A Hopf Bifurcation Theorem for Singular Differential-Algebraic Equations

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Abstract

We prove a Hopf bifurcation result for singular differential-algebraic equations (DAE) under the assumption that a trivial locus of equilibria is situated on the singularity as the bifurcation occurs. The structure that we need to obtain this result is that the linearisation of the DAE has a particular index-2 Kronecker normal form, which is said to be *simple index-2*. This is so-named because the nilpotent mapping used to define the Kronecker index of the pencil has the smallest possible non-trivial rank, namely one. This allows us to recast the equation in terms of a singular normal form to which a local centre-manifold reduction and, subsequently, the Hopf bifurcation theorem applies.

Key words: Hopf bifurcation, singularity, differential-algebraic equations

1 Introduction

We begin with some definitions: let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a C^∞ map where $n \geq 3$ and suppose $F(0, \mu) = 0$ for all $\mu \in \mathbb{R}$. Suppose also that $A : \mathbb{R}^{n+1} \to \mathcal{L}(\mathbb{R}^n)$ is a smooth mapping where $\mathcal{L}(X)$ denotes the space of linear maps from X to itself. A *singular solution* of a quasi-linear differential-algebraic equation (DAE) is encapsulated in the following definition which is given in the context of a one-parameter family of DAEs parameterised by a real parameter μ.

Definition 1 *The quasi-linear DAE*

$$A(x, \mu)\dot{x} = F(x, \mu), \qquad ((x(0), \dot{x}(0)) \text{ given}), \tag{1}$$

is said to have a singular solution $x_0(\cdot)$ at $\mu = \mu_0$ if $x_0 : (\alpha, \omega) \to \mathbb{R}^n$ satisfies

$$A(x_0(t), \mu_0)\dot{x}_0(t) = F(x_0(t), \mu_0), \qquad (\forall\ t \in (\alpha, \omega)),$$

and there are $t_1, t_2 \in (\alpha, \omega)$ *such that*

$$n = \operatorname{rank}(A(x_0(t_1), \mu_0)) \neq \operatorname{rank}(A(x_0(t_2), \mu_0)). \tag{2}$$

The purpose of this note is to demonstrate the existence of conditions under which one can formulate a Hopf bifurcation theorem for such quasi-linear differential-algebraic equations (DAEs) with a singularity of the form (2), hence yielding DAEs with smooth periodic orbits that intersect the singularity

$$S_\mu := \{x \in \mathbb{R}^n : \det(A(x, \mu)) = 0\}.$$

Motivation for this work can found in [3] where discrete-time implicit systems are shown to possess a family of quasi-invariant circles when the appropriate Neimark-Sacker conditions are satisfied by the linearisation of the problem. The main bifurcation result of [3] is proven in the absence of the existence of unique forward orbits by extending the centre manifold theorem appropriately which then allows one to apply standard bifurcation theorems. This note establishes an analogous Hopf bifurcation result for continuous-time systems of DAEs by imposing a simple index-2 structure on the linearisation which ensures that we can rely on the existing centre manifold theorem for smooth differential equations [6].

Hopf bifurcation theorems for DAEs in various guises do already exist [8,7,10] but none of the results of those articles apply to the class of DAE studied here. *Singular Hopf bifurcations* also arise in the context of singularly perturbed, ordinary differential equations [1,2] and these give often rise to relaxation oscillations but what we have in mind here is quite different, despite the similar nomenclature.

It is known [11] that almost-all singular solutions of (1) are non-smooth on their domains of definition and that they terminate at the singularity S_μ at either *forward* or *backward impasse points*. This means that *smooth* singular periodic solutions, meaning C^1 or greater on their interval of existence, can only exist if the set of *pseudo-equilibria* $P_\mu = \{x \in S_\mu : F(x, \mu) \in \operatorname{im}(A(x, \mu))\}$ is non-empty as any singular periodic orbit Γ must satisfy $\Gamma \cap S_\mu \subset P_\mu$. This is clear from equation (1) for if $\Gamma \cap S_\mu \subset P_\mu$ fails somewhere and $x(t_s) \in (\Gamma \cap S_\mu) \backslash P_\mu$, $A(x(t_s), \mu)\chi = F(x(t_s), \mu)$ has no solution χ and so the derivative $\dot{x}(t_s)$ must fail to exist at some point along the solution $x(t)$.

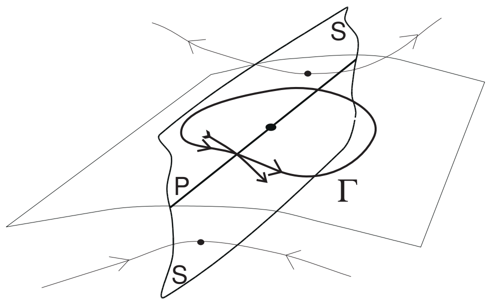
From a geometric point of view, the singular periodic orbit Γ must form *two* connections between *two* distinct points on P_μ and as the singularity is generically a codimension-1 submanifold of $\mathbb{R}^n$, $\Gamma \backslash S_\mu$ will contain at least two components, one either side of the singularity. This is illustrated in Figure 1 where P is a manifold of pseudo equilibria and S is the singularity. As pictured, solution uniqueness typically breaks down at pseudo equilibria [5] and one can only claim as a result that a singular periodic orbit Γ forms a *quasi*-invariant set, defined for completeness as follows.

Suppose that $\{\phi(t,x)\}_{x\in\mathfrak{M}}$ for $\alpha(x) \le t \le \omega(x)$ is a set of solution trajectories of the DAE $F(x,\dot{x}) = 0$ for each $x \in \mathfrak{M} \subset \mathbb{R}^n$ in the sense that $x(t) = \phi(t, x(0))$ are solutions:

$$F\left(\phi(t,x), \frac{\partial \phi}{\partial t}(t,x)\right) = 0$$

for all (t,x); the mapping ϕ satisfies $\phi(0,x) = x$ for all x for which ϕ is defined. A set $\mathcal{Q} \subset \mathfrak{M}$ is said to be *quasi-invariant* if **there exists at least one** $\bar{x} \in \mathcal{Q}$ such that $\phi(t,\bar{x}) \in \mathcal{Q}$ for all $\alpha(\bar{x}) \le t \le \omega(\bar{x})$. An *invariant set* $\Omega \subset \mathfrak{M}$ is one such that $\phi(t,x) \in \Omega$ for all $t \in [\alpha(x), \omega(x)]$ and all $x \in \Omega$.

