Frequency Domain Wild Bootstrap for Dependent Data

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The bootstrap is a technique for making a random draw from the empirical distribution defined by a sample of data.

Commonly applied to regression residuals, to construct test distributions by Monte Carlo simulation

- The original idea (Efron 1979) assumes an i.i.d. random sample – randomly drawing sample coordinates with replacement reproduces the frequency distribution of the original data.

- The basic property of the bootstrap distribution is that the mean, variance, skewness and other moments match the *empirical* moments of the original series.

  - In large samples, the bootstrap and sample distributions are therefore increasingly close to one another.

  - If the sample data obey the central limit theorem, so does the bootstrap draw.

  - We can therefore use the Monte Carlo method to generate consistent tests.

  - The well-known benefit of the bootstrap is that in small samples, the error in rejection probability under the null hypothesis is $O_p(1/T)$, smaller than that of the asymptotic test based on the normal tables, $O_p(1/\sqrt{T})$.

Suppose the series to be bootstrapped is serially independent but not identically distributed?

In particular, suppose it is heteroscedastic.

- The *wild bootstrap* is a method suited to heteroscedastic data – preserves the variances.

  1. Take the sample $X_1, \ldots, X_n$ in the original order.

  2. Create the bootstrap draw by multiplying each data point X_t by an independent drawing Z_t, from an auxiliary distribution with $E(Z_t) = 0$ and $\text{Var}(Z_t) = 1$.

     - Various auxiliary distribution have beens proposed.

     - The simplest is the two-point *Rademacher*, setting $Z_t = +1$ and $Z_t = -1$ with equal probabilities of 0.5.

     - In effect, randomly flip the sign of X_t.

- If the distribution of X_t is symmetric about zero, the distribution of the Rademacher bootstrap series reproduces the original distribution and preserves any heteroscedastic pattern.

If the sample is autocorrelated...

... reproducing the distribution is much harder!

- Proposed methods have included:

  1. The block bootstrap (several variants): draw short blocks of successive data points, with replacement, concatenate blocks to get the bootstrap draw.

     - Not very good!

  2. The sieve autoregression: filter the data with a fitted AR process, do Efron draw from the residuals, then "recolour".

     - Works well if an AR process did generate the series – otherwise, properties not so clear.

Our basic method for autocorrelated data is as follows:

1. Compute the discrete Fourier transform (DFT) of the series.
   - It's well-known that if a series is autocorrelated, the DFT is asymptotically uncorrelated, but heteroscedastic: the variances are the values of the spectral density at each frequency.
2. Apply the Rademacher wild bootstrap to the DFT.
3. Compute the inverse DFT of the resulting bootstrap draw.
   - Minor complication – the inverse DFT is complex-valued.
   - We simply add the real and imaginary parts of this series together to get the real-valued bootstrap draw.

Properties of our bootstrap.

We show the following in the paper:

1. Our bootstrap draw has the *same periodogram as the original sample*.

2. The autocovariances under the bootstrap distribution equal the empirical autocovariances of the original sample.

3. The draws are jointly Gaussian in large samples.

4. There are also some peculiar features:

   a. The mean of the bootstrap draw equals that of the original sample apart from a random sign flip. If the original sample is in mean deviation form (e.g. residuals) the distribution of the sample mean is degenerate... = 0 with probability 1.

   b. The distribution of the variance is also degenerate, equals that of the original sample with probability 1.

- Property 4a is not a problem for significance tests in regression, since in these statistics depend on the residuals in mean deviation form.

- It does rule out tests of location (significance of the intercept) and unit root tests, which involve the distribution of the sample mean.

A related method: TFT Bootstrap

The so-called "time-frequency toggle" is a class of methods due to Kirsch and Politis (2011).

- Resamples the DFT similarly, but (in one case) uses a spectral density estimate to normalize the series before Efron-style resampling.
- Hence,
  - depends on a choice of kernel and bandwidth.
  - In contrast, the AFB is strictly nonparametric.
  - The distribution of the bootstrap mean is not degenerate in this case: but is not correct, either.

