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A Wald test of restrictions on the cointegrating space based on Johansen's estimator

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Abstract

A test is derived of the hypothesis that the cointegrating space of a collection of $I(1)$ variables contains a vector subject to a set of linear restrictions. Applications to the problem of testing for irreducible cointegrating relations, and also structural hypotheses, are discussed. © 1998 Elsevier Science S.A.

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1. Introduction

This article derives a test of restrictions on the cointegrating vectors estimated by the Johansen (1988), (1991) maximum likelihood estimator. Let the DGP for the $I(1)$ process x_t ($m \times 1$) be written as

$$\Delta x_t = \sum_{j=1}^{k-1} \Gamma_j \Delta x_{t-j} + \alpha\beta' x_{t-k} + u_t, \tag{1.1}$$

for $t=1,\ldots,T$, where α and β are $m \times s$ matrices of rank s (assumed known). Recall that in Johansen's procedure $\hat{\beta}$, the MLE of β, is calculated as the set of eigenvectors corresponding to the s largest eigenvalues of $S_{0k}'S_{00}^{-1}S_{0k}$ with respect to S_{kk}, where in the usual notation, S_{00}, S_{kk} and S_{0k} are the moment matrices formed from the residuals from regressing Δx_t and x_{t-k}, respectively, onto the Δx_{t-j}. Accordingly, $\hat{\beta}$ is normalised to satisfy $\hat{\beta}'S_{kk}\hat{\beta} = I_s$ and $\hat{\beta}'S_{0k}'S_{00}^{-1}S_{0k}\hat{\beta} = \hat{D}$, where $\hat{D}$ is the diagonal matrix formed from the s largest eigenvalues. As is well known, these vectors span the cointegrating space, but have no interpretation as structural economic parameters.

Johansen and Juselius (1990, Section 5.2) show how to construct Wald tests for hypotheses of the form $H\beta = 0$, where H is a known $p \times m$ matrix of constants, using the result shown in Johansen (1991) that $T(\hat{\beta} - \beta)$ is locally asymptotically mixed normal (LAMN) under standard assumptions.

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While this type of hypothesis is invariant to the normalisation since all the columns of β satisfy the same restrictions under the null, it is not widely applicable. By contrast, the hypothesis we consider here is that the cointegrating space *contains* a vector which satisfies restrictions H. This may be expressed as

$$\mathrm{H}_0\text{: } \exists a(s \times 1) \text{ such that } H\beta a = 0. \tag{1.2}$$

The chief motivation of this approach is to give direct tests of structural hypotheses on the cointegrating relationships of the model. As a byproduct, estimates of identified structural parameters can be solved from $\hat{\beta}$. The test is described in Section 2, and the applications to structural modelling are briefly outlined in Section 3.

2. The test

Assume $p \geq s$, since otherwise H_0 is trivially true. Further, assume that the vector a specified in (1.2) is unique up to a normalisation. (Both of these assumptions will be motivated in Section 3.) Then, under H_0 the matrix $\beta'H'H\beta$ $(s \times s)$ has rank of $s-1$, its smallest eigenvalue is 0, and a equals the eigenvector corresponding to this simple eigenvalue, up to a normalisation. On the other hand, if H_0 is false the minimal eigenvalue is positive. H_0 can therefore be restated as $H\beta a = 0$ where a is the eigenvector corresponding to the smallest eigenvalue of $\beta'H'H\beta$. The vector a so defined is a continuous function of β, differentiable to all orders. If $\hat{a}$ denotes the eigenvector corresponding to the smallest eigenvalue of $\hat{\beta}'H'H\hat{\beta}$, then $\hat{a}$ is consistent for a. Since $T(\hat{\beta} - \beta) = O_p(1)$, we may write, when H_0 is true

$$TH\hat{\beta}\hat{a} = TH(\hat{\beta} - \beta)a + TH\beta(\hat{a} - a) + o_p(1). \tag{2.1}$$

To derive the asymptotic distribution of this vector we can use the fact (see Magnus and Neudecker, 1988, Chapt. 8, Th. 7) that the differential with respect to β of an eigenvector of $\beta'H'H\beta$, corresponding to a simple eigenvalue λ, has the form

$$da = P(d\beta'H'H\beta + \beta'H'Hd\beta)a, \tag{2.2}$$

where P denotes the Moore–Penrose inverse of the singular matrix $\lambda I_s - \beta'H'H\beta$. Letting h_i' denote the ith row of H, we can write

$$\begin{aligned} h_i'\beta da &= a'\beta'H'Hd\beta P\beta'h_i + h_i'\beta P\beta'H'Hd\beta a \\ &= (h_i'\beta P \otimes a'\beta'H'H + a' \otimes h_i'\beta P\beta'H'H)\mathrm{Vec}\, d\beta, \end{aligned}$$

and hence

$$\begin{aligned} h_i'd(\beta a) = h_i'd\beta a + h_i'\beta da &= (a \otimes h_i + P\beta'h_i \otimes H'H\beta a + a \otimes H'H\beta P\beta'h_i)'\ \mathrm{Vec}\, d\beta \\ &\equiv k_i'\ \mathrm{Vec}\, d\beta \end{aligned}$$