Fig. 1. A periodic orbit Γ connects pseudo equilibria to each other and encircles an equilibrium. Forward uniqueness of solutions typically breaks down at P, even though smooth solutions do exist there.



Forward non-uniqueness properties of singular periodic solutions are observed in very simple DAEs such as

$$\dot{x} = y, \tag{3}$$
$$0 = x^2 + y^2 - 1, \tag{4}$$

which has a plethora of periodic orbits. Clearly $(x(t), y(t)) = (\cos(t), -\sin(t))$ represents a periodic solution, but so too does the 4π periodic extension of

$$(x(t), y(t)) = \begin{cases} (\cos(t), -\sin(t)) : & 0 \le t \le 2\pi, \\ (1,0) & : 2\pi \le t \le 4\pi, \end{cases}$$

to $\mathbb{R}$ which can be constructed due to the presence of the equilibrium point $(1,0)$ that lies on a periodic orbit. It is easy to construct infinitely many *Canard*-like periodic solutions for (3-4) whereby $y \in L^\infty(\mathbb{R})$ and y is piecewise smooth but not continuous. However, the types of solutions identified in such simple systems, while interesting, are not tractable using the techniques of this paper as they result from global connections between different quasi-invariant sets.

In the remainder of the paper we shall also state results in terms of the so-called semi-explicit DAE form

$$\dot{x} = f(x, y, \mu), \tag{5}$$
$$0 = g(x, y, \mu), \tag{6}$$

such that $x \in \mathbb{R}^n, y \in \mathbb{R}^m, f(0,0,\mu) = 0, g(0,0,\mu) = 0$ for all $\mu \in \mathbb{R}$ where $n \geq 2$ and $m \geq 1$. In terms of (5-6) we shall define objects associated with the singularity using **bold** font in order to distinguish them from the related concepts of the quasilinear form (1). So, the constraint manifold is

$$\mathbf{C}_\mu := \{(x, y) \in \mathbb{R}^{n+m} : g(x, y, \mu) = 0\},$$

the singularity is

$$\mathbf{S}_\mu := \{(x, y) \in \mathbf{C}_\mu : \det(d_y g(x, y, \mu)) = 0\},$$

and the set of pseudo-equilibria is

$$\mathbf{P}_\mu := \{(x, y) \in \mathbf{S}_\mu : d_x g[f] \in \operatorname{im}(d_y g)\},$$

where functions and their derivatives are evaluated at (x, y, μ) in the latter definition. A *singular equilibrium* is an equilibrium point of (5-6) in $\mathbf{S}_\mu$.

1.1 Notation

Here and throughout for real τ and integer k, $\mathcal{O}_k^\tau(x)$ is the set of smooth maps $f(x, \tau)$ such that $\|f(x, \tau)\| \leq \kappa(\tau)\|x\|^k$ for a continuous family of constants κ and $f \in \mathcal{O}_k(x)$ means that $\|f(x)\| \leq \kappa\|x\|^k$ for all x in some neighbourhood of zero. Also, $\mathcal{O}_k(x, y)$ is a synonym for $\mathcal{O}_k((x, y))$ and so, for example, $\tau x \in \mathcal{O}_1^\tau(x) \cap \mathcal{O}_2(\tau, x)$. By $T_p(\mathcal{M})$ we mean the tangent space of the manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$ and both symbols d and ∇ represent the derivative of a smooth map, similarly d_x and $\frac{\partial}{\partial x}$ both represent a partial derivative with respect to x.

The spectrum of a matrix pencil (A, B) is the set

$$\sigma(\mathsf{A}, \mathsf{B}) = \{\alpha \in \mathbb{C} : \det(\alpha\mathsf{A} - \mathsf{B}) = 0\}.$$

Suppose that $\det \mathsf{A} = 0$ in the matrix pencil (A, B) acting on $\mathbb{R}^r$ and suppose this pencil is regular, meaning that $\det(\alpha\mathsf{A} - \mathsf{B}) \neq 0$ for some $\alpha \in \mathbb{C}$, then (A, B) can be transformed into Kronecker Normal Form (KNF) through a change of coordinates:

$$P\mathsf{A}Q = \begin{pmatrix} I_p & 0 \\ 0 & \mathsf{N} \end{pmatrix}, P\mathsf{B}Q = \begin{pmatrix} \mathsf{C} & 0 \\ 0 & I_q \end{pmatrix}. \tag{7}$$

Here $\mathsf{N} \neq 0$ and $\mathsf{N}^\nu = 0$ for some $\nu \geq 1$, $p, q \geq 1$ are integers such that $p+q = r$ and P, Q are real $r \times r$ matrices.

If $i\omega \in \sigma(\mathsf{A}, \mathsf{B})$ for a real $\omega > 0, z \in \mathbb{C}^r$ satisfies $i\omega \mathsf{A} z = \mathsf{B} z$ and we set $z = Qw$, then $i\omega P\mathsf{A}Qw = P\mathsf{B}Qw$ and so the real and imaginary part of the first p-components of w (with associated projection Π, so that $\Pi(w_p, w_q) = w_p$ for $(w_p, w_q) \in \mathbb{C}^{p+q}$) forms a 2-dimensional real space

$$\mathrm{span}\{\Re(\Pi[w]), \Im(\Pi[w])\}$$

which is invariant for C. Similarly, we shall call the 2-dimensional space given by $\mathrm{span}\{\Re(z), \Im(z)\}$ the *real eigenspace* of (A, B) associated with $i\omega$.

Throughout the paper we shall use M for the operator matrix

$$M \equiv \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

acting on a space of the form $\mathbb{R}^{n+m}$ for integers n and m.

2 Preliminary: the *simplest possible* index-2 linear structure

Our aim is to employ the Hopf bifurcation theorem to create smooth, singular periodic solutions of DAEs by suitably restricting the nature of the singularity of the linearisation at an equilibrium point and by subsequently controling the unfolding of that singularity as a parameter is varied. Therefore a restricted definition, the so-called *simple index-2* singularity will be invoked to ensure that singularities of (1) and (5-6) are manifested in their linearisations in as mild a fashion as possible.

Definition 2 *A regular matrix pencil* (A, B) *on* $\mathbb{R}^r$ *is said to be* simple index-2 *if in its Kronecker normal form (7), the matrix* $\mathsf{N} \in \mathcal{L}(\mathbb{R}^q)$ *is nilpotent with* $\mathrm{rank}(\mathsf{N}) = 1$.