Augmented Fourier Bootstrap

For tests of location we propose an augmentation of the basic bootstrap (AFB):

- We add a Gausssian random variable, with zero mean and the appropriate long-run variance to each coordinate of the bootstrap series.
- We assume the sample is centred. Hence, the bootstrap distribution of the mean is not degenerate, but is centred on zero and the variance (if consistently estimated) matches that of the sample distribution in large samples by construction.
- Kernel variance estimators are biased. To implement the AFB, we use a response suface to estimate an optimal correction.
- The augmentation affects only tests of location, and also unit root tests.
- Significance tests on slope coefficients use centred data, so augmentation has no effect here.

Monte Carlo: Significance tests in a regression model with autocorrelation:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + U_t, \; U_t = \rho U_{t-1} + E_t, \qquad t = 1, \ldots, n$$

Absolute 5% ERPs and mean square deviations of p-value distributions from uniform, ×100.
Table shows average performances over 12 cases $n = 50, 200, 800$ and $\rho = 0, 0.3, 0.6, 0.9$.

Gaussian and non-Gaussian distributions for E_t compared, also TFT and block bootstraps.

	Size Distortion						Power (%)		
	β_0		β_1		β_2		β_0	β_1	β_2
	5%	CvM	5%	CvM	5%	CvM			
AFB (N)	8.24	5.94	1.19	2.0	2.46	2.81	42.6	64.2	58.7
AFB (χ^2_1)	19.9	7.68	3.77	3.87	1.79	4.09	46.5	65.6	59.8
AFB (t_3)	11.0	9.89	3.03	3.81	2.22	5.07	45.6	68.0	61.9
TFT (N)	165	69.8	2.81	2.42	5.73	4.50	79.8	63.4	60.4
MBB (N)	40.7	19.5	1.77	2.55	4.53	3.87	51.0	63.6	59.1
SBB (N)	37.6	19.5	1.38	2.22	4.13	3.81	51.4	63.8	59.8
Asy (N)	100.1	38.4	17.4	8.28	41.5	18.3	66.5	74.6	72.3
Asy (t_3)	100.0	40.4	12.8	9.45	36.2	19.5	68.3	73.8	74.0

Same regression model with moving average errors, $U_t = E_t + E_{t-m}$

Averages over nine cases, $n = 50, 200, 800$ and $m = 1$, $m = 2$, $m = 4$.

	Size Distortion						Power		
	β_0		β_1		β_2		β_0	β_1	β_2
	5%	CvM	5%	CvM	5%	CvM			
AFB (N)	29.6	17.1	2.40	2.07	5.19	3.92	41.1	51.7	57.0
TFT (N)	120	64.5	2.79	2.19	4.64	3.61	71.8	51.9	56.5
SAR (N)	29.3	17.1	3.82	1.99	3.47	3.05	39.9	49.9	55.8
Asy (N)	62.6	8.22	17.9	7.55	33.5	7.65	54.6	62.3	69.5

(SAR - sieve autoregressive bootstrap.)

Same regression model with fractionally integrated errors, $U_t = (1 - L)_+^{-d} E_t$

Averages over six cases, $n = 50, 200, 800$ and $d = 0.1$, $d = 0.3$.

	Size Distortion						Power		
	β_0		β_1		β_2		β_0	β_1	β_2
	5%	CvM	5%	CvM	5%	CvM			
AFB (N)	227	83.9	1.81	1.60	2.42	2.91	78.2	76.6	76.6
TFT (N)	367	126	2.03	2.04	2.74	2.56	90	77.2	76.9
SAR (N)	202	77.4	2.95	2.25	1.32	2.11	75.3	75.2	75.5
Asy (N)	62.6	8.22	17.9	8.42	33.5	14.7	54.7	62.3	69.5

Unit Root Tests (Phillips-Perron Statistic)

Null model:

$$Y_t = Y_{t-1} + U_t, \;\; U_t = \rho U_{t-1} + E_t, \;\; t = 1, \ldots, n, \qquad Y_0 = U_0 = 0$$

Table shows averages over $n = 50,\ 200,\ 800$ and $\rho = 0,\ 0.3,\ 0.6,\ 0.9$.

	Size Distortion		Power
	5%	CvM	
AFB (N)	9.1	8.5	90.9
AFB (χ_1^2)	9.6	10.0	90.8
AFB (t_3)	7.6	11.4	91.2
TFT (N)	16.2	42.0	86.5
MBB (N)	10.3	16.8	88.5
SBB (N)	10.5	16.1	88.9
Asy (N)	39.1	115.5	86.9
Asy (t_3)	39.8	113.6	87.8