Re: Davidson and de Jong (2000) Theorem 3.1. The uniform integrability of Y_{1n} in (B.33).

It has been pointed out by a reader that a step in the proof of Theorem 3.1 of Davidson and de Jong (2000) is obscure. This note seeks to expand and clarify the reasoning at this step. The first thing to point out is that there is unfortunately a typographic error in the published form of (B.33). $a_{nt}(\xi, \xi' + \delta)$ should read $a_{nt}(\xi, \xi')$. Since the definition only makes sense if $\xi \leq \xi' + \delta$, the need for a correction here is evident.

The issue is to establish the uniform square-integrability of the sequence $\{Y_{1n}\}$ defined in (B.33), which is merely asserted in the published proof. Begin by fixing n, and noticing that

$$\begin{aligned} &X_n(\xi) - X_n(\xi') - R_n(\xi, \xi') \\ &\quad = \sigma_n^{-1} \sum_{t=[n\xi']+1}^{[n\xi]} a_{nt}(\xi, \xi')u_t + \sigma_n^{-1} \sum_{t=-N_n}^{[n\xi']} a_{nt}(\xi, \xi')u_t \\ &\quad = T_1(\xi, \xi') + T_2(\xi, \xi'). \end{aligned} \tag{1}$$

According to equation (3.2) of the paper, the weights $a_{nt}(\xi, \xi')$ in T_1 consist of partial sums of the fractional lag coefficients b_j, for j running from a maximum of $[n\xi] - [n\xi']$, down to 0. Arguing as in (B.32) of the paper, also note that

$$\begin{aligned} &\sup_{\xi:\xi-\xi'<\delta} |T_1(\xi, \xi') + T_2(\xi, \xi')|\nu_n^{-1}(\xi', \delta) \\ &\quad \leq \sup_{\xi:\xi-\xi'<\delta} |T_1(\xi, \xi')|\nu_n^{-1}(\xi', \delta) + \sup_{\xi:\xi-\xi'<\delta} |T_2(\xi, \xi')|\nu_n^{-1}(\xi', \delta). \end{aligned} \tag{2}$$

Consider the majorant terms in (2), starting with T_1. There are three points to note. First, since ξ' and δ are fixed, Corollary 16.14 of Davidson (1994) applies to the case

$$T_1^*(\xi, \xi') = \sigma_n^{-1} \sum_{t=[n\xi']+1}^{[n\xi]} a_{nt}(\xi' + \delta, \xi')u_t. \tag{3}$$

Second, in view of Assumption 1(b) of the paper, Corollary 16.14 continues to hold under arbitrary permutations of the weights in (3). These quantities correspond to the mixingale constants c_{nt} specified in the Corollary, and their ordering is irrelevant. Letting $p([n\xi'] + 1), \ldots, p([n(\xi' + \delta)])$ denote a permutation of the integers $[n\xi'] + 1, \ldots, [n(\xi' + \delta)]$, $a_{nt}(\xi' + \delta, \xi')$ in (3) can be replaced by $a_{np(t)}(\xi' + \delta, \xi')$ as the weight on u_t without affecting the implicit application of Corollary 16.14. Third, the sequence of weights appearing in $T_1(\xi, \xi')$ in (2) corresponds to one of these permutations, because for any ξ in the relevant range,

$$\begin{aligned} &\{a_{nt}(\xi, \xi'), t = [n\xi'] + 1, ..., [n\xi]\} \\ &\quad \subseteq \{a_{nt}(\xi' + \delta, \xi'), t = [n\xi'] + 1, ..., [n(\xi' + \delta)]\}. \end{aligned} \tag{4}$$

(That is to say, the sets in question contain the partial sums of the b_j from 0 up to the indicated limits, by (3.2) of the paper.)

Next, letting $T_{1p}^*(\xi, \xi')$ denote the partial sum corresponding to (3) under permutation p, note that there exists a p such that

$$\sup_{\xi:\xi-\xi'<\delta} |T_1(\xi, \xi')|\nu_n^{-1}(\xi', \delta) \leq \sup_{\xi:\xi-\xi'<\delta} |T_{1p}^*(\xi, \xi')|\nu_n^{-1}(\xi', \delta). \tag{5}$$

To see this, consider any case p where the weights in T^*_{1p} match those of T_1 up to time $[n\xi^*]$, where ξ^* is the value of ξ defined by the sup for T_1 on the left. We get equality in the case where ξ^* defines the sup on the right-hand side also, and can do no worse by letting ξ vary on the right.

Now let n increase, and while the sets of permutations p with n observations (call these p_n) also increases, this just corresponds to all the different ways of ordering the sets $\{c_{nt}, t = 1, \ldots, n\}$ specified in Corollary 16.14, for $n = 1, 2, 3, \ldots$ As noted, since $\{u_t\}$ has uniformly bounded r-norms, the corollary holds uniformly with respect to these orderings. Therefore, we can construct the random sequence

$$\max_{p_n} \sup_{\xi:\xi-\xi'<\delta} |T^*_{1p_n}(\xi,\xi')|\nu_n^{-1}(\xi',\delta), \quad n \geq 1 \tag{6}$$

of which the nth term dominates $\sup_{\xi:\xi-\xi'<\delta} |T_1(\xi,\xi')|\nu_n^{-1}(\xi',\delta)$ according to (5). Each term of sequence (6) is drawn from a uniformly square-integrable sequence, and hence it is itself uniformly square-integrable.

Next consider T_2. The number of terms in this sum is fixed for fixed n. All we have to observe is that

$$T^*_2(\xi,\xi') = \sigma_n^{-1} \sum_{t=-N_n}^{[n\xi']} a_{nt}(\xi'+\delta,\xi')u_t \tag{7}$$

is uniformly square-integrable in the usual way. For the case $d > 0$, note that $0 \leq a_{nt}(\xi,\xi') \leq a_{nt}(\xi'+\delta,\xi')$ for $\xi' \leq \xi \leq \xi'+\delta$, for all $-N_n \leq t \leq [n\xi']$. For the case $d < 0$ all the weights in T_2 are negative, so consider instead the case of $-T_2$.

Davidson, James (1994) *Stochastic Limit Theory.* Oxford: Oxford University Press

Davidson, James, and Robert M. de Jong (2000) The functional central limit theorem and weak convergence to stochastic integrals II: fractionally integrated processes *Econometric Theory* 16, 5 643-666