(say). The p-vector in (2.1) is therefore approximated linearly, element by element, by

$$Th_i'\hat{\beta}\hat{a} = Tk_i' \operatorname{Vec}(\hat{\beta} - \beta) + o_p(1), \; i = 1, \ldots, p. \tag{2.5}$$

The vectors k_i ($sm \times 1$) are continuous functions of β. Under H_0, (2.5) can be treated as exact if k_i is evaluated at a point on the line segment joining $\hat{\beta}$ and β, say β^*. Whether evaluated at $\hat{\beta}$ or at β^*, consistency of $\hat{\beta}$ for β implies that the k_i converge to fixed limits at β as $T \to \infty$.

According to Theorem 5.1 of Johansen (1991), $T \operatorname{Vec}(\hat{\beta} - \beta)$ ($sm \times 1$) is LAMN under standard assumptions, and the limiting conditional covariance matrix[1] is consistently estimated by

$$T(\hat{D}^{-1} - I_s) \otimes \hat{M}_\beta \hat{\nu}\hat{\nu}'\hat{M}_\beta \equiv T\hat{A} \otimes \hat{B}, \tag{2.6}$$

where the identity defines $\hat{A}$ and $\hat{B}$, $\hat{\nu}$ ($m \times (m - s)$) is the matrix of the eigenvectors corresponding to the $m - s$ smallest eigenvalues of $S_{0k}'S_{00}^{-1}S_{0k}$ with respect to S_{kk} (i.e., those not included in $\hat{\beta}$), and $\hat{M}_\beta = I_m - \hat{\beta}(\hat{\beta}'\hat{\beta})^{-1}\hat{\beta}'$. Johansen's covariance matrix formula is given for $\hat{\beta}$ normalised by a matrix c, or in other words for $\hat{\beta}_c = \hat{\beta}(c'\hat{\beta})^{-1}$ such that $c'\beta$ is a matrix of full rank (Johansen, 1991, Section 5). Here, we choose $c = \beta(\beta'\beta)^{-1}$, so that $\beta_c = \beta$. The formula is also derived for the general case where β is subject to restrictions of the form $\beta = H\varphi$ where H is a known matrix and φ is a matrix of parameters, but we consider the case $H = I_m$, so that this matrix disappears from the formula. With these substitutions, and the replacement of β by the consistent estimate $\hat{\beta}$, (2.6) corresponds to Johansen's expression (5.6). For the case of the model with deterministic trends constrained to zero, the corresponding version of Johansen's expression (5.8) is used.

Combining (2.6) with (2.5) evaluated at the estimate $\hat{\beta}$, the asymptotic conditional covariance matrix of $TH\hat{\beta}\hat{a}$ is consistently estimated by the matrix $T\hat{V}$ ($p \times p$) where (omitting hats for clarity) the typical element of $\hat{V}$ is

$$\begin{aligned} v_{ij} = {} & a'Aa.h_i'Bh_j + h_i'\beta PAa.a'\beta'H'HBh_j + a'Aa.h_i'BH'H\beta P\beta'h_j + a'AP\beta'h_j.h_i'BH'H\beta a \\ & + h_i'\beta PAP\beta h_j.a'\beta H'HBH'H\beta a + h_i'\beta Pa.a'\beta'H'HBH'H\beta P\beta'h_j + a'Aa.h_i'\beta P\beta'H'HBh_j \\ & + a'AP\beta'h_j.h_i'\beta P\beta'H'HBH'H\beta a + a'Aa.h_i'\beta P\beta'H'HBH'H\beta P\beta'h_j. \end{aligned} \tag{2.7}$$

Under H_0 the statistic

$$T^2\hat{a}'\hat{\beta}H'\hat{V}^{-1}H\hat{\beta}\hat{a} \tag{2.8}$$

is asymptotically chi-squared with p degrees of freedom, provided $\hat{V}$ is of rank p.

