by order 1. The limit of geometrical optics is where $n k_0 a \gg 1$ throughout the medium; i.e. the local wavelength λ is always such that $a/\lambda \gg 1$. To take this limit we start from the Helmholtz equation

$$[\boldsymbol{\nabla}^2 + k_0^2 n^2(\boldsymbol{x})]\phi(\boldsymbol{x}) = 0 \tag{1}$$

and translate this into an equation for the amplitude $R(\boldsymbol{x})$ (real and positive) and phase $k_0 S(\boldsymbol{x})$ (real) of the function $\phi(\boldsymbol{x})$

$$\phi(\boldsymbol{x}) = R(\boldsymbol{x}) e^{i k_0 S(\boldsymbol{x})} \tag{2}$$

Substituting (2) into (1) we find a complicated equation for R and S

$$\boldsymbol{\nabla} \cdot [e^{i k_0 S(\boldsymbol{x})} \boldsymbol{\nabla} R(\boldsymbol{x}) + i k_0 R(\boldsymbol{x}) e^{i k_0 S(\boldsymbol{x})} \boldsymbol{\nabla} S(\boldsymbol{x})] + k_0^2 n^2(\boldsymbol{x}) R(\boldsymbol{x}) e^{i k_0 S(\boldsymbol{x})} = 0$$

$$\rightarrow \quad \boldsymbol{\nabla}^2 R(\boldsymbol{x}) + 2 i k_0 \boldsymbol{\nabla} R(\boldsymbol{x}) \cdot \boldsymbol{\nabla} S(\boldsymbol{x}) + i k_0 R(\boldsymbol{x}) \boldsymbol{\nabla}^2 S(\boldsymbol{x}) - k_0^2 R(\boldsymbol{x}) \boldsymbol{\nabla} S(\boldsymbol{x})^2 + k_0^2 n^2(\boldsymbol{x}) R(\boldsymbol{x}) = 0$$

Separating this equation into real and imaginary parts we find two equations for the amplitude and the phase of the wave

$$\begin{aligned} [\boldsymbol{\nabla} S(\boldsymbol{x})]^2 - n^2(\boldsymbol{x}) - \frac{\boldsymbol{\nabla}^2 R(\boldsymbol{x})}{k_0^2 R(\boldsymbol{x})} &= 0 \\ \boldsymbol{\nabla} \cdot [R(\boldsymbol{x})^2 \boldsymbol{\nabla} S(\boldsymbol{x})] &= 0 \end{aligned} \tag{3}$$

These equations have a relatively simple physical interpretation. The first is of the same form as the Hamilton–Jacobi equation of mechanics (see Landau and Lifshitz "*Mechanics*"), where we can identify $\boldsymbol{\nabla} S$ with the momentum of particle, and $-n^2-\boldsymbol{\nabla}^2 R\,/\,k_0^2 R$ as proportional to the potential energy minus the total energy of the particle. The second equation is stating that there are no sources or sinks of $R^2\boldsymbol{\nabla} S$. We can therefore think of our wave in new terms: the gradient of the phase $\boldsymbol{\nabla} S$ traces out a trajectory in space, with the density of trajectories given by R^2. These trajectories cannot be created or destroyed, hence $\boldsymbol{\nabla}\cdot[R^2\boldsymbol{\nabla} S]=0$. It is this change in perspective (from wave to particle) that is the origin of our understanding of light as a collection of rays: in the cases treated here *a ray is a line traced out by the gradient of the phase of a wave*.

The two equations (3) are at the moment entirely equivalent to the Helmholtz equation, so we've not made anything easier. But it is now clear in a different way why solving the Helmholtz equation is difficult. The two equations (3) are coupled through the term $-\boldsymbol{\nabla}^2 R\,/\,k_0^2 R$. This term means that as the density of trajectories varies rapidly in space (e.g. as the wave is focussed into a small volume so that $k_0^3 V\sim 1$), the particle momentum $\boldsymbol{\nabla} S$ is changed, and the ray changes direction. We can simplify things if we take the limit $k_0\to\infty$ (i.e. the phase of the wave varies much more rapidly than anything else). Having done this, the problematic term disappears from (3) and $\boldsymbol{\nabla} S$ can be determined without reference to the wave amplitude R

$$\begin{aligned}[\boldsymbol{\nabla} S(\boldsymbol{x})]^2-n^2(\boldsymbol{x}) &= 0\\ \boldsymbol{\nabla}\cdot[R(\boldsymbol{x})^2\boldsymbol{\nabla} S(\boldsymbol{x})] &= 0\end{aligned}\tag{4}$$