It is straightforward to prove a Hopf bifurcation theorem for (1) under the assumption that $A(0, \mu)$ is an invertible mapping: one simply rewrites the problem as an ODE near $x = 0$ and utilises the well-known Hopf bifurcation theorem for smooth ODEs. In order to relax such an invertibility restriction in order to allow singularities, we study certain classes of DAE whose linearisation is *simple* index-1 at the bifurcation point.

From [4] we have the following lemma that gives a concrete case in which simple index-2 matrix pencils can be identified.

Lemma 3 *Suppose that* $A : \mathbb{R}^n \to \mathbb{R}^n, B : \mathbb{R}^m \to \mathbb{R}^n, C : \mathbb{R}^n \to \mathbb{R}^m, D : \mathbb{R}^m \to \mathbb{R}^m$ *are linear maps satisfying* $\ker(D) = \operatorname{span}\{k\} \neq \{0\}, CBk \notin \operatorname{im}(D)$ *and*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0,$$

then the matrix pencil

$$\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right)$$

is simple index-2.

Lemma 3 provides a link to the Singularity-Induced Bifurcation (SIB) theorem of Venkatasubramanian *et al.* [12,13] where the authors study solution trajectories of DAEs when equilibria collide with singularities under a one-parameter variation. Lemma 3 above follows from the proof of [4, Theorem 7] and the assumptions of that theorem are equivalent to the original SIB theorem of [12].

2.1 A Centre-Manifold Reduction

In order to address bifurcation issues regarding (1) we seek an appropriate centre-manifold for this type of DAE, leading us to initially consider problems without parameters. So, consider the quasi-linear differential equation

$$A(x)\dot{x} = F(x), \qquad (x \in \mathbb{R}^n), \tag{9}$$

which we may write in semi-explicit form as

$$\dot{x} = y, \; A(x)y = F(x), \tag{10}$$

where $(x(0), \dot{x}(0)) = (x(0), y(0)) \in \mathbb{R}^{2n}$ is given. The basis for this paper begins with the following theorem which shows not only that the singularity of a DAE can support smooth solutions, those solutions can be extended to form a smooth quasi-invariant manifold.

Theorem 4 *Suppose that* $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *and* $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$ *are* C^∞ *maps such that* $f(0,0) = 0, g(0,0) = 0$ *and*

$$\left(M, \begin{pmatrix} d_x f(0,0) & d_y f(0,0) \\ d_x g(0,0) & d_y g(0,0) \end{pmatrix} \right)$$

is a simple index-2 *matrix pencil over* $\mathbb{R}^{n+m}$. *Then, for each* $r \geq 1$ *there is a codimension-1, quasi-invariant* C^r *graph* $\mathcal{M}^{(r)} \subset g^{-1}\{0\}$ *of*

$$\dot{x} = f(x,y), 0 = g(x,y), \tag{11}$$

such that $(0,0) \in \mathcal{M}^{(r)}$. *Moreover, for each open ball* $\mathcal{M}' \subset \mathcal{M}^{(r)}$ *containing* $(0,0)$ *there is a* $\mathcal{T} \subset \mathbb{R}$ *and a local dynamical system* $\boldsymbol{\phi} : \mathcal{T} \times \mathcal{M}' \to \mathcal{M}^{(r)}$ *such that* $(x(t), y(t)) = \boldsymbol{\phi}(t; (x_0, y_0))$ *satisfies (11) for all* $t \in \mathcal{T}$.

PROOF. Expand (11) using Taylor's theorem as

$$\dot{x} = Ax + By + \mathcal{O}_2(x,y), \tag{12}$$
$$0 = Cx + Dy + \mathcal{O}_2(x,y), \tag{13}$$

and form the matrix pencil (M, L), where $A = d_x f(0,0)$ and the remaining entries of L are analogously defined:

$$L \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The pencil (M, L) is regular and *simple index-2* by assumption.

We can set (11) in terms of a nonlinear perturbation of the Kronecker normal form of (M, L) by putting $Z \equiv Q(u,v) = (x,y)$ for $(u,v) \in \mathbb{R}^{p+q}$ so that $x = \alpha u + \beta v$. This is a problem of the form $M\dot{z} = L\dot{z} + \mathcal{O}_2(z)$, whence $PMQ(\dot{u}, \dot{v}) = PLQ(u,v) + \mathcal{O}_2(u,v)$ and we may finally write this in the form

$$\dot{u} = \mathsf{C}u + \mathcal{F}(u,v), \tag{14}$$
$$\mathsf{N}\dot{v} = v + \mathcal{G}(u,v), \tag{15}$$

where $\mathrm{im}(\mathsf{N}) = \mathrm{span}\{n_0\}$ for some vector $n_0 \in \mathbb{R}^q$ of unit norm. Here $\mathcal{F}$ and $\mathcal{G}$ are both maps of the form $\mathcal{O}_2(u,v)$.

Next we shall write $v(t) = \lambda(t)n_0 + w(t)$ where $\lambda(t) \in \mathbb{R}$ and $w(t) \in \mathrm{span}\{n_0\}^{\perp}$, for all values of t for which these functions are defined. Hence, from (15) we find that

$$\mathsf{N}[\dot{\lambda(t)} n_0 + \dot{w}(t)] = \lambda(t)n_0 + w(t) + \mathcal{G}(u, \lambda(t)n_0 + w(t)),$$

from where

$$\mathsf{N}[\dot{w}(t)] = \lambda(t)n_0 + w(t) + \mathcal{G}(u, \lambda(t)n_0 + w(t)), \tag{16}$$

as $\mathsf{N}[n_0] = 0$. Because $\mathsf{N}[\dot{w}] \in \text{span}\{n_0\}$, by projecting (16) onto span$\{n_0\}$ and its orthogonal complement we are led to the following coupled differential and algebraic equation:

$$n_0^T \mathsf{N}[\dot{w}] = \lambda + n_0^T \mathcal{G}(u, \lambda n_0 + w), \tag{17}$$
$$0 = w + \Pi_0 \mathcal{G}(u, \lambda n_0 + w), \tag{18}$$

where $\Pi_0 : \mathbb{R}^q \to \text{span}\{n_0\}^\perp$ is an orthogonal projection.

