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Applied Mathematics Letters 17 (2004) 561–567
Applied Mathematics Letters
www.elsevier.com/locate/aml

Finite Time Extinction in Nonlinear Diffusion Equations

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(Received and accepted March 2003)

Communicated by B. Peletier

Abstract—We consider a class of degenerate diffusion equations where the nonlinearity is assumed to be singular (non-Lipschitz) at zero. It is shown that solutions with compactly supported initial data become identically zero in finite time. Such extinction follows by comparison with newly constructed finite travelling waves connecting two stable equilibria. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Finite travelling waves, Degenerate diffusion, Singular, Extinction.

1. INTRODUCTION

This paper is concerned with the quasilinear parabolic equation

$$u_t = d\left(u^{m+1}\right)_{xx} + f(u), \qquad (x,t) \in Q := \mathbb{R} \times (0,\infty), \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \quad 0 \le u_0 \le 1, \tag{2}$$

where $m > 0$, $d = 1/(m+1)$, $u_0 \in C(\mathbb{R})$, and f satisfies the following.

(A1) $f \in C[0,1] \cap C^1(0,1]$, $f(0) = f(1) = 0$, and $f'(1) < 0$. Moreover, $\exists\, \alpha \in (0,1)$ such that $f < 0$ on $(0,\alpha)$ and $f > 0$ on $(\alpha, 1)$.
(A2) $\exists\, p \in (0,1)$ and $\beta > 0$ such that $N := m + p \ge 1$ and $u^{-p} f(u) \to -\beta$ as $u \to 0^+$.
(A3) $\int_0^1 v^m f(v)\, dv < 0$.

0893-9659/04/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.aml.2003.03.008

Assumptions (A1)–(A3) will collectively be referred to as (A). Such diffusion equations arise in many applications, including population genetics, signal propagation in nerve axons, and combustion theory [1–4].

In this paper, we establish the existence of a unique (modulo translation) finite travelling wave (FTW) $v_\omega(z)$ of (1) satisfying $v_\omega(-\infty) = 1$ and $v_\omega(z) = 0$ for all $z \geq \omega$, for any $\omega \in \mathbb{R}$ (see Section 3). In particular, v_ω has negative velocity. By an FTW, we mean any travelling wave (TW) solution $u(x - ct)$ of (1) satisfying $u(z) = 0$ for all $z \geq \omega$ (or $z \leq \omega$) for some $\omega \in \mathbb{R}$. Utilising v_ω and its reflection as upper solutions, we then deduce the finite time extinction of compactly supported solutions of (1),(2). This follows from an existence-uniqueness-comparison result in Section 2. Finally, we apply our results to a singular bistable nonlinearity and present some numerical simulations.

In the case of a smooth bistable nonlinearity $f \in C^2[0,\infty)$, Hosono [5] proved the existence and stability of a unique FTW with nonnegative velocity satisfying $u(-\infty) = 1$ and $u(z) = 0$ for all $z \geq 0$, provided $\int_0^1 u^m f(u)\,du \geq 0$. However, when the reverse integral inequality holds in [5] there are no FTW solutions connecting the equilibria. It is precisely the regularity of f at the degenerate point $u = 0$ which excludes the existence of such waves in the nonsingular case.

We remark that while finite time extinction phenomena are known to exist in absorptive heat equations of the form

$$u_t = d\left(u^{m+1}\right)_{xx} - Cu^p, \qquad 0 < p < 1 \tag{3}$$

(see [6] and the references therein), we are unaware of any results in this direction for sign-changing nonlinearities. In particular, for initial data satisfying $f(u_0(x)) > 0$ for some x, one cannot deduce the finite time extinction property in the general case via comparison with solutions of (3).

2. PRELIMINARIES

Before proceeding to the study of TW solutions, we first settle the question of existence, uniqueness, and comparison of solutions for the Cauchy problem (1),(2). Here and throughout, $Q_T := \mathbb{R} \times (0,T)$.

DEFINITION 2.1. *A nonnegative function u is said to be a weak solution of (1),(2) if and only if for every $r, T > 0$, $u \in C(Q_T) \cap L^\infty(Q_T)$ and*

$$\int_{Q_T} u\phi_t + du^{m+1}\phi_{xx} + f(u)\phi\,dx\,dt = \int_{-r}^{r} u(x,T)\phi(x,T) - u_0(x)\phi(x,0)\,dx + d\int_0^T u^{m+1}(r,t)\phi_x(r,t) - u^{m+1}(-r,t)\phi_x(-r,t)\,dt,$$

for all $\phi \in C^{2,1}(\bar{Q}_T)$ such that $\phi \geq 0$ and $\phi(\pm r, t) = 0$ for all $t \in [0,T]$.