These are the equations of geometrical optics (the first of which is known as the *eikonal equation*), and are valid whenever the curvature of the wave amplitude $-\boldsymbol{\nabla}^2 R\,/\,R$ is vanishingly small in comparison to the square of the free space wave–vector k_0^2. It is worth metioning for a second time that this procedure can be carried out for any wave equation, and one can switch between the wave and a particle pictures. This takes some of the mystery out of quantum mechanics, where the 'rays' of the Schrodinger equation are the particles of classical mechanics. We'll now spend some time exploring these approximate equations (4).

1.1 Geometrical optics

We'll begin with the simplest case. We'll ignore the second of the two equations (4) and multiply the first by $1/2$.

$$\frac{\boldsymbol{p}^2}{2}+\frac{1}{2}(1-n^2(\boldsymbol{x}))=\frac{1}{2}\tag{5}$$

where I wrote $\boldsymbol{\nabla} S=\boldsymbol{p}(\boldsymbol{x})$. This equation is of the same form as the conservation of energy in mechanics, for a particle of unit mass $m=1$, in a potential $V(\boldsymbol{x})=\frac{1}{2}[1-n^2(\boldsymbol{x})]$ and with fixed energy $1/2$. Note that I added and substracted $1/2$ from both sides of the first of (4) to obtain (5). I did this so that at infinity (where we assume $n=1$), the momentum is a unit vector $|\boldsymbol{p}|=1$. Having associated (5) with energy conservation, the Hamiltonian can be identified as

$$H(\boldsymbol{x},\boldsymbol{p})=\frac{\boldsymbol{p}^2}{2}+\frac{1}{2}(1-n^2(\boldsymbol{x}))\tag{6}$$

and we can use Hamilton's equations of motion (see e.g. Landau and Lifshitz "*Mechanics*") to find the trajectories of the rays

$$\begin{aligned}\dot{\boldsymbol{x}} &= \frac{\partial H}{\partial \boldsymbol{p}}\\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{x}}\end{aligned}\tag{7}$$

which for our Hamiltonian (6) take the form

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{p}\\ \dot{\boldsymbol{p}} &= \frac{1}{2}\frac{\partial n^2(\boldsymbol{x})}{\partial \boldsymbol{x}}\end{aligned} \tag{8}$$

which are the same equations that govern a classical particle of unit mass in a potential $V(\boldsymbol{x}) = \frac{1}{2}[1-n^2(\boldsymbol{x})]$. Given a particular refractive index profile, we can integrate these equations, finding the trajectories of the rays, and this in turn tells us about the phase of the wave through $\boldsymbol{p} = \boldsymbol{\nabla} S$ and the amplitude of the wave through the density of rays. In the attached notebook "*ray–tracing.ipynb*" I have written a simple program to integrate these equations in an arbitrary two dimensional refractive index profile.

1.2 A couple of useful refractive index profiles

There are a few 'famous' refractive index profiles that perform useful operations on bundles of rays. Here we look at two of these and solve for the motion of the rays.

1.2.1 The Luneburg lens:

Perhaps the simplest and most useful is the 'Luneburg lens', which focusses a parallel bundle of rays to a point

$$n(r) = \begin{cases}\sqrt{2-\left(\frac{r}{a}\right)^2} & r \leqslant a\\ 1 & r > a\end{cases} \tag{9}$$

where r is equal to $\sqrt{x^2+y^2}$ in the two dimensional Luneburg lens, and $\sqrt{x^2+y^2+z^2}$ in the three dimensional device. Within this index profile the equations of motion (8) are those for a radial harmonic oscillator

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{p}\\ \dot{\boldsymbol{p}} &= \ddot{\boldsymbol{x}} = -\frac{1}{2}\boldsymbol{\nabla}\left(\frac{r}{a}\right)^2\end{aligned} \tag{10}$$