We can use the implicit function theorem to remove the algebraic equation (18) and locally solve for $w = W(u, \lambda)$, where $W(0,0) = 0$ and $dW(0,0) = 0$. However, equations (17) and (18) remain coupled and we now find that the differential relation

$$\frac{d}{dt}\omega(u, \lambda) = \lambda + \gamma(u, \lambda) \tag{19}$$

holds, where $\omega(u, \lambda) := n_0^T \mathsf{N}[W(u, \lambda)]$ and $\gamma(u, \lambda) = n_0^T \mathcal{G}(u, \lambda n_0 + W(u, \lambda))$.

If we assume that one can differentiate (19) for the moment, we obtain

$$d_u\omega(u, \lambda)[\dot{u}] + d_\lambda\omega(u, \lambda)[\dot{\lambda}] = \lambda + \gamma(u, \lambda).$$

From here

$$d_u\omega(u, \lambda)[\mathsf{C}u + \mathcal{F}(u, \lambda)] + d_\lambda\omega(u, \lambda)[\dot{\lambda}] = \lambda + \gamma(u, \lambda), \tag{20}$$

and so we can write (14-15) in the form

$$\dot{u} = \mathsf{C}u + \mathcal{O}_2(u, \lambda), \tag{21}$$
$$\sigma(u, \lambda)\dot{\lambda} = \lambda + \mathcal{O}_2(u, \lambda), \tag{22}$$

where $\sigma(u, \lambda) \equiv d_\lambda\omega(u, \lambda)$ carries the location of the singularity. Note that $\sigma(0,0) = 0$ follows from $\frac{d\omega}{d\lambda}(u, \lambda) = n_0^T \mathsf{N} d_\lambda W(u, \lambda)$ and the fact that $dW(0,0) = 0$.

By rescaling the time variable we can put (21-22) in the form of an ordinary differential which is smooth and orbit equivalent to the quasi-linear equation (21-22):

$$u' = \sigma(u, \lambda)(\mathsf{C}u + \mathcal{O}_2(u, \lambda)) = \mathcal{O}_2(u, \lambda), \tag{23}$$
$$\lambda' = \lambda + \mathcal{O}_2(u, \lambda), \tag{24}$$

where a prime ($'$) denotes differentiation with respect to s and $\frac{dt}{ds} = \sigma(u, \lambda)$.

We can now apply the centre manifold theorem to (23-24) as its linearisation at $(u, \lambda) = (0, 0)$ is the map $(u, \lambda) \mapsto (0, \lambda)$ which has a single, algebraically simple non-zero eigenvalue associated with eigenvector $(0, 1)$. This application yields the existence of an invariant C^r graph, $\mathcal{M}_0^{(r)} \subset \mathbb{R}^{p+1}$, of (23-24) as the graph of a function $\lambda = \lambda(u)$ such that $\lambda(0) = 0$ and $d\lambda(0) = 0$. This graph is quasi-invariant for (21-22) which possesses a local flow on $\mathcal{M}_0^{(r)}$ that derives from the local flow of the differential equation $\dot{u} = \mathsf{C}u + \mathcal{F}(u, \lambda(u))$. This flow is mapped under the geometric transformation constructed in this proof into a quasi-invariant manifold of (10); explicitly this transformation is $(x, y) = Q[u, \lambda n_0 + W(u, \lambda(u))] =: \mathcal{H}[u, \lambda]$. Hence we finally define $\mathcal{M}^{(r)} = \mathcal{H}(\mathcal{M}_0^{(r)}) = \mathcal{H}\{(u, \lambda(u)) : \|u\| \text{ small}\}$.

Corollary 5 *Suppose that $A : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n)$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ are smooth maps such that $F(0) = 0, \det(A(0)) = 0$ and*

$$\left(M, \begin{pmatrix} 0 & I \\ -dF(0) & A(0) \end{pmatrix} \right) \tag{25}$$

is a simple index-2 *matrix pencil over $\mathbb{R}^{2n}$. Then, for each $r \geq 1$ there is a codimension-1, quasi-invariant C^r graph $\mathcal{M}^{(r)} \subset \mathbb{R}^{2n}$ of (10) such that $(0, 0) \in \mathcal{M}^{(r)}$. Moreover, for each open ball $\mathcal{M}' \subset \mathcal{M}^{(r)}$ containing $(0, 0)$ there is a $\mathcal{T} \subset \mathbb{R}$ and a local dynamical system $\boldsymbol{\phi} : \mathcal{T} \times \mathcal{M}' \to \mathcal{M}^{(r)}$ such that $(x(t), y(t)) = \boldsymbol{\phi}(t; (x_0, y_0))$ satisfies (10) for all $t \in \mathcal{T}$.*

PROOF. Apply Theorem 4 with $f(x, y) = y$ and $g(x, y) = A(x)y - F(x)$ as then (25) is the linearisation of the resulting DAE $\dot{x} = y,\ 0 = A(x)y - F(x)$ at $(x, y) = (0, 0)$.

Applying Lemma 3, we note that the matrix pencil in (25) is simple index-2 if

$$\det(d_x F(0)) \neq 0, \ker(A(0)) = \operatorname{span}\{k\} \quad \text{and} \quad d_x F(0)[k] \notin \operatorname{im}(A(0));$$

these conditions will be utilised later in Theorem 8.

The main tenet of Theorem 4, namely the existence of a quasi-invariant manifold of a nonlinear DAE when its linearisation is simple index-2, relies essentially on the following observation that is buried within the proof of Theorem 4.

Lemma 6 *If* $\mathsf{C} \in \mathcal{L}(\mathbb{R}^p)$ *is any matrix and* $\mathsf{N} \in \mathcal{L}(\mathbb{R}^q)$ *is nilpotent with rank 1, then the semilinear DAE, with* $(a, b) \in \mathbb{R}^{p+q}$*,*

$$\dot{a} = \mathsf{C}a + \mathcal{F}(a, b), \tag{26}$$

$$\mathsf{N}\dot{b} = b + \mathcal{G}(a, b), \qquad (\text{where } \mathcal{F}, \mathcal{G} \in \mathcal{O}_2(a, b)) \tag{27}$$

has a local, quasi-invariant manifold $\mathcal{M} \subset \mathbb{R}^{p+q}$ *that contains the point* $(a, b) = (0, 0)$*. Solution trajectories on* $\mathcal{M}$ *satisfy* $b(t) = h(a(t))$ *where* h *is a differentiable function defined in a neighbourhood of* $a = 0$ *with* $h(0) = 0, dh(0) = 0$*.*

PROOF. The details of this result are a repetition of the proof of Theorem 4 that follows the presentation of (14)-15). The basis of the proof of both lemmas is that a smooth partial differential equation of the form

$$h(a) = \mathsf{N}dh(a)[\mathsf{C}a + \mathcal{F}(a, h)] + \mathcal{G}(a, h), \qquad h(0) = 0, dh(0) = 0, \tag{28}$$

has a solution given by a certain centre manifold if N is nilpotent with rank 1.

