A Simple Proof of the SIB Theorem

R. Beardmore

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Abstract

This supersedes a previous version of the SIB theorem [1] (which contains an error) and gives a proof of SIB in the context of a 1-parameter family of operator matrix-pencils (M, L) on a pair of Hilbert spaces. The method of proof generalises to show that for suitable Fredholm pencils, there is a finite-dimensional space and a matrix pencil (E, F) on that space whose finite eigenvalues determine the number of diverging eigenvalues of (M, L).

1 The SIB Theorem

Theorem 1 (SIB) *Suppose that X and Y are Hilbert spaces and let $A(\lambda) : X \to X, B(\lambda) : Y \to X$ and $C(\lambda) : X \to Y$ be a C^1-parameterised family bounded, linear operators and suppose that $D(\lambda) : Y \to Y$ is a C^2 family of bounded, linear operators. Suppose also that*

1. *$D(\lambda_0)$ is Fredholm of index zero with* $\ker(D(\lambda_0)) = \langle k \rangle$,
2. *$D'(\lambda_0)k \notin \operatorname{ran}(D(\lambda_0))$ and*
3. *$C(\lambda_0)B(\lambda_0)k \notin \operatorname{ran}(D(\lambda_0))$.*

Now define the operator pencil

$$(M, L(\lambda)) := \left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \right).$$

There is a non-zero $\mu \in \mathbb{R}$, a neighbourhood $N \subset \mathbb{R}$ containing λ_0 and C^1 mappings $(x, y, \alpha) : N \backslash \{\lambda_0\} \to X \times Y \times \mathbb{R}$ such that

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)\alpha(\lambda_0) = \mu,$$

and $\alpha(\lambda) \in \sigma(M, L(\lambda))$ is an eigenvalue *for all $\lambda \in N \backslash \{\lambda_0\}$.*

Proof. By assumption we can find a closed subspace $K \subset X$ such that $Y = \langle k \rangle \oplus K$ and $Y = \langle u \rangle \oplus \mathrm{ran}(D(\lambda_0))$, where $\ker(D(\lambda_0)^*) = \langle u \rangle$. Now let

$$y = \theta k/(\lambda - \lambda_0) + v \in \langle k \rangle \oplus K,$$

and set $\beta = (\lambda - \lambda_0)\alpha$. The eigenvalue problem

$$\alpha M z = L(\lambda) z,\ \|Mz\|^2 = 1, \qquad z \in X \times Y,$$

becomes

$$\begin{aligned} \beta x &= (\lambda - \lambda_0) A(\lambda) x + B(\lambda)(\theta k + (\lambda - \lambda_0) v), & (1) \\ 0 &= P[C(\lambda) x + \theta \Delta(\lambda) k + D(\lambda) v], & (2) \\ 0 &= Q[C(\lambda) x + \theta \Delta(\lambda) k + D(\lambda) v], & (3) \\ 1 &= \|x\|^2, & (4) \end{aligned}$$

where Δ is the C^1 family of operators given by

$$\Delta(\lambda) = \frac{D(\lambda) - D(\lambda_0)}{\lambda - \lambda_0}, \qquad \Delta(\lambda_0) = D'(\lambda_0).$$

In addition, $Q + P$ is the identity on Y and $Q : Y \to \mathrm{ran}(D(\lambda_0))$ is the projection operator along $\langle u \rangle$.

Solving (1-4) when $\lambda = \lambda_0$ leads to

$$x_0 = B(\lambda_0)k/\|B(\lambda_0)k\|, \beta_0 = -\frac{P[C(\lambda_0)B(\lambda_0)k]}{[PD'(\lambda_0)k]}, \theta_0 = \beta_0/\|B(\lambda_0)k\|,$$

and then we obtain $v_0 \in \mathrm{ran}(D(\lambda_0))$ by solving

$$Q[D(\lambda_0)v_0] = -Q[\beta_0^{-1} C(\lambda_0) B(\lambda_0) k + \theta_0 D'(\lambda_0) k].$$

It is now straightforward to show that if (1-4) is denoted by a C^1 mapping of Hilbert spaces $F : X \times \mathrm{ran}(D(\lambda_0)) \times \mathbb{R}^3 \to X \times \mathrm{ran}(D(\lambda_0)) \times \langle u \rangle \times \mathbb{R}$, with $F = F(x, v, \theta, \beta, \lambda)$, then $d_{x,v,\theta,\beta} F(x_0, v_0, \theta_0, \beta_0, \lambda_0) \in BL(X \times \mathrm{ran}(D(\lambda_0)) \times \mathbb{R}^3, X \times \mathrm{ran}(D(\lambda_0)) \times \langle u \rangle \times \mathbb{R})$ is an isomorphism. The result now follows from the implicit function theorem. □

In [1] it was shown that

$$\mu = -\frac{P[C(\lambda)B(\lambda_0)k]}{P[D'(\lambda_0)k]},$$

where P is a projection taken from the above proof. We show below that this value of μ arises as an eigenvalue of a particular matrix pencil.

