A General Bound for the Limiting Distribution of Breitung’s Statistic*

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Let $x_1, x_2, \ldots, x_n$ represent an arbitrary sequence of random variables, and define the statistic

$$\xi_n := \frac{\sum_{t=1}^{n} \left( \sum_{j=1}^{t} (x_j - \bar{x}_n) \right)^2}{n^2 \sum_{t=1}^{n} (x_t - \bar{x}_n)^2}$$

where $\bar{x}_n$ denotes the sample mean. Note that $\xi_n$ is the statistic for Breitung’s (2002) nonparametric test of the I(1) hypothesis. Let $\xi$ denote the limit in distribution of $\xi_n$ as $n \to \infty$. We prove the following theorem.

**Theorem:** $\xi$ is supported on the interval $[0, 1/\pi^2]$.

Note that this result holds under any assumption on $x_1, \ldots, x_n$ whatsoever. It is trivially true when the process is I(0) because then, as Breitung shows, the distribution is degenerate at 0. Under Breitung’s null hypothesis, where $x_t$ is an I(1) process subject to the usual regularity conditions, $\xi$ would correspond to the functional

$$\frac{\int_0^1 \left( \int_0^t W \, dt \right)^2 \, dt}{\int_0^1 W^2 \, dt - \left( \int_0^1 W \, dt \right)^2}$$

where $W$ is standard Brownian motion. However, the theorem also holds when $x_t$ is I($d$), for any finite $d$. Some simulations are shown in Figure 1, for cases with $1 \leq d \leq 2$.

To prove the theorem, we let $x := (x_1, x_2, \ldots, x_n)'$ and define the $n \times n$ lower triangular “cumulation” matrix

$$C := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

as in Tanaka (1996, Equation 1.3). Then,

$$\xi_n = \frac{x' M C' C M x}{n^2 x' M x},$$

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where $M := I_n - (1/n)uu'$, and $u := (1, 1, \ldots, 1)'$. Let $A := (1/n)C$ and let $M = QQ'$, where $Q$ is the $n \times (n-1)$ matrix containing the eigenvectors associated with the $n-1$ unit eigenvalues of $M$. Letting $y := Q'x$ we have

$$
\xi_n = \frac{x' M A' A M x}{x'Mx} = \frac{y' Q' A' AQ y}{y'y},
$$

and hence

$$
\xi_n \leq \lambda_{\max}(Q' A' AQ) = \lambda_{\max}(AQQ'A') = \lambda_{\max}(AMA').
$$

Hence, the theorem is true if and only if

$$
\lambda_{\max}(AMA') \to \frac{1}{\pi^2} \text{ as } n \to \infty.
$$

We note that the matrix $A'A$ is positive definite and its eigenvalues are given by $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, where

$$
\lambda_j = \left(2n \sin \frac{\pi(2j-1)}{4n+2}\right)^{-2}, \quad j = 1, \ldots, n;
$$

see Rutherford (1946) and Dickey and Fuller (1979). Since the sine-function is monotonic on the interval $(0, \pi/2)$, the matrix $A'A$ has no multiple eigenvalues.

The matrix $AMA'$ is positive semidefinite and has rank $n - 1$. By Theorem 11.11 of Magnus and Neudecker (1988, p. 210) we know that its eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ satisfy

$$
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.
$$

In particular,

$$
\left(2n \sin \frac{\pi}{4n+2}\right)^{-2} \geq \mu_1 \geq \left(2n \sin \frac{3\pi}{4n+2}\right)^{-2}.
$$
Since none of the eigenvalues of $AMA'$ can be eigenvalues of $A'A$, the maximal eigenvalue $\mu_1$ of $AMA'$ is found as the unique solution of

$$i'(I_n - \mu_1(A'A)^{-1})^{-1} = n, \quad \lambda_1 > \mu_1 > \lambda_2;$$

see Lemma A3 in the Appendix.

The eigenvectors of $A'A$ are known (Dickey and Fuller, 1979). Let $s_j$ be the normalized eigenvector associated with $\lambda_j$. The $i$-th element of $s_j$ is given by

$$s_{ij} = \frac{2}{\sqrt{2n+1}} \cos \frac{\pi(2i-1)(2j-1)}{4n+2}, \quad i = 1, \ldots, n.$$ 

If we define $S := (s_1, s_2, \ldots, s_n)$, then $S'A'AS = \Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Letting $q := S't$, with components $q_1, \ldots, q_n$, such that $q_j = \sum_{i=1}^n s_{ij}$, note that $i'(I_n - \mu_1(A'A)^{-1})^{-1}t = n$ if and only if $q' (I_n - \mu_1 \Lambda^{-1})^{-1}q = n$. To show that $\mu_1 \to 1/\pi^2$ as $n \to \infty$ we therefore need to show that

$$\frac{1}{n} \sum_{j=1}^n q_j^2 \left( \frac{\lambda_j \pi^2}{\lambda_j \pi^2 - 1} \right) \to 1.$$