It may be instructive to compare Lemma 6 with [4, Theorem 2.5], [5, conditions (A1-A5)] and [12]. It is shown in these references that that singular equilibria of DAEs possess certain invariant manifolds under restrictions on *both* the matrix pencil obtained from linearising the DAE and on the geometric nature of the constraint in (11). Lemma 6 is a slightly stronger result because it states that one of the invariant manifolds described in these references can be found if the DAE only satisfies a condition on its linearisation that does not require any information of the geometric properties of the constraint manifold.

Lemma 6 may also be considered a continuous-time analogy of a result in [3] which studies the functional equation

$$h(a) = \mathsf{N}h(\mathsf{C}a + \mathcal{F}(a, h)) + \mathcal{G}(a, h), \quad h(0) = 0. \tag{29}$$

Equation (29) arises in the context of discrete-time implicit systems, whether obtained from a numerical discretisation of (1) or from the first-order optimality equations of infinite-horizon optimal control (see [9] for example). It is noteworthy that discrete-time problems are simpler than their continuous time counterparts in the sense that one can establish the existence of a fixed point h of (29) independently of the index of the matrix N using a contraction argument. It is an open problem to establish the existence of solutions of (28) when N has rank greater than 1, this is an important question as its resolution would quickly lead to new bifurcation theorems for singular DAEs.

3 A Singular Hopf Bifurcation Theorem

With Lemma 6 in place we are in a position to prove the following main result regarding systems of the form (5-6).

Theorem 7 *Suppose that the one-parameter family of DAEs (5-6) satisfies* $f(0,0,\mu) = 0, g(0,0,\mu) = 0$ *for all* μ *and write* $L(\mu)$ *for the matrix of partial derivatives* $d_{x,y}(f,g)(0,0,\mu)$. *If*

(1) $\ker(d_y g(0,0,0)) = \operatorname{span}\{k\}$,
(2) $d_x g(0,0,0) d_y f(0,0,0) k \notin \operatorname{im}(d_y g(0,0,0))$,

and $(M, L(\mu))$ *has a* transverse Hopf point *in the sense that*

(3) $\pm\omega_0 i \in \sigma(M, L(0))$, $\pm\ell\omega_0 i \notin \sigma(M, L(0))$ *for each* $\ell \in \{0, 2, 3, ...\}$ *and*
(4) $\lambda(\mu) \in \sigma(M, L(\mu))$ *satisfies*

$$\operatorname{Im}(\lambda(0)) = \omega_0 > 0, \operatorname{Re}(\lambda(0)) = 0 \quad \text{and} \quad \left.\frac{d}{d\mu}\operatorname{Re}(\lambda(\mu))\right|_{\mu=0} \neq 0,$$

then

(i) there is a half-open interval J *containing* $\mu = 0$ *in its closure such that for all* $\mu \in J$*, (5-6) possesses a periodic orbit.*

(ii) It follows from assumptions (1) and (2) of the theorem that $\mathbf{S}_\mu$ *is a codimension-1 submanifold of the* n*-dimensional manifold* $\mathbf{C}_\mu$ *for sufficiently small* $|\mu|$*. As a result, if* $0 \in \mathbf{S}_\mu$ *for small* $|\mu|$ *and the two-dimensional real eigenspace* $E := \operatorname{span}\{e_R, e_I\}$ *associated with purely imaginary eigenvalues* $\pm\omega_0 i \in \sigma(M, L(0))$ *intersects the singularity* $\mathbf{S}_\mu$ *transversally at* $\mu = 0$*, then the periodic orbit from (i) is singular.*

PROOF. It follows from the assumptions and Lemma 3 that $(M, L(0))$ is *simple index-2*. Now re-write (5-6) in the form

$$\begin{aligned} \dot{\mu} &= 0, && (30)\\ \dot{x} &= A(\mu)x + B(\mu)y + \mathcal{O}_2^\mu(x,y), && (31)\\ 0 &= C(\mu)x + D(\mu)y + \mathcal{O}_2^\mu(x,y), && (32) \end{aligned}$$

that can be re-cast as $\dot{\mu} = 0, \quad M\dot{z} = L(\mu)z + \mathcal{O}_2^\mu(z)$, or

$$\dot{\mu} = 0, \quad M\dot{z} = L(0)z + \mu L'(0)z + \mathcal{O}_2(z,\mu), \tag{33}$$

where a prime ($'$) denotes $d/d\mu$. Putting the linear terms of equation (33) into KNF upon setting $Qw = z$ yields

$$PMQ\dot{w} = PL(0)Qw + \mu PL'(0)Qw + \mathcal{O}_2(Qw, \mu)$$

and this can be written in the form

$$\begin{aligned} \dot{\mu} &= 0, && (34)\\ \dot{a} &= \mathsf{C}a + \mathcal{O}_2(\mu, a, b), && (35)\\ \mathsf{N}\dot{b} &= b + \mathcal{O}_2(\mu, a, b). && (36) \end{aligned}$$

From Lemma 6, (34-36) possesses a differentiable, quasi-invariant manifold on which $b = h(a, \mu)$ and $h(0, 0) = 0, dh(0, 0) = (0, 0)$.

Now consider the resulting differential equation

$$PMQ\dot{w} = PL(\mu)Qw + \mathcal{O}_2^{\mu}(Qw), \tag{37}$$

noting that M is independent of μ. Equation (37) can be rewritten as

$$\begin{aligned} \dot{\mu} &= 0, && (38)\\ \dot{a} &= \mathsf{C}(\mu)a + \mathcal{E}_1(\mu)b + \mathcal{O}_2^{\mu}(a, b), && (39)\\ \mathsf{N}\dot{b} &= \mathcal{E}_2(\mu)a + I(\mu)b + \mathcal{O}_2^{\mu}(a, b), && (40) \end{aligned}$$

where $\mathcal{E}_i(0) = 0$ for $i = 1, 2$ and $\mathsf{C}(0) = \mathsf{C}, I(0) = I$. From the fact that $b = h(a, \mu)$ holds on a quasi-invariant manifold, it follows that the following PDE is satisified by h

$$\mathsf{N}d_a h(a, \mu)[\mathsf{C}(\mu)a + \mathcal{E}_1(\mu)h + \mathcal{O}_2^{\mu}(a, h)] = \mathcal{E}_2(\mu)a + I(\mu)h + \mathcal{O}_2^{\mu}(a, h)$$

and on setting $a = 0$ we obtain an equation for $H := h(0, \mu)$:

$$\mathsf{N}d_a h(0, \mu)[\mathcal{E}_1(\mu)H + \mathcal{O}_2^{\mu}(H)] = I(\mu)H + \mathcal{O}_2^{\mu}(H). \tag{41}$$

Equation (41) has solution $H = 0$ for all small $|\mu|$ and because $I(0)$ is the identity, it follows from the implicit function theorem that H is identically zero in μ, meaning that $h(0, \mu) = 0$ for all $|\mu|$ small.

