Top trading cycle

- When all preferences are strict the procedure yields a unique outcome.
- The procedure also ensures truthful preference revelation.
- Examples: Housing, students residence, extended to kidneys.
If we assume that kidneys get exchanged only between donors and recipients then the housing market result will carry over to the kidney market.

All surgeries occur simultaneously.

But a key difference is that kidney markets have recipients/patients without donors.
Consider allocation of student housing
- There are students with occupied rooms.
- New students arrive
- And there are fixed number of vacant rooms.

Existing tenants are entitled to keep their room but also apply for others.

Mechanism: *you request my house-* I get your turn.
For any given preference and ordering:

1. Assign the first student his top choice, the second his from the remaining rooms and so on until someone requests the room of an existing tenant.

2. When that happens, modify the procedure by moving *the existing tenant* to beginning of the line and proceed with the procedure.

3. If at any point a cycle is formed, then the cycle is exclusively formed by existing tenants.
For constructing the Kidney exchange, the key difference between the student room allocation problem and the kidney allocation problem is that in the kidney allocation problem the number of new kidneys are not known but the fixed number of vacant rooms are known.
There is a donor and recipient pair \((k_i, t_i)\), where \(k\) is the donor/kidney and \(t\) is the patient/recipient.

\(K\) is the set of kidneys.

- A patient will most likely be incompatible with a part of the set of available kidneys \(K\).

Patients will have preferences over compatible kidneys.

If only direct exchanges are allowed then *top trading cycles mechanism* be used to allocate kidneys.
Problem is that getting on to the waiting list implies receiving a lottery.

For a patient $t_i$, let $K_i$ be the set of compatible kidneys.

Let $w$ be the option of entering the waiting list.

So patient $t_i$ has preferences over $K_i \cup \{k_i, w\}$. 
Cycles and chains

- A *cycle* is an ordered list of kidneys and patients
  - \((k'_1, t'_1, k'_2, t'_2, \ldots, k'_m, t'_m)\),
  - \(k'_1\) points to \(t'_1\), \(t'_1\) points to \(k'_2\), \(k'_2\) points to \(t'_2\) and so on and finally \(t'_m\) is assigned kidney \(k'_1\)
- Note: each kidney or patient can at most be part of one cycle.

- **w-chain** is an ordered pair of kidneys and patients
  - \((k'_1, t'_1, k'_2, t'_2, ..., k'_m, t'_m)\)
  - \(k'_1\) points to \(t'_1\), \(t'_1\) points to \(k'_2\), \(k'_2\) points to \(t'_2\) and so on and finally \(t'_m\) is assigned kidney from the wait list \(w\).

- Crucially a kidney or a patient can be part of several w-chains.
Consider a graph where there are distinct nodes of kidney and patient pair. Each patient either points toward a kidney or the waiting list $w$, and each kidney points to a paired recipient then either a cycle or $w$-chain exists.
1. Each active patient points to his most preferred remaining unassigned kidney or to the wait list option.
2. There is either a cycle or w-chain or both.
3. Locate cycles, and assign the kidneys and then remove all the patients and then continue with next set of cycles.
Example - 12 pairs

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Final matching is
- t1,k9
- t2,k11
- t3,k2
- t4,k8
- t5,k7
- t6,k5
- t7,k6
- t8,k4
- t9,w
- t10,k1
- t11,k3
- t12,k10
Cycle 1 (k11, t11, k3, t3, k2, t2)
Cycle 2 (k7, t7, k6, t6, k5, t5)
No new cycle can be formed after this
  But two w-chains can be formed
  \( W1=(k8,t8, k4,t4, k9,t9) \) and \( W2=(k10,t10, k1, t1, k9,t9) \)
  Since the high priority patient is in \( W2 \),
  implement \( W2 \). \( k10 \) is available for next round.
Cycle 3 (k4, t4, k8, t8)
Allocation without transfer payments

- One of the basic problems in economics is to allocate scarce resources

- Markets are not always feasible or desirable
  - Kidney markets
  - Sports tickets
  - Metropolitan Opera
23 Flats being allocated in “first come first serve” basis at Princesshay

Since 2007, school places in Brighton were to be allocated by lottery.

Parents were upset.

Todd and Surajeet allocated papers to read, on the basis of first come first serve basis, where the queue was to start at 8 am.

Students complained to the University.
Many people want scarce goods. Individuals know how much they want the good, but the one determining the allocation may not. Typical solution is to use markets.
Goods are often allocated not to agents who are willing to pay most, but willing to work the most.

Examples
1. Olympics.
2. Sport and concert ticket.

Another mechanism observed is random allocation or lottery

Examples
1. Baseball or College bowl tickets
2. Charity
And we also observe the mix of the two
- Wimbledon
- Running marathons

Pseudo-markets
- Example: class allocation in MBA schools and law schools in US
Compare lottery and queue.

- Valuation of money and time are homogeneous.
- Time and money valuations are negatively correlated.
- Objects are scarce.

No transfer payments
Allocation problem

- The designer’s problem is to allocate $M$ homogeneous, not-necessarily-divisible goods among $N$ agents (bidders) where $M < N$.
- The designer is benevolent and wishes to maximize the social surplus.
- Each agent $i$ has a privately known type (signal) $\theta_i \in \mathbb{R}_+$ that is drawn independently from cumulative distribution $F$.
- Agent $i$ has value $v(\theta_i) \geq 0$ for at most one object, such that $v'(\theta_i) \geq 0$.
- Each agent $i$ is able to send a range of costly messages $x_i \in \mathbb{R}_+$ to the designer. The cost to the agent of sending message $x_i$ depends upon his type and equals $c(x_i) \cdot g(\theta_i) \geq 0$.
- We have the assumption $v'(\theta_i)/v(\theta_i) > g'(\theta_i)/g(\theta_i)$ for $\theta_i > 0$. 
• The designer then receives these costly signals \((x_1, \ldots, x_N)\) and uses them to allocate the \(M\) goods by rule \(a : \mathbb{R}_+^N \rightarrow [0, 1]^N\) where \(\sum_i a_i(x_1, \ldots, x_N) \leq M\) guarantees feasibility.

• Given allocation rule \(a \in A\), the agents form a Bayes-Nash equilibrium by choosing a strategy \(x_i(\theta_i, a)\) to maximize their expected surplus given the strategies of other agents. The designer’s problem is to choose rule \(a\) to maximize the equilibrium social surplus of the agents given the future Bayes-Nash equilibria of the agents, that is, the designer solves

\[
\max_{a \in A} \sum_i E[v(\theta_i) \cdot a_i(x_1(\theta_1, a), \ldots, x_N(\theta_N, a)) - c(x_i(\theta_i, a)) \cdot g(\theta_i)]
\]