If we concentrate on the two dimensional case, there is a very nice trick that enables these equations to be solved quite simply (essentially it amounts to the replacement $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rightarrow 1, i$. We re–write our coordinate on the ray, and the ray momentum as complex numbers

$$\begin{aligned}z &= x + iy\\ p &= p_x + ip_y\end{aligned}$$

similarly the gradient operation as

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z^*}$$

then our equations of motion within the Luneburg lens (10) become the differential equation for two scalar variables

$$\begin{aligned}\dot{z} &= p\\ \dot{p} &= -\frac{\partial}{\partial z^*}\frac{zz^*}{a^2} = -\frac{z}{a^2}\end{aligned} \tag{11}$$

or

$$\ddot{z} = -\frac{z}{a^2}$$

which has the general solution

$$z(t) = A e^{\frac{it}{a}} + B e^{-\frac{it}{a}} \tag{12}$$

where A and B are arbitrary constants i.e. the motion of the complex number z within the lens in the x, y plane is the sum of two circulating motions. The values of A and B are determined by the boundary conditions at the edge of the lens ($r = a$), which are fixed by the motion of the incoming rays. Suppose we have a set of rays travelling to the right, parallel to the $x-$ axis. When one of these rays hits the edge of the lens (at $t = 0$) it will have a position

$$z(0) = x(0) + i y(0) = a e^{i\theta(0)} \tag{13}$$

(where $\theta \in [\pi/2, 3\pi/2]$ is the angle made between the position at the edge of the lens and the origin) and momentum

$$p(0) = \dot{z}(0) = 1 \tag{14}$$

Enforcing (13) and (14) on (12) we find the values of A and B. First finding

$$\begin{aligned} A + B &= a e^{i\theta(0)} \\ A - B &= -i a \end{aligned}$$

so that

$$\begin{aligned} A &= \frac{a}{2}[e^{i\theta(0)} - i] \\ B &= \frac{a}{2}[e^{i\theta(0)} + i] \end{aligned} \tag{15}$$

and the motion of our rays is given by

$$z(t) = \begin{cases} t + a e^{i\theta(0)} & t \leqslant 0 \\ a\left[\cos\left(\frac{t}{a}\right) e^{i\theta(0)} + \sin\left(\frac{t}{a}\right)\right] & t > 0 \text{ (inside lens)} \end{cases}$$

The thing to notice now is that at the time $t = \pi a/2$ all rays have the same position $z = a$, which is the furthest point on the right of the lens: i.e. all the rays arrive in parallel, but are focussed to the same point on the right of the lens. This shows that, in the limit of geometrical optics the Luneburg lens profile will take an incident plane wave and focus it to a point. The behaviour is illustrated in the accompanying notebook: "*ray–tracing.ipynb*".

1.2.2 The Eaton lens:

The Eaton lens is another refractive index profile that performs a useful function. It takes any ray incident onto the surface of the device (at $r = a$), and forces it on a trajectory that bends around the origin ($r = 0$), leaving the device parallel to the incident ray but moving in the opposite direction. It is a perfect 'retroreflector'. The index profile is closely related to that of the Luneburg lens (9), but with a refractive index that diverges at $r = 0$.

$$n(r) = \sqrt{\frac{2a}{r} - 1} \tag{16}$$

The reason for the divergence at $r=0$ is clear if we consider those rays that enter the profile and are heading almost directly for the origin. They must be bend on a very small circle around the origin, which requires a very large index gradient.

In this index profile the equations of motion (8) are those for a particle in the Coulomb potential

$$\dot{\boldsymbol{v}} = \boldsymbol{\nabla}\left(\frac{a}{r}\right) \tag{17}$$

In order to solve this equation of motion we again consider the two dimensional version of the problem and use complex numbers to represent the position and momentum of the ray. However, it's a bit tricky to do this in the z coordinate system we used before, and is much simpler if we change from the complex variable $z = x + iy$ to a new one $w = w_1 + iw_2$ (this is a particular kind of *conformal transformation*—we shall return to these in detail below):

$$z = z(w) = \frac{1}{2a}w^2 \tag{18}$$

under such a change of coordinates the eikonal equation of geometrical optics (4) changes to

$$[\boldsymbol{\nabla}S]^2 = \left|\frac{\partial S}{\partial z}\right|^2 = n^2(z, z^*) \rightarrow \left|\frac{\partial S}{\partial w}\right|^2 = \left|\frac{dw}{dz}\right|^{-2} n^2(w, w^*)$$