Through this series of transformations, if we can show that (38-40) restricted to the graph of h has periodic solutions, then so too will the DAE (5-6) and this restriction is given by an ODE of the form

$$\dot{\mu} = 0, \tag{42}$$
$$\dot{a} = \mathsf{C}(\mu)a + \mathcal{E}_1(\mu)h(a,\mu) + \mathcal{O}_2^{\mu}(a, h(a,\mu)). \tag{43}$$

Now, $\Delta(\mu) := d_a h(0,\mu)$ is a linear map that satisfies the quadratic equation

$$\mathsf{N}\Delta[\mathsf{C}(\mu) + \mathcal{E}_1(\mu)\Delta] = \mathcal{E}_2(\mu) + I(\mu)\Delta \tag{44}$$

for small $|\mu|$. When $\mu = 0$ the matrix equation (44) is the linear problem

$$\mathsf{N}\Delta\mathsf{C} = \Delta \tag{45}$$

and since N is nilpotent, (45) has the unique solution $\Delta = 0$. The implicit function theorem then shows that (44) can be solved locally to $\mu = 0$ for $\Delta = \Delta(\mu)$ such that $\Delta(0) = 0$.

The linearisation of (43) at $a = 0$ is $\mathsf{C}(\mu) + \mathcal{E}_1(\mu)\Delta(\mu)$ and one can show that the spectrum of this linear map coincides with the *non-diverging* spectrum of $(M, L(\mu))$ for small $|\mu|$. To see this, suppose $\lambda \in \sigma(M, L(\mu))$, then $\lambda \in \sigma(PMQ, PL(\mu)Q)$ and

$$\lambda \begin{pmatrix} I & 0 \\ 0 & \mathsf{N} \end{pmatrix} \begin{pmatrix} w \\ \Delta(\mu)w \end{pmatrix} = \begin{pmatrix} \mathsf{C}(\mu) & \mathcal{E}_1(\mu) \\ \mathcal{E}_2(\mu) & I(\mu) \end{pmatrix} \begin{pmatrix} w \\ \Delta(\mu)w \end{pmatrix}$$
$$\iff \qquad \lambda w = (\mathsf{C}(\mu) + \mathcal{E}_1(\mu)\Delta(\mu))w$$

by the construction of $\Delta(\mu)$. The fact that $(M, L(\mu))$ has a transverse Hopf point at $\mu = 0$ ensures that $\mathsf{C}(\mu) + \mathcal{E}_1(\mu)\Delta(\mu)$ satisfies the conditions of the Hopf bifurcation theorem yielding the existence of a branch of periodic solutions of (43) emanating from $\mu = 0$ and part (i) follows.

Part (ii) is true for the following reasons. Given that $i\omega_0 \in \sigma(M, L(0))$ there is complex vector z such that $i\omega_0 Mz = L(0)z$. The real vectors $e_R = \mathrm{Re}(z)$ and $e_I = \mathrm{Im}(z)$ then yield a two-dimensional eigenspace of $(M, L(0))$ and we define $E := \mathrm{span}\{e_R, e_I\}$. In order to establish that the branch of period solutions from part (i) are singular it suffices that the two-dimensional centre space associated with the Hopf bifurcation point $\mu = 0$, namely E, intersects the singularity transversally at that point. However, this property is one of the assumptions of part (ii). (Note that one has to ensure that $\dim(T_0(\mathbf{S}_0) \oplus E) = n$, but since $\mathbf{S}_0$ is a codimension-1 submanifold of $\mathbf{C}_0$ from the assumptions of the theorem, in practice we actually only have to verify that either $e_R \notin T_0(\mathbf{S}_0)$ or $e_I \notin T_0(\mathbf{S}_0)$.)

A Hopf bifurcation theorem for quasi-linear problems of the form (1) now follows.

Corollary 8 *Suppose that $F(0,\mu)=0$ for all $\mu\in\mathbb{R}$. If*

(1) $\ker(A(0,0))=\operatorname{span}\{k\}$,
(2) $d_xF(0,0)[k]\notin \operatorname{im}(A(0,\mu_0))$,
(3) $\pm i\omega_0\in\sigma(A(0,0),d_xF(0,0))$ *and* $\pm i\ell\omega_0\notin\sigma(A(0,0),d_xF(0,0))$ *for all* $n\in\{0,2,3,...\}$,
(4) $\lambda(\mu)\in\sigma(A(0,\mu),d_xF(0,\mu))$ *satisfies*

$$\operatorname{Im}(\lambda(0))=\omega_0>0, \operatorname{Re}(\lambda(0))=0 \quad \text{and} \quad \frac{d}{d\mu}\operatorname{Re}(\lambda(\mu))\bigg|_{\mu=0}\neq 0,$$

then

(i) there is a half-open interval J containing $\mu=0$ in its closure such that for all $\mu\in J$ the DAE (1) possesses a periodic orbit.

(ii) If, in addition, S_μ is a codimension-1 submanifold of $\mathbb{R}^n, 0\in S_\mu$ for all μ and the real eigenspace associated with the purely imaginary eigenvalues intersects S_μ transversally at $\mu=0$, then periodic orbits formed in this bifurcation are singular.

PROOF. This is immediate from Lemma 3 and Theorem 7 and follows from setting $f(x,y,\mu):=y$ and $g(x,y,\mu):=A(x,\mu)y-F(x,\mu)$. As a result, the matrix pencil $(M,L(\mu))$ from the statement of Theorem 7 is given by

$$(M,L(\mu))=\left(\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & I\\ -d_xF(0,\mu) & A(0,\mu).\end{pmatrix}\right) \tag{46}$$

It is clear from the entries of $(M,L(\mu))$ in (46) that conditions (1) and (2) of Theorem 7 are satisfied.