LEMMA 2.1. *If (A1),(A2) hold then there exists a unique weak solution u of (1),(2). Moreover, if u and v denote the solutions of (1),(2) with initial data satisfying $0 \leq u_0 \leq v_0 \leq 1$ in $\mathbb{R}$, then $0 \leq u \leq v \leq 1$ in Q.*

PROOF. The proof is identical to [7, Theorems 2.5 and 2.10] except for a minor modification to allow for the sign-changing nonlinearity f considered here. The vital ingredient which allows us to generalise the result in [7] is the upper Lipschitz condition

$$\exists\, K > 0 \text{ such that } f(v) - f(u) \leq K(v - u), \qquad \text{for all } 0 \leq u \leq v \leq 1, \tag{4}$$

which holds due to (A1),(A2). Below, we outline the modification required.

Existence follows via the following well-known construction. For $k \in \mathbb{N}$ denote by (P_k) the problem

$$u_t = d\left(u^{m+1}\right)_{xx} + f(u) - f(\epsilon_k), \qquad (x,t) \in Q_{k,T} := (-k,k) \times (0,T),$$
$$u(\pm k, t) = M := \sup_{x \in (-\infty,\infty)} u_0(x), \qquad t \in [0,T],$$
$$u(x,0) = \psi_{0,k}(x), \qquad x \in [-k,k].$$

Here, $\epsilon_k \to 0$ as $k \to \infty$ and $\psi_{0,k}$ is a monotone decreasing sequence converging to u_0 (see [7, Section 4.A]). By classical results for uniformly parabolic equations, the solution sequence u_k is monotone decreasing and bounded below by $\epsilon_k > 0$. The pointwise limit function $u \in L^\infty(Q_T)$ then satisfies the integral identity in Definition 2.1.

To obtain uniqueness, suppose that $\hat{u}$ is any other solution of (1),(2). By comparison for (P_k), $\hat{u} \le u_k$ for all k so that $\hat{u} \le u$. Hence, it is sufficient to prove that for any fixed $t \in (0,T]$, $\chi \in C_0^\infty(\mathbb{R})$, $0 \le \chi \le 1$ and $\epsilon > 0$, there exists an $r > 0$ such that

$$\int_{-r}^{r} (u(x,t) - \hat{u}(x,t))\chi(x)\,dx < \epsilon$$

(see [7, equation (4.3)]). We define a_k as in [7, Section 3.B] by

$$a_k(x,t) = \begin{cases} \dfrac{d\left(\hat{u}^{m+1} - u_k^{m+1}\right)}{(\hat{u} - u_k)}, & u_k \ne \hat{u}, \\ \hat{u}^m, & u_k = \hat{u} \end{cases}$$

and note that since $u_k \ge \epsilon_k$, the bound $a_k \ge C(k) := \epsilon_k^m$ holds as in [7, equation (3.10)]. The only modification we need to make is in the choice of b_k in [7, Section 4.B]. Due to the lack of monotonicity of f in our case, we take

$$b_k(x,t) = \begin{cases} \dfrac{(f(u_k) - f(\hat{u}))}{(\hat{u} - u_k)} + K, & u_k \ne \hat{u}, \\ K - f'(\hat{u}), & u_k = \hat{u}. \end{cases}$$

By (4), it follows that $0 \le b_k \le C_3(k)$ as in [7, equation (4.5)]. The integral identity [7, equation (4.6)] then has the extra term $\iint K(u_k - \hat{u})\phi_{k,n}$ on the right-hand side. Letting $n \to \infty$, then $k \to \infty$ and noting that $\phi_{k,n} \le 1$, one then has the inequality

$$\int_{-r}^{r} (u(x,t) - \hat{u}(x,t))\chi(x)\,dx < \epsilon + K\int_0^t \int_{-r}^{r} (u(x,s) - \hat{u}(x,s))\,dx\,ds$$

for r sufficiently large. Hence, by Gronwall's lemma

$$\int_{-r}^{r} (u(x,t) - \hat{u}(x,t))\chi(x)\,dx < \epsilon e^{Kt}.$$

Since t is fixed and ϵ is arbitrary, this gives the required result.