The latter requirement presents a difficulty when $p > m - s$, since the matrix $\hat{B} = \hat{M}_\beta\hat{\nu}\hat{\nu}'\hat{M}_\beta$ in (2.6) is of rank $m - s$. Note that the distribution of $T \operatorname{Vec}(\hat{\beta} - \beta)$ is singular, of rank $s(m - s)$, in view of the s^2 restrictions to which $\hat{\beta}$ is subject. This difficulty can be overcome by employing the Moore–Penrose inverse of the variance matrix. Diagonalising $\hat{V}$ as $Q_1'\Lambda Q_1$, where Λ ($r \times r$) is the diagonal matrix of the positive eigenvalues, with $r = \min\{p, m - s\}$, the test statistic

$$T^2\hat{a}'\hat{\beta}'H'Q_1'\Lambda^{-1}Q_1H\hat{\beta}\hat{a} \tag{2.9}$$

is asymptotically chi-squared with r degrees of freedom on H_0. The implication of having $p > m - s$

[1]Note that in this model the matrix specified is random even asymptotically. It can be called the covariance matrix because test statistics formed from it in the conventional manner have the standard asymptotic distributions under H_0. See Johansen (1991) for further details.

restrictions is that we are unable to test them all independently because of the way restrictions are imposed on β for estimation. We can however test the $m-s$ independent restrictions represented by Q_1H, which in general will hold only if (1.2) is true.

3. Applications

Consider the case of zero (exclusion) restrictions. After suitable re-ordering, let $H=[0,I_p]$ for $s \le p \le m-2$. This corresponds to the hypothesis that the cointegrating space contains a vector with only $m-p$ nonzero elements, and equivalently that the first $m-p$ elements of x_t form a cointegrated subset. Moreover, the cointegrating vector for this subset is directly estimated by the vector

$$\hat{b} = G\hat{\beta}\hat{a} \quad ((m-p)\times 1), \tag{3.1}$$

where $G=[I_{m-p},0]$, and $\hat{b}$ is LAMN with asymptotic conditional covariance matrix estimated by letting G replace H in (2.7).

The assumptions of Section 2 can be conveniently motivated with reference to this case. Note that if we have s linearly independent cointegrating vectors for m variables, we can always construct a cointegrating vector containing $s-1$ zeros, by forming appropriate linear combinations. The test statistic corresponding to this case is identically zero by construction. Next, consider the case where $\beta'H'H\beta$ has more than one zero eigenvalue. Then, there exist two or more linearly independent cointegrating vectors for the same subset of $m-p$ variables. By the preceding argument, this means there exists a cointegrating vector with *more* than p zero elements, and hence a more restrictive hypothesis than that under test is also true. So while the derivation of the null distribution is not valid for multiple zero eigenvalues, we note that there exists a test statistic computed for at least $p+1$ restrictions which *does* have the stated distribution. On heuristic grounds we should expect the test size in the case of multiple zeros to be if anything lower than the nominal size, so that the test should perform correctly in practice. Monte Carlo experiments support this conjecture.

The phenomenon of a cointegrating subset, in the context of a structural model of cointegration, is analysed in Davidson (1994) and Davidson (1997). The latter paper utilises the present test to implement the algorithm MINIMAL, which determines all the irreducible cointegrating subsets of a data set (those from which no variable can be dropped without ‘losing’ cointegration) by an exhaustive sequence of exclusion tests. It is shown that an identified structural cointegrating relation is always irreducible. Moreover, LAMN estimates of the structural parameters are obtained directly from the Johansen matrix by formula (3.1), without the need for instruments or iterative methods.[2]

A test of ‘long run’ structural restrictions can always be cast in the form of (1.2). Thus, suppose that the economic model generating the data according to (1.1) has structural parameters α_0 and β_0. In (1.1) $\beta=\beta_0 d$ and $\alpha=\alpha_0 d^{-1}$, where d $(s\times s)$ is an arbitrary nonsingular matrix to which Johansen’s procedure assigns a value different from I_s, in general. Prior restrictions on α_0 and β_0 are required to identify the remaining elements from β and α, according to the usual rank condition. The structural

[2]This assumes that no further restrictions on the vector are accepted. A vector which is not irreducible is either not structural, or not identified. Also, note that an irreducible cointegrating relation is not necessarily structural. See the above-cited papers for details.

hypothesis that β_{01} (a column of β_0, labelled 1 without loss of generality) is subject to p linear restrictions can be written as $H\beta_{01}=0$. But if β_{01} is identified (i.e., such that the restrictions are overidentifying) it has the representation $\beta_{01}=\beta a$, where a is unique up to normalisation. In other words, the structural hypothesis always has the representation (1.2), in which the uniqueness assumption of Section 2 must hold.

Finally, we note that Pesaran and Shin (1994) have constructed likelihood ratio tests of structural restrictions on β_0 by estimating (1.1) subject to the constraints, in effect setting $d=I_s$ by imposition of identifying restrictions. This provides an alternative to the present procedure, but numerical optimisation is required to compute the constrained estimates, whereas our procedure requires only the estimation of the standard Johansen model. (Gauss code to compute the statistics described in this paper is available from the author.)

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