Now, note that $\lambda\in\sigma(M,L(\mu))$ if and only if $\lambda Mz=L(\mu)z$ for some non-zero vector $z=(x,y)^T$, which is the eigenvalue equation

$$\lambda x-y=0,\quad d_xF(0,\mu)x-A(0,\mu)y=0. \tag{47}$$

It is clear that (47) holds if and only if $d_xF(0,\mu)x-A(0,\mu)\lambda x=0$ for a non-zero x and finally therefore $\lambda\in\sigma(A(0,\mu),d_xF(0,\mu))$. Thus conditions (3) and (4) of Theorem 7 are satisfied.

If assumptions (3-4) of Theorem 8 are satisfied then we say that (1) possesses a transverse Hopf point at $\mu = 0$.

4 Examples

Let us probe the results of this paper with a series of related examples. First consider the fully nonlinear differential equation in $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$\dot{x}_1 = \mu x_1 - x_2 + \mu\eta(\mathbf{x}), \tag{48}$$
$$\dot{x}_2 = x_1 + \mu x_2 + \mu\zeta(\mathbf{x}), \tag{49}$$
$$\frac{d}{dt}\psi_\mu(\mathbf{x}) = \sigma x_3 + x_\tau^2, \tag{50}$$

where ψ_μ is a smooth one-parameter family of functions with $\nabla\psi_0(0,0,0) = (0,0,0)$ and σ is a parameter that is fixed at either 0 or 1, $\tau \in \{1,2,3\}$ and we also assume that $\eta, \zeta \in \mathcal{O}_2(\mathbf{x})$. Let us assume for simplicity that

$$\zeta(0,0,x_3) = 0 \Leftrightarrow x_3 = 0$$

and $\eta(0,0,x_3) \equiv 0$ so that for a sufficiently small $\delta > 0$,

$$\lim_{x_2 \to 0} \frac{\eta(0,x_2,x_3)}{x_2} = \frac{\partial\eta}{\partial x_2}(0,0,x_3) =: \beta(x_3)$$

is a continuous function satisfying $\mu\beta(x_3) \neq 1$ for all $\mathbf{x} \in B_\delta(\mathbf{0}) \subset \mathbb{R}^3$ and $|\mu| < \delta$.

This set of assumptions ensure that (48-50) has a transverse Hopf point at $\mu = 0$, but is this sufficient to ensure that a branch of period solutions emanates from $\mu = 0$?

The singularity for (48-50) is the set

$$S_\mu = \left\{ \mathbf{x} \in \mathbb{R}^3 : \frac{\partial\psi_\mu}{\partial x_3}(\mathbf{x}) = 0 \right\}$$

and the Hopf point that occurs at $\mu = 0$ is associated with the two-dimensional real eigenspace

$$E := \mathrm{span}\{(1,0,0),(0,1,0)\}$$

arising from the two purely imaginary finite eigenvalues $\pm i$ of the matrix pencil

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}\right). \tag{51}$$

We now consider three examples that derive from (48-50).

Example 9 *This example is constructed so that Theorems 7 and 8 are inapplicable to (48-50) and no periodic solutions exist: set $\sigma = 0$ in (50) and note that the matrix pencil (51) is singular.*

If $\sigma = 0$ and $\tau = 1$ there is a neighbourhood of zero that is independent of μ such that the differential system (48-50) has no periodic orbits on which $x_1 \not\equiv 0$. This can be seen by integrating (50) over a period, from $t = 0$ to $t = T$ say, yields $\int_0^T x_1^2(t)dt = 0$. However, on the corresponding space $\mathcal{I}_\epsilon := \{\mathbf{x} \in B_\epsilon(\mathbf{0}) : x_1 = 0\}$, any solution of (48-50) would have to satisfy

$$0 = \dot{x_1} = -x_2 + \mu\eta(0, x_2, x_3), \;\; \dot{x_2} = \mu x_2 + \mu\zeta(0, x_2, x_3).$$

But then $\dot{x_1} = x_2(-1+\mu\eta(0, x_2, x_3)/x_2)$ and one can therefore choose ϵ so that $\dot{x_1} \neq 0$ in $\mathcal{I}_\epsilon$ unless $x_2 \equiv 0$ along that solution, but then $\zeta(0, 0, x_3(t)) = 0$ on some time interval and so $x_3(t) = 0$ too by the assumption on ζ.

As a result, $\mathcal{I}_\epsilon \backslash \{\mathbf{0}\}$ cannot be quasi-invariant in the sense that every non-trivial solution starting in $\mathcal{I}_\epsilon$ near zero must leave $\mathcal{I}_\epsilon$ immediately. This set cannot therefore contain a periodic orbit near zero and so $\mu = 0$ cannot be a Hopf bifurcation point as ϵ can be chosen independently of μ. In this case Theorem 8 cannot be applicable, indeed it is not because conditions (2) and (3) fail with $\ell = 0$ for any nonlinear map ψ_μ with the stated restrictions.

Example 10 *This example shows that even if Theorems 7 and 8 are inapplicable to (48-50), a Hopf bifurcation to singular periodic solutions may still occur.*

To achieve this, set $\sigma = 0$ and $\tau = 3$ in (50) and note again that conditions (2) and (3) of Theorem 8 fail. Let

$$\psi_\mu(\mathbf{x}) = \frac{1}{2}(x_1^2 + (x_2 + x_3 - \mu)^2)$$

so that $S_\mu = \{\mathbf{x} \in \mathbb{R}^3 : x_2 + x_3 = \mu\}$ and the tangent space of S_0 at $\mathbf{x} = \mathbf{0} \in \mathbb{R}^3$

is

$$T_{\mathbf{0}}(S_0) = \text{span}\{(1,0,0)^T, (0,1,-1)^T\}$$

which is transverse to the eigenspace E at their point of intersection.