Comparison and continuity of u follow from [7, Theorems 2.10 and 2.5]. ∎

3. FINITE TRAVELLING WAVES AND EXTINCTION

Let us write (1) in the divergence form

$$u_t = (u^m u_x)_x + f(u), \qquad (x,t) \in Q.$$

Setting $z = x - ct$, u is (formally) a TW solution of (1), with velocity c, if and only if u satisfies the quasilinear ordinary differential equation (ODE)

$$-cu' = (u^m u')' + f(u), \qquad z \in \mathbb{R}, \tag{5}$$

where $' = \frac{d}{dz}$. Following [8,9], if we rescale the 'time' variable z according to

$$\frac{ds}{dz} = \frac{1}{u^m(z)}, \tag{6}$$

and set $U(s) = u(z)$, then $u(z)$ is a (weak) TW solution of (1) if and only if $U(s)$ is a (classical) TW solution of

$$U_t = U_{xx} + U^m f(U), \qquad (x,t) \in Q, \tag{7}$$

or equivalently,

$$-c\dot{U} = \ddot{U} + U^m f(U), \qquad s \in \mathbb{R}, \tag{8}$$

where $\dot{} = \frac{d}{ds}$. We call (8) the *desingularised ODE* since by (A1),(A2), $U^m f(U) \in C^1[0,1]$.

Initially, we seek TW solutions of (7) for $c > 0$ connecting the equilibria $U = 0$ and $U = 1$. The following result can be found in [2, Theorem 2.4(b) and equation (2.7)].

LEMMA 3.1. *If (A) holds, then there exists a unique wave speed $c^* > 0$ such that (8) has a positive solution $U^*(s)$ satisfying $U^*(-\infty) = 0$ and $U^*(\infty) = 1$. Moreover, U^* is monotone in s.*

Now let $c > 0$ and write (8) as the first-order system

$$\dot{U} = V, \tag{9}$$

$$\dot{V} = -cV - U^m f(U). \tag{10}$$

System (9),(10) possesses equilibria at $(0,0)$ and $(1,0)$. Linearisation about these points then yields the local flow. (Of course, technically one would need to extend the functions U^m and $f(U)$ smoothly to include $U \leq 0$ in order to define a smooth vector field in an open neighbourhood of the origin, but the flow in the right-half plane would remain unaltered.) The equilibrium $(1,0)$ is a hyperbolic saddle point. The topological type of $(0,0)$ depends on the value of N as follows.

If $N > 1$, the origin is nonhyperbolic and has a one-dimensional stable manifold $W^s(0,0)$ tangent to the eigenvector $(1,-c)^T$ with corresponding eigenvalue $\lambda = -c$, and a one-dimensional centre manifold $W^c(0,0)$ tangent to the eigenvector $(1,0)^T$ with corresponding eigenvalue $\lambda = 0$. (Note that the superscript in $W^c(0,0)$ signifies the *centre* manifold and not its dependence on the wave speed c.) A straightforward centre manifold reduction [10, Theorem 3, p. 25] gives the local representation of $W^c(0,0)$, restricted to $U \geq 0$, as a graph over U given by

$$V = \frac{\beta}{c} U^N + o\left(U^N\right), \qquad \text{as } U \to 0^+. \tag{11}$$

Consequently, the local flow of (9),(10) restricted to $W^c(0,0)$ for $U \geq 0$, is unstable.

If $N = 1$, the origin is a hyperbolic saddle with corresponding eigenvalues

$$\lambda_\pm(c) := \frac{1}{2}\left(-c \pm \sqrt{c^2 + 4\beta}\right) \tag{12}$$

and stable and unstable manifolds $W^s(0,0)$ and $W^u(0,0)$ tangent to the eigenvectors $(1,\lambda_-)^T$ and $(1,\lambda_+)^T$, respectively.

Candidates for FTWs of (1) are rescaled solutions of (9),(10) satisfying $(U(s),V(s)) \to (0,0)$ as $s \to -\infty$ along $W^c(0,0)$ when $N > 1$, or $W^u(0,0)$ when $N = 1$. We now show that the departure times along these invariant manifolds are finite in the original z time scale while establishing sufficient regularity required of a weak solution. In what follows, we define the departure time $\omega \in [-\infty, \infty]$ by $u(\omega) = 0$.

