Are the Treasures of Game Theory Ambiguous?

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Abstract  Goeree and Holt (2001) experimentally study a number of games. In each case they initially find strong support for Nash equilibrium, however by changing an apparently irrelevant parameter they find results which contradict Nash equilibrium. In this paper, we study the five normal form games from Goeree and Holt (2001). We argue that their results may be explained by the hypothesis that subjects view their opponents’ behaviour as ambiguous. Ambiguity-aversion causes players to avoid strategies which give low out of equilibrium pay-offs. Similarly ambiguity-preference can make strategies with high pay-offs more attractive.

Keywords Ambiguity, coordination games, experiments, traveller’s dilemma, matching pennies, optimism.

JEL Classification C72, D81.

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1 INTRODUCTION

Nash equilibrium (henceforth NE) is the most widely used equilibrium concept in game theory. A large and growing number of experimental studies indicate its weaknesses. Goeree and Holt (2001) (henceforth GH) published an article with the provocative title “Ten Little Treasures of Game Theory and Ten Intuitive Contradictions” in which they claim that “for each of these ten games there is an experimental treatment in which behaviour conforms nicely to predictions of NE” but where “a change in the payoff structure produces a large inconsistency between theoretical predictions and observed behaviour”.

In the present paper we restrict attention to the five one-shot games, which GH studied. We argue that many of the “inconsistencies” can be explained by strategic ambiguity.

Ambiguity describes situations where individuals cannot or do not assign subjective probabilities to uncertain events. This may be because the decision problem is complex or unfamiliar. There is by now considerable experimental evidence which shows that individuals treat ambiguous decisions differently from risks with known probabilities. The best known example is the Ellsberg paradox, Ellsberg (1961). There is also experimental evidence that behaviour in games is affected by ambiguity. Colman and Pulford (2007) present experimental evidence that ambiguity affects behaviour in games but do not test any particular theories. Eichberger and Kelsey (2002) show that the comparative static effects of changes in ambiguity in games of strategic complements are the opposite of those in games of strategic substitutes. Eichberger et al (2008) test this prediction experimentally and find the evidence is broadly supportive.

Although there are many other explanations of the deviations from NE in the GH experiments, among others Quantal Response Equilibrium, McKelvey and Palfrey (1995), and level-k rationality, Stahl and Wilson (1995), it is interesting to consider an explanation in terms of ambiguity. Firstly it is intuitively appealing that ambiguity influences behaviour both in the economy and laboratory experiments. Secondly our theory is related to the large decision theoretic literature on ambiguity. This establishes a connection with a rigorous axiomatic decision theory. As a result one can work with a theory whose general properties are well known. It suffices to modify standard rationality conditions in individual decisions and games to allow for ambiguity. The usual independence axiom has been relaxed to only apply to comonotonic acts. Moreover the link to the extant experimental literature on ambiguity enables us to use robust experimental results in order to make predictions about strategic behaviour under ambiguity.

In GH’s experiments each game was only played once, which means that subjects were unlikely to be familiar with the game or the behaviour of their opponents. In such conditions subjects are more likely to perceive ambiguity about the behaviour of the other players. In addition, some of these experi-

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2 This has been confirmed by the subsequent experimental literature, see for instance Camerer and Weber (1992) and Cohen et al (1985).
ments concerned coordination games with multiple Nash equilibria. Thus for any given player, there will typically be two or more strategies associated with different equilibria which his/her opponent can play. Uncertainty about which NE will be played is another possible source of ambiguity.

In this paper, each player views the strategy choice by his/her opponents as potentially ambiguous. We use a model of uncertainty axiomatized by Chateauneuf et al (2007), (henceforth CEG), in which ambiguity is represented by non-additive beliefs. In the context of games, this implies that players will maximize a weighted average of the equilibrium pay-off, the best pay-off and the worst pay-off for any given strategy. Thus players may react to ambiguity partly in an optimistic way by over-weighting good outcomes and partly in a pessimistic way by over-weighting bad outcomes. Using estimates of the parameters relating to ambiguity and ambiguity-attitude from the experimental literature, we can calibrate our model of the players. Thus, on the one hand, relaxing the independence axiom and on the other hand, constraining the ambiguity parameters by restricting them to robust values found in the experimental literature we are able to explain the GH results for normal form games.