One can extend Theorem 1 to the case where the dimension of the null-space of $D(\lambda_0)$ is not simple. In this case, one can reduce the problem to a matrix pencil over a finite-dimensional space, (E, F) say, the finite eigenvalues of which determine how many of the eigenvalues of $(M, L(\lambda))$ diverge at λ_0.

In the following, a hat on a Hilbert space represents an admissible complexification. A $\mu_0 \in \mathbb{C}$ is said to be an algebraically simple, finite eigenvalue of a matrix pencil (E, F) if the real-valued function

$$\mu \mapsto \det(E + \mu F),$$

has a transverse zero at $\mu = \mu_0$. This is equivalent to

$$\det(E + \mu_0 F) = 0, \;\; \ker(E + \mu_0 F) = \langle k \rangle \,, \;\; Fk \notin \operatorname{ran}(\mu_0 E + F).$$

Theorem 2 *Suppose that* $A(\lambda) : X \to X, B(\lambda) : Y \to X$ *and* $C(\lambda) : X \to Y$ *is a* C^1*-parameterised family bounded, linear operators and suppose that* $D(\lambda) : Y \to Y$ *is a* C^2 *family of bounded, linear operators. Suppose also that* $D(\lambda_0)$ *is Fredholm of index zero with* $\ker(D(\lambda_0)) = \langle k_1, k_2 \rangle$ *and let*

$$Y = \operatorname{sp}\{u_1, u_2\} \oplus \operatorname{ran}(D(\lambda_0)).$$

Let $P : Y \to \operatorname{sp}\{u_1, u_2\}$ *be the projection operator along* $\operatorname{ran}(D(\lambda_0))$ *and let* $P_{1,2} : Y \to \mathbb{R}$ *be defined so that* $P[y] = u_1 P_1[y] + u_2 P_2[y]$*. Now take the real matrices* E, F *given by*

$$(E_{ij}) := P_i[C(\lambda_0)B(\lambda_0)k_j], \quad (F_{ij}) := P_i[D'(\lambda_0)k_j], \qquad (i, j = 1, 2).$$

Suppose that $\mu_{1,2} \in \mathbb{C}$ *satisfy* $\det(E + \mu F) = 0$ *and are algebraically simple.*

Then there is a neighbourhood $N \subset \mathbb{R}$ *containing* λ_0 *and two* C^1 *mappings* $(x_{1,2}, y_{1,2}, \alpha_{1,2}) : N \backslash \{\lambda_0\} \to \hat{X} \times \hat{Y} \times \mathbb{C}$ *such that*

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)\alpha_{1,2}(\lambda_0) = \mu_{1,2},$$

and $\alpha_{1,2}(\lambda) \in \sigma(M, L(\lambda))$ *is an* eigenvalue *for all* $\lambda \in N \backslash \{\lambda_0\}$.

This theorem is an example of the following more general result.

Theorem 3 *Suppose that* $A(\lambda) : X \to X, B(\lambda) : Y \to X$ *and* $C(\lambda) : X \to Y$ *be a* C^1*-parameterised family bounded, linear operators and suppose that* $D(\lambda) : Y \to Y$ *is a* C^2 *family of bounded, linear operators. Suppose also* *that* $D(\lambda_0)$ *is Fredholm of index zero with* $K := \ker(D(\lambda_0))$ *and* $\dim K \geq 1$ *and let* W *be a finite-dimensional space with*

$$Y = W \oplus \operatorname{ran}(D(\lambda_0)).$$

If $P : Y \to W$ *are the projection operators along* $\operatorname{ran}(D(\lambda_0))$*, let us define finite-dimensional, linear mappings* $E, F \in BL(W)$ *by*

$$E := P[C(\lambda_0)B(\lambda_0)]|_W, \quad F := P[D'(\lambda_0)]|_W,$$

now let $\mu \in \mathbb{C}$ *satisfy* $\det(E+\mu F) = 0$ *and be algebraically simple. There is a neighbourhood* $N \subset \mathbb{R}$ *containing* λ_0 *and a* C^1 *mapping* $(x, y, \alpha) : N\backslash\{\lambda_0\} \to \hat{X} \times \hat{Y} \times \mathbb{C}$ *such that*

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)\alpha(\lambda_0) = \mu,$$

and $\alpha(\lambda) \in \sigma(M, L(\lambda))$ *is an* eigenvalue *for all* $\lambda \in N\backslash\{\lambda_0\}$.

Proof. The proof of this result is almost identical to that of Theorem 1. The algebraic simplicity of μ is used to demonstrate that the linearisation operator of the problem which corresponds to (1-4) is an isomorphism. □

References

[1] R. Beardmore, Proc. Royal Soc. A(2001) **457**, pp.1295-1305