For $N = m + p = 1$, $W^u(0,0)$ has the local form $V = \lambda_+ U + o(U)$, by Hartman-Grobman. Hence,

$$\frac{du}{dz} = \frac{dU}{ds}\frac{ds}{dz} = \frac{\lambda_+ u + o(u)}{u^m}, \qquad \text{as } u \to 0^+ \tag{13}$$

by (6) and (9). Integrating (13) from $z = \omega$ to z, one obtains $m\lambda_+(z-\omega) = u^m + o(u^m)$. Hence, ω is finite and we have the regularity result

$$u = O\left((z-\omega)^{1/(1-p)}\right), \qquad \text{as } z \to \omega^+. \tag{14}$$

When $N > 1$ and we consider solutions departing along $W^c(0,0)$, the local form (11) together with (6) yield

$$\frac{du}{dz} = \frac{dU}{ds}\frac{ds}{dz} = \frac{(\beta/c)u^N + o\left(u^N\right)}{u^m} = \frac{\beta}{c}u^p + o\left(u^p\right).$$

Integrating from $z = \omega$ to z, one obtains $\beta(z-\omega) = cu^{1-p}/(1-p) + o(u^{1-p})$ and again ω is finite. Consequently, we have the regularity result $u(z) = O((z-\omega)^{1/(1-p)})$ as $z \to \omega^+$, just as in (14).

Using the finite time departure in z along $W^u(0,0)$ ($N = 1$) or $W^c(0,0)$ ($N > 1$), we may construct the FTWs of (1) connecting $u = 0$ and $u = 1$. By Lemma 3.1, there exists a unique TW (modulo translation) $U^*(s)$ of (7) for $c = c^* > 0$ satisfying $U^*(s) \to 0$ as $s \to -\infty$ and $U^*(s) \to 1$ as $s \to \infty$. Clearly, this solution corresponds to a trajectory (U^*, V^*) of (9),(10) leaving $(0,0)$ along $W^u(0,0)$ ($N = 1$) or $W^c(0,0)$ ($N > 1$) at $s = -\infty$ and arriving at $(1,0)$ at $s = \infty$. For any $\omega \in \mathbb{R}$, the rescaling (6) now gives rise to a function $u^*(z)$, defined for all $z \geq \omega$, satisfying $u^*(z) \to 0$ as $z \to \omega^+$ and $u^*(z) \to 1$ as $z \to \infty$. Furthermore, the regularity estimate (14) holds for u^*. Defining the extended function $u_\omega(z)$ by

$$u_\omega(z) = \begin{cases} 0, & z \leq \omega, \\ u^*(z), & z > \omega, \end{cases}$$

it then follows that $u_\omega(z)$ is a weak FTW solution of (1). Hence, we have the following.

THEOREM 3.1. *Assume (A) holds and let $\omega \in \mathbb{R}$. There exists a unique $c^* > 0$ such that (1) has a finite travelling wave solution $u_\omega(z)$ satisfying $u_\omega(z) = 0$ for all $z \leq \omega$ and $u_\omega(\infty) = 1$. Furthermore, $u_\omega(z)$ is monotone in z and the regularity estimate (14) holds.*

Note that by setting $v_\omega(z) = u_\omega(-z)$, we obtain the negative velocity FTW referred to in the Introduction. Our finite time extinction result for (1),(2) now easily follows.

COROLLARY 3.1. *If (A) holds and u_0 has compact support, then the solution u of (1),(2) has compact support for all $t > 0$ and there exists a $T \geq 0$ such that $u(x,t) \equiv 0$ for all $t \geq T$.*

PROOF. First, observe that u is a subsolution of the linear porous medium equation $v_t = d(v^{m+1})_{xx} + Cv$ with the same initial data (for sufficiently large $C > 0$) and so by standard theory u has compact support for all $t \geq 0$. Second, since $u_0 \not\equiv 1$ and the solution u is classical away from $u = 0$, it follows that $u < 1$ in Q. Hence, for any $\epsilon > 0$, we may bound $u(x,\epsilon)$ above by suitable translates $u_{\omega_1}(z)$ and $v_{\omega_2}(z)$. Extinction then follows by comparison for all $t \geq \epsilon$ and the fact that $\min\{u_{\omega_1}, v_{\omega_2}\} \equiv 0$ after some finite time T. ∎

REMARK 3.1. Clearly, Corollary 3.1 holds for more general f provided there exists a solution of (8) satisfying $U^*(-\infty) = 0$ and $U^*(\infty) = 1$ for some $c^* > 0$. If f has more than one zero in $(0,1)$, then (A3) is a necessary, but not a sufficient condition for the existence of such a TW. For such cases, sufficient conditions are given in [2, Theorem 2.7]. For example, suppose f has three simple zeros $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$ such that $P(\alpha_1) < P(\alpha_2) < P(1) < 0$ and $P(\alpha_3) < P(\alpha_2)$, where $P(u) := \int_0^u v^m f(v)\,dv$. Three applications of [2, Theorem 2.7] over $[0,\alpha_2]$, $[\alpha_2, 1]$, and $[0,1]$ then yields the required TW and Corollary 3.1 applies.