We do not view our results in this paper as the exclusive explanation for the observations about the behaviour in games reported by GH. We believe however that ambiguity about strategic behaviour is an important feature of interactive situations which are one-shot and/or are not repeated often enough to establish firm patterns of behaviour. Ambiguity is modelled by relaxing the independence condition which creates additional free parameters. These can be estimated in experiments on individual decision-making under ambiguity. In this respect, we were surprised to find that restricting the estimated parameters would allow us to reproduce the observations which GH report for behaviour in their experiments in games.

Ideally, one would want to measure individual characteristics like ambiguity-attitudes and the degree of ambiguity in a particular game directly for the individuals who will play the games. The way in which the ambiguity parameters have been estimated in the literature show how one can run experiments in which one determines first the ambiguity attitudes of the participants and then observes these people’s behaviour in games. Performing such experiments is beyond the scope of this paper. Our results demonstrate the potential of strategic ambiguity to contribute to the explanation of behaviour in games and they suggest experimental designs for testing these hypotheses.

Organization of the Paper The next section describes our model of ambiguity in games. Subsection 2.1 will introduce these preferences, provide some intuition for the parameters of this model and point to the experimental literature, which tries to estimate them. Subsection 2.3 defines an Equilibrium under Ambiguity (EUA), the equilibrium notion which we shall use. In section 3 we argue that GH’s results on one-shot games can be explained by ambiguity. In Section 4 we discuss competing theories and section 5 concludes. The appendix contains proofs of those results not proved in the text.
2 STRATEGIC AMBIGUITY

This section introduces our model of ambiguity and uses it as the basis of a solution concept for normal form games.

2.1 Non-additive beliefs

In the present paper we restrict attention to ambiguity in 2-player games, which requires the following notation. A 2-player game \( G = \langle \{1, 2\} ; S_1, S_2, u_1, u_2 \rangle \) consists of players, \( i = 1, 2 \), finite pure strategy sets \( S_i \) and payoff functions \( u_i(s_i, s_{-i}) \) for each player. The space of all strategy profiles is denoted by \( S \). The notation, \( s_{-i} \), denotes the strategy chosen by \( i \)'s opponent. The set of all strategies for \( i \)'s opponent is \( S_{-i} \). We shall adopt the convention that female pronouns (she, her etc.) denote player 1 and male pronouns denote player 2.

Beginning with Schmeidler (1989), ambiguous beliefs have been modelled as capacities, which are similar to subjective probabilities except that they are not necessarily additive. We shall use a model of ambiguity from CEG, which has the advantage that it is parsimonious in the number of parameters. This theory represents beliefs by a neo-additive capacity \( \nu \) defined by:

\[
\nu(A|\alpha, \delta, \pi) = \begin{cases} 
1 & \text{for } A = S_{-i}, \\
\alpha \delta + (1 - \delta) \pi(A) & \text{for } \emptyset \subset A \subset S_{-i}, \\
0 & \text{for } A = \emptyset,
\end{cases}
\]

where \( \alpha, \delta \in [0, 1] \), \( \pi \) is an additive probability distribution on \( S_{-i} \) and \( \pi(A) := \sum_{s_{-i} \in A} \pi_i(s_{-i}) \). They show that preferences may be represented in the form:

\[
V_i(s_i; \alpha, \delta, \pi) = \delta \left[ \alpha \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1 - \alpha) \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \right] + (1 - \delta) \cdot E_{\pi} u_i(s_i, s_{-i}),
\]

where \( E_{\pi} u_i(s_i, s_{-i}) \), denotes a conventional expectation taken with respect to the probability distribution \( \pi \).

One can interpret \( \pi \) as the decision-maker’s belief. However (s)he may not be fully confident in this belief. Thus it is an ambiguous belief. His/her confidence is modelled by the weight \( (1 - \delta) \) given to the expected payoff \( E_{\pi} u_i(s_i, s_{-i}) \). Or equivalently \( \delta \) can be interpreted as a measure of the ambiguity which the decision-maker perceives. The highest (resp. lowest) possible level of ambiguity corresponds to \( \delta = 1 \) (resp. \( \delta = 0 \)). Ambiguity-attitude is measured by \( \alpha \), which represents the optimism/pessimism of the decision maker. Purely optimistic preferences are given by \( \alpha = 1 \