For the resulting system

$$\dot{x}_1 = \mu x_1 - x_2 + \mu\eta(\mathbf{x}), \tag{52}$$

$$\dot{x}_2 = x_1 + \mu x_2 + \mu\zeta(\mathbf{x}), \tag{53}$$

$$\frac{d}{dt}\psi_\mu(\mathbf{x}) = x_3^2, \tag{54}$$

on setting $\mu = 0$, (52-54) simplifies somewhat to

$$\dot{x}_1 \;=\; -x_2, \quad \dot{x}_2 = x_1, \tag{55}$$

$$(x_2 + x_3)\ \dot{x}_3 = x_3\ (x_3 - x_1). \tag{56}$$

We therefore observe that (52-54) possesses a so-called vertical *Hopf bifurcation at $\mu = 0$ with periodic solutions on the quasi-invariant set $\{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\}$ of the form*

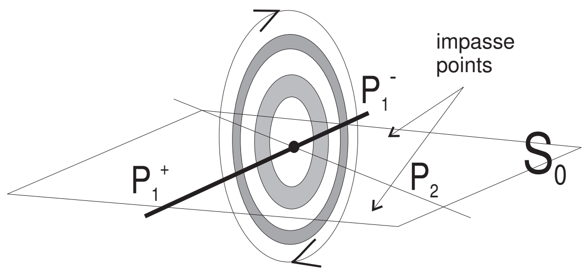
$$\mathbf{x}_{s_1,s_2}(t) = (s_1\cos(t) + s_2\sin(t), s_1\sin(t) - s_2\cos(t), 0)$$

for any non-zero, real s_1 and s_2. This family of periodic solutions contains trajectories that are singular as the condition $\mathbf{x}_{s_1,s_2}(t) \in S_0$ is met when $s_1\sin(t) - s_2\cos(t) = 0$, a solution of which arises twice per period whereupon the rank of the matrix

$$A(\mathbf{x}_{s_1,s_2}(t), 0) = \text{diag}\{1, 1, x_2(t) + x_3(t)\} = \text{diag}\{1, 1, s_1\sin(t) - s_2\cos(t)\}$$

changes from 3 to 2 and back.

Fig. 2. A *vertical* Hopf bifurcation to singular solutions: a schematic of the phase portrait of Example 2 when $\mu = 0$ showing the two lines of pseudo-equilibria that intersect at a singular equilibrium and a two-dimensional family of periodic orbits.



Many of the aspects of the phase portrait of (52-54) pertinent to this example can be determined from the ODE that results from (55-56) on rescaling time *according to $ds/dt = x_2 + x_3$, where a prime (′) now denotes d/ds:*

$$x_1' \;=\; -(x_2 + x_3)x_2, \;\; x_2' = (x_2 + x_3)x_1, \;\; x_3' = x_3\;(x_3 - x_1). \tag{57}$$

Firstly, note that there are two lines of pseudo-equilibria of (52-54) given by

$$\mathsf{P}_1 = \{\mathbf{x} \in \mathbb{R}^3 : x_2 + x_3 = 0, x_3 = 0\} \text{ and } \mathsf{P}_2 = \{\mathbf{x} \in \mathbb{R}^3 : x_2 + x_3 = 0, x_3 = x_1\},$$

*and $\mathsf{P}_1 \cap \mathsf{P}_2 = \{\mathbf{0}\}$ is a singular equilibrium point. (*Pseudo-equilibria *are equilibria of (57) that are not equilibria of (55-56)). The two-dimensional family of periodic orbits $\{\mathbf{x}_{s_1,s_2}\}_{s_1>0,s_2>0}$ intersects the singularity S_0 at an element of P_1 and the periodic orbits are formed from the union of two heteroclinic trajectories between pseudo-equilibria in*

$$\mathsf{P}_1^+ = \{\mathbf{x} \in \mathsf{P}_1 : x_1 > 0\} \text{ and } \mathsf{P}_1^- = \{\mathbf{x} \in \mathsf{P}_1 : x_1 < 0\}.$$

See Figure 2 for a schematic of this structure.

Example 11 *We can utilise the results of this paper to demonstrate that the periodic singular solutions of Example 2 persist to nearby singular systems by applying Theorem 8(i & ii).*

If we now have in mind $\sigma \neq 0$ in (50) but, as per Example 2 put $\psi_\mu(\mathbf{x}) = \frac{1}{2}(x_1^2 + (x_2 + x_3 - \mu)^2)$, then Theorem 8 applies in its entirety to the system

$$\dot{x}_1 = \mu x_1 - x_2 + \mu\eta(\mathbf{x}), \tag{58}$$

$$\dot{x}_2 = x_1 + \mu x_2 + \mu\zeta(\mathbf{x}), \tag{59}$$

$$\frac{d}{dt}\psi_\mu(\mathbf{x}) = \sigma x_3 + x_\tau^2, \qquad (\text{for } \tau = 1, 2 \text{ or } 3) \tag{60}$$

as $S_\mu = \{\mathbf{x} \in \mathbb{R}^3 : x_2 + x_3 = \mu\}$ is a codimension-1 subspace of $\mathbb{R}^3$ and $T_{\mathbf{0}}(S_0) = \mathrm{span}\{(1,0,0)^T, (0,1,-1)^T\}$. Since $\dim(T_{\mathbf{0}}(S_0) \oplus E) = 3$, the set S_0 transversally intersects the two-dimensional centre manifold that carries the bifurcating periodic orbit that is associated with the eigenspace E and, as a result, the periodic orbits obtained in this Hopf bifurcation are singular.

Acknowledgements

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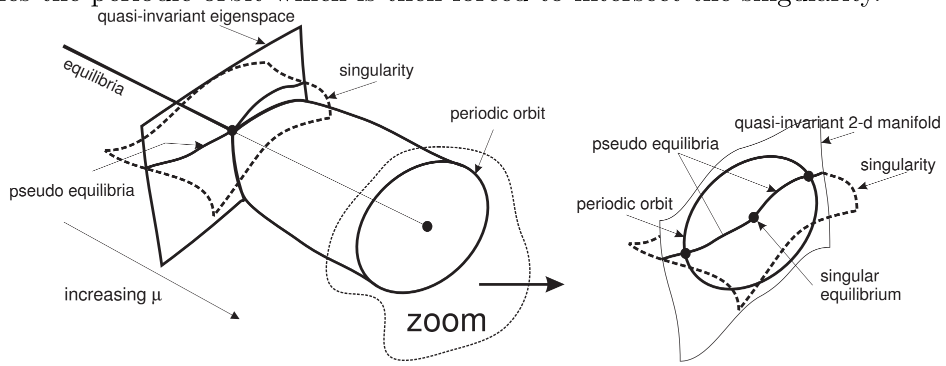
Fig. 3. Sketch of the creation of singular periodic orbits: provided the eigenspace intersects the singularity transversally at the bifurcation point, a path of singular periodic orbits is created because the eigenspace yields an invariant manifold that carries the periodic orbit which is then forced to intersect the singularity.



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