4. AN EXAMPLE: NAGUMO'S EQUATION

Consider the special case where $f(u) = u^p(u-\alpha)(1-u)$, where $0 < \alpha,\ p < 1$. For $p = 1$, (1) is commonly known as Nagumo's equation, used in modelling electrical pulse propagation in nerve axons and in population genetics to model the allele effect [1,4]. As far as we are aware, all existing literature on Nagumo's equation assumes that $p \geq 1$. The singularity assumption $p < 1$ appears to be new.

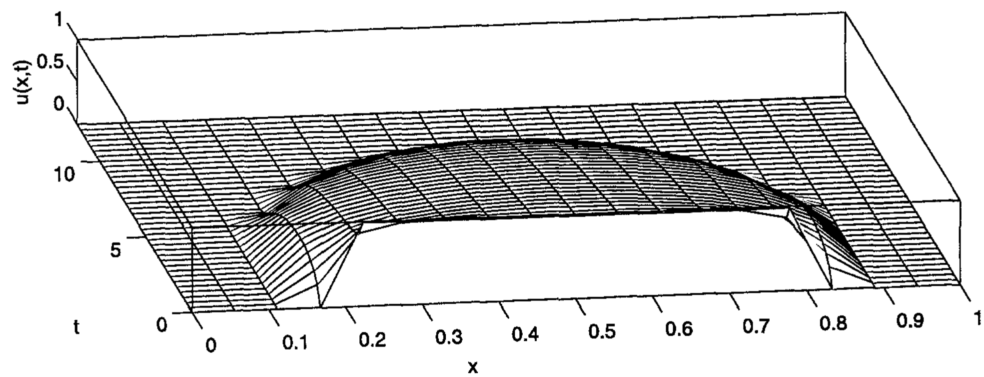
A simple calculation shows that (A) holds if and only if $\alpha > \alpha^* := (N+1)/(N+3)$. Figure 1a shows the finite time extinction of solutions proved in Corollary 3.1 when $\alpha > \alpha^*$. Figure 1b depicts convergence to the FTW u_ω of Theorem 3.1 and its reflection v_ω. Note the formation of a region Ω_0 in Q where the solution is identically zero even though the initial data is everywhere positive. Such a region is commonly known as a *dead core* [11].

In the special case $N = 1$, one can in fact verify that the FTW u_ω of Theorem 3.1 is given by the solution of the ODE

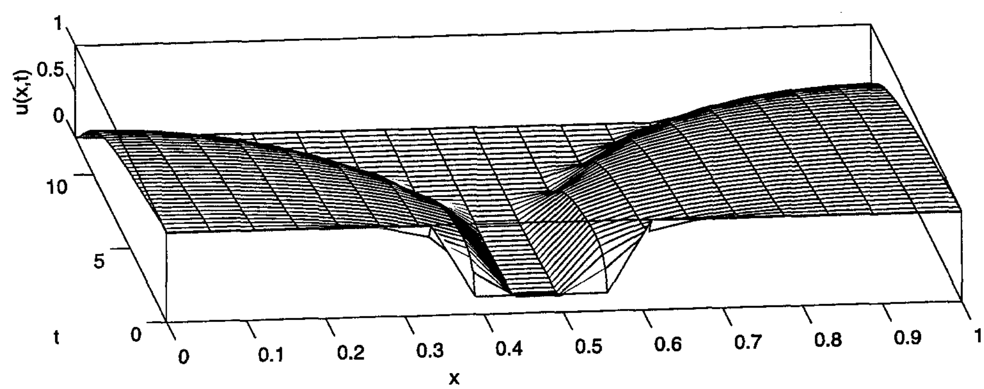
$$\frac{du}{dz} = \frac{1}{\sqrt{2}} u^p(1-u), \qquad u(\omega) = 0, \qquad u(\infty) = 1. \tag{15}$$

Furthermore, when $m = p = 1/2$, (15) admits the explicit solution

$$u = \left(\left[\tanh\left(\frac{z-\omega}{2\sqrt{2}} \right) \right]^+ \right)^2 .$$



(a)



(b)

Figure 1. Numerical solutions of (1) for $f(u) = u^p(u-a)(1-u)$ with $m = 3$, $p = 0.5$, and $a = 0.85 > a^* = 9/13$. The top figure illustrates finite time extinction while the bottom figure shows the emergence of a dead core.

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