Note that

$$\lambda_j = \frac{1}{\pi^2(j - 1/2)^2} + O\left(\frac{1}{n^2}\right)$$

and

$$\frac{q_j}{\sqrt{n}} = \frac{2}{\sqrt{n}\sqrt{2n+1}} \sum_{i=1}^n \cos \frac{\pi(2i-1)(2j-1)}{4n+2} = \frac{\sqrt{2}\sin \pi(j - 1/2)}{\pi(j - 1/2)} + O\left(\frac{1}{n}\right).$$

Using the fact that $\sin^2 \pi(j - 1/2) = 1$, we obtain

$$\frac{1}{n} \sum_{j=1}^n q_j^2 \left( \frac{\lambda_j \pi^2}{\lambda_j \pi^2 - 1} \right) = \frac{2}{\pi^2} \sum_{j=1}^n \frac{1}{(j - 1/2)^2(1 - (j - 1/2)^2)} + O\left(\frac{1}{n}\right)$$

where

$$\frac{2}{\pi^2} \sum_{j=1}^n \frac{1}{(j - 1/2)^2(1 - (j - 1/2)^2)} = \frac{2}{\pi^2} \left( \sum_{j=1}^n \frac{1}{1 - (j - 1/2)^2} + \sum_{j=1}^n \frac{1}{(j - 1/2)^2} \right) = \frac{2}{\pi^2} \left( 0 + \frac{\pi^2}{2} \right) + O\left(\frac{1}{n}\right) \to 1.$$

This concludes the proof.

1 Appendix

We consider a positive semidefinite $n \times n$ matrix $V$, and an $n \times 1$ vector $\omega$.

Lemma A1:

$$|V - \omega\omega'| = \begin{cases} |V|(1 - \omega'V^{-1}\omega) & \text{if } \text{rk}(V) = n; \\ -(\omega'x)^2 \cdot p(V) & \text{if } \text{rk}(V) = n - 1, Vx = 0, x'x = 1; \\ 0 & \text{if } \text{rk}(V) \leq n - 2, \end{cases}$$
where \( p(V) \) denotes the product of the nonzero eigenvalues of \( V \).

**Proof:** We prove the lemma first for a diagonal \( n \times n \) matrix \( \Lambda \) with nonnegative diagonal elements, and an \( n \times 1 \) vector \( a \).

(i) If all diagonal elements of \( \Lambda \) are nonzero (hence positive), then
\[
|\Lambda - aa'| = |\Lambda^{1/2}(I_n - \Lambda^{-1/2}aa'\Lambda^{-1/2})\Lambda^{1/2}| = |\lambda|(1 - a'\Lambda^{-1}a).
\]

(ii) If one of the diagonal elements of \( \Lambda \) is zero (say, the \( n \)-th), then we partition
\[
\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.
\]
If \( a_2 \neq 0 \), then
\[
|\Lambda - aa'| = \begin{vmatrix} \Lambda_1 - a_1a_1' & -a_2a_1 \\ -a_2a_1' & -a_2^2 \end{vmatrix} = -a_2^2|\Lambda_1|,
\]
see Exercise 5.30(b) in Abadir and Magnus (2005). The result remains true when \( a_2 = 0 \), because both sides of the equality are then zero.

(iii) If two or more diagonal elements of \( \Lambda \) are zero, then
\[
\text{rk}(\Lambda - aa') = \text{rk}(\Lambda) + \text{rk}(aa') \leq (n - 2) + 1 = n - 1,
\]
and hence \( |\Lambda - aa'| = 0 \).

In the general case, we diagonalize \( V \) as \( S'VS = \Lambda \) and define \( a := S'\omega \). The results (i) and (ii) follow immediately. For (iii) we partition \( S = (S_1, x) \), so that \( \omega'x = a'S'x = a_1'S_1'x + a_2x'x = a_2 \).

This completes the proof.

**Lemma A2:** \( |V - \omega\omega'| = 0 \) if and only if
\[
\begin{cases}
\text{rk}(V) = n, \quad \omega'V^{-1}\omega = 1; \\
\text{rk}(V) = n - 1, \quad \omega'x = 0, \quad Vx = 0; \\
\text{rk}(V) \leq n - 2.
\end{cases}
\]

**Proof:** This follows directly from Lemma A1.

**Lemma A3:** Let \( C \) be a nonsingular \( n \times n \) matrix, and let \( M := I_n - (1/n)\mu' \). Then, \( \mu \) is an eigenvalue of \( CMC' \) if and only if
\[
\begin{cases}
\mu \text{ is not an eigenvalue of } CC', \quad \mu'(I_n - \mu(C'C)^{-1})^{-1}I = n; \\
\mu \text{ is a simple eigenvalue of } CC' \text{ with associated eigenvector } x, \quad \mu'C'x = 0; \\
\mu \text{ is a multiple eigenvalue of } CC'.
\end{cases}
\]

**Proof:** The eigenvalues of \( CMC' \) are given by
\[
|CC' - \mu I_n - \frac{1}{n}Cu'C| = 0.
\]
Let \( V := CC' - \mu I_n \) and \( \omega := Ci/\sqrt{n} \), and apply Lemma A2.
References