implying that the effect of the change of coordinate system (18) is to replace the refractive index squared with $n^2|dw/dz|^{-2}$. In terms of the complex variable w the equations of geometrical optics therefore take a similar form to (11)

$$\begin{aligned} p &= \frac{dw}{dt} \\ \frac{dp}{dt} &= \frac{\partial}{\partial w^*}\left(\left|\frac{dw}{dz}\right|^{-2} n^2(w, w^*)\right) \end{aligned}$$

substituting the Eaton lens profile (16) and the coordinate transformation (18) we see that the above equation reduces to

$$\begin{aligned} p &= \frac{dw}{dt} \\ \frac{dp}{dt} &= \frac{\partial}{\partial w^*}\left(\left|\frac{w}{a}\right|^2\left[\frac{2a}{\sqrt{zz^*}} - 1\right]\right) \\ &= \frac{\partial}{\partial w^*}\left(\left|\frac{w}{a}\right|^2\left[\frac{4a^2}{|w|^2} - 1\right]\right) \\ &= -\frac{w}{a^2} \end{aligned}$$

which is exactly the same equation of motion as we had for the Luneburg lens (11), just in terms of this new variable w. We therefore know the form of the solution is the same as (12)

$$w(t) = Ae^{\frac{it}{a}} + Be^{-\frac{it}{a}}$$

which in terms of the real space coordinates is

$$z(t) = \frac{1}{2a}w(t)^2 = \frac{1}{2a}\left[Ae^{\frac{it}{a}} + Be^{-\frac{it}{a}}\right]^2 \tag{19}$$

All that's left is to apply the boundary condition. We assume that we have a parallel bundle of rays travelling to the right, parallel to the $x-$axis, so that at the boundary of the device (at time $t=0$) we have

$$\begin{aligned} z(0) &= a e^{i\theta(0)} \\ \dot{z}(0) &= 2 \end{aligned}$$

(the reason that $\dot{z}=2$ on the boundary of the device rather than $\dot{z}=1$ is because the 'time' parameterizing the motion inside the device was introduced in the w coordinate system. In the w system the index equals $\sqrt{2}$ at the edge of the device (where $w=\sqrt{2}a$), and hence $\dot{z}=w\dot{w}/a=2$). We now demand that the constants A and B in equation (19) are such that these boundary conditions are satisfied at $t=0$

$$\begin{aligned} (A+B)^2 &= 2a^2 e^{i\theta(0)} \\ \frac{i}{a^2}(A-B)(A+B) &= 2 \end{aligned}$$

and we can then use these equations to find A and B

$$\begin{aligned} B &= -A\left(\frac{1-i e^{i\theta(0)}}{1+i e^{i\theta(0)}}\right) \\ A &= \frac{i a}{\sqrt{2}} e^{-i\theta(0)/2}(1+i e^{i\theta(0)}) \end{aligned} \tag{20}$$

Applying conditions (20) to our general solution (19) we find the motion of our rays

$$z(t)=\frac{1}{2a}w(t)^2=\begin{cases} t+a e^{i\theta(0)} & t\leqslant 0 \\ \frac{a}{4}\left(e^{i\theta(0)/2}+\frac{e^{-i\theta(0)/2}}{i}\right)^2\left[e^{\frac{it}{a}}+\frac{e^{i\theta(0)/2}-\frac{e^{-i\theta(0)/2}}{i}}{e^{i\theta(0)/2}+\frac{e^{-i\theta(0)/2}}{i}}e^{-\frac{it}{a}}\right]^2 & t>0\,(\text{inside lens}) \end{cases}$$

Our rays start at the position $a e^{i\theta(0)}$ (with $\theta(0)\in[\pi/2,3\pi/2]$) on the edge of the device. After the time $t=\pi a/2$, the above equation shows that they are at the position

$$\begin{aligned} z(\pi a/2) &= \frac{a}{4}\left(e^{i\theta(0)/2}+\frac{e^{-i\theta(0)/2}}{i}\right)^2\left[i-\frac{e^{i\theta(0)/2}-\frac{e^{-i\theta(0)/2}}{i}}{e^{i\theta(0)/2}+\frac{e^{-i\theta(0)/2}}{i}}i\right]^2 \\ &= a e^{-i\theta(0)} \end{aligned}$$

which is again on the edge of the device, but at the negative of the original angle (i.e. flipped across the $y-$axis) compared to the incoming ray. The ray is also propagating in the reverse direction

$$\dot{z}(\pi a/2) = -2$$

Therefore the Eaton lens acts as a perfect retroreflector. The analytic and numerical results for the trajectories in this profile are plotted in the notebook that goes with this lecture "*ray–tracing.ipynb*".

1.3 The WKB approximation

To go a step beyond the approximation of the previous section, we keep *both* equations (4). In general it is still very difficult to find both the amplitude R and the phase S of the wave. That is, except in one–dimensional problems, where we can find both the phase and the amplitude exactly. The form of the wave that one finds is called the WKB approximation (after Wentzel, Kramers and Brillouin, none of whom invented the method, according to John Heading in "*Phase Integral Methods*"—its basics were put in place by Rayleigh).

In one dimension the two equations of geometrical optics (4) reduce to

$$\begin{aligned} \left(\frac{dS}{dx}\right)^2 &= n^2(x) \\ \frac{d}{dx}\left(R(x)^2\frac{dS}{dx}\right) &= 0 \end{aligned} \tag{21}$$

From the second of these we can find $R(x)$ exactly in terms of the gradient of S

$$R = \frac{A}{\sqrt{\left|\frac{dS}{dx}\right|}} \tag{22}$$

where A is a constant. From the first of equations (21) we can also find the phase $S(x)$.

$$\frac{dS}{dx} = \pm n(x) \to S(x) = \int^x n(x')dx'$$

combining (21) and (22) we find an approximate form of the solution to the 1D Helmholtz equation in an inhomogeneous medium

$$\phi(x) = \frac{A}{\sqrt{n(x)}} e^{\pm i k_0 \int^x n(x')dx'} \tag{23}$$

this is known as the WKB approximation, and is valid for 1D problems where the refractive index varies on a length scale that is much larger than the local wavelength. The solution (23) has a clear physical interpretation. As the wave travels through the medium it has a local wave–vector $k_0 n(x)$, so the accumulated phase is $k_0\int^x n(x')\,dx'$. Furhtermore, as seen in lecture 5, the time average of the power carried by a wave is proportional to $\text{Im}[\phi^*\,d\phi/dx]$. To keep this power flow constant (so energy is neither created nor destroyed), the amplitude of the wave must decrease as the phase varies more rapidly, hence the pre–factor of $1/\sqrt{n}$.

Notice that each of the two solutions (23) propagates either entirely to the right, or entirely to the left. This indicates that the WKB approximation neglects the phenomenon of reflection. One needs to add in some extra physics if reflection is to be included in the approximation. How to do this is interesting but we do not have space here—see e.g. "*Phase integral methods*" by John Heading.

The WKB approximation is useful for finding the change in the amplitude and phase of a wave, after it has passes through a slowly varying refractive index profile. In the accompanying notebook "*wkb-approximation.ipynb*" we compare (23) to a numerical solution of the one–dimensional Helmholtz equation.

2 Conformal mapping and transformation optics:

So far we have explored the geometrical optics approximation for solving the Helmholtz equation in inhomogeneous materials. This is very useful when the material properties vary on a length scale that is much larger than the local wave–length, and geometrical optics has long been the theory of choice for the design of gradient index lenses. However, the approximation of geometrical optics breaks down when the material properties vary on a length scale that is comparable to or smaller than the wavelength. We are actually very familiar with this in everyday life: all reflection is evidence of the break–down of geometrical optics. As we saw in the WKB approximation (23), there is no reflection when we drop the term $-\boldsymbol{\nabla}^2 R/k_0^2 R$ from equations (3).

In comparatively recent times there has been some effort to go beyond geometrical optics, to provide an exact theory for designing materials that manipulate waves rather than rays. One branch of this effort is called '*transformation optics*'. Transformation optics is based on the fact that performing a coordinate transformation changes the wave equation in the same way that a material changes the wave equation. One can therefore start with empty space where the wave is (say) $\exp(ik_0 x)$ and perform a coordinate transformation $x, y, z \to u, v, w$. In the new system the wave can look arbitrarily complicated as a function of u, v, w: $\exp(ik_0 x(u, v, w))$. But we can use the equivalence between the coordinate transformation and a material to read off the material parameters that will give rise to such a wave. This method has been applied most famously to design '*invisibility cloaks*' (J. B. Pendry, D. Schurig and D. R. Smith *Science* **312** 5781 2006). For more details see Leonhardt and Philbin, "*Geometry and Light: The Science of Invisibility*" Dover (2010).

Here we shall restrict ourselve to 2D problems and derive the simplest form of transformation optics, which is sometimes called 'optical conformal mapping' (see U. Leonhardt *Science* **312** 1777 2006). In solving for the propagation of rays through the Luneburg and Eaton lenses we made use of a single complex number z to represent the position of the ray in the $x-y$ plane. When it came to solving for motion in the Eaton lens, we used a coordinate transformation (18) to change between the w and z complex planes, so that the index profile was simplified to something we had solved before. This ability to change complex variables, and in the process exchange one refractive index profile for another is not unique to geometrical optics, but also holds exactly for the 2D Helmholtz equation.

To see how complex numbers help us out, let's start with the Laplacian

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{24}$$

if we write $z = x + iy$, and $z^* = x - iy$ then the derivatives with respect to x and y in (24) can be replaced with derivatives with respect to z and z^*

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} \\ \frac{\partial}{\partial y} &= i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right) \end{aligned} \tag{25}$$

Substituting (25) into (24) we see that the 2D Laplacian takes a rather simple form in terms of the complex numbers z and z^*

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right)^2 - \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)^2 = 4\frac{\partial^2}{\partial z \partial z^*}$$

Using this result we find our 2D Helmholtz equation can be written in the simpler form

$$\left[4\frac{\partial^2}{\partial z \partial z^*} + k_0^2 n^2(z, z^*)\right]\phi(z, z^*) = 0 \tag{26}$$

Now, all we've done so far is re–write our 2D Helmholtz equation in a different form. There is no advantage to doing this yet. But suppose we perform a coordinate transformation to a new complex variable w

$$z \to z(w) \tag{27}$$

It is assumed that $z = x + iy$ depends only on $w = \mathrm{Re}[w] + i\,\mathrm{Im}[w]$ and *not* on $w^* = \mathrm{Re}[w] - i\,\mathrm{Im}[w]$. Such transformations are called *conformal maps*, and have the rather nice property that the lines of constant $\mathrm{Re}[w]$ are orthogonal to the lines of constant $\mathrm{Im}[w]$, which we can see quite straightforwardly

$$\boldsymbol{\nabla} w = \frac{dz}{dw}\boldsymbol{\nabla} z \to \left.\begin{array}{l}\boldsymbol{\nabla}\mathrm{Re}[w] = \mathrm{Re}[z']\hat{\boldsymbol{x}} - \mathrm{Im}[z']\hat{\boldsymbol{y}}\\ \boldsymbol{\nabla}\mathrm{Im}[w] = \mathrm{Im}[z']\hat{\boldsymbol{x}} + \mathrm{Re}[z']\hat{\boldsymbol{y}}\end{array}\right\} \to \boldsymbol{\nabla}\mathrm{Re}[w]\cdot\boldsymbol{\nabla}\mathrm{Im}[w] = 0$$

Performing the conformal transformation (27), the Laplacian becomes

$$\boldsymbol{\nabla}^2 = 4\frac{\partial^2}{\partial z\partial z^*} = \frac{dw}{dz}\frac{\partial}{\partial w}\left(\frac{dw^*}{dz^*}\frac{\partial}{\partial w^*}\right) = \left|\frac{dw}{dz}\right|^2\frac{\partial^2}{\partial w\partial w^*}$$

where we should reiterate that e.g. z^* depends only on w^* and not on w. The Helmholtz equation (26) thus becomes

$$\left[4\frac{\partial^2}{\partial w\partial w^*} + k_0^2 n^2(w, w^*)\left|\frac{dw}{dz}\right|^{-2}\right]\phi(w, w^*) = 0$$

Which is also a Helmholtz equation of exactly the same form, but in the new refractive index profile $n^2(w, w^*)\left|\frac{dw}{dz}\right|^{-2}$. Therefore, if we can solve (26) exactly then we can simply perform as many conformal transformations as we like and get the exact solution to the Helmholtz equation in an arbitrary number of other inhomogneous refractive index profiles. An example of such a method of solution is given in the notebook that accompanies this lecture: "*conformal-maps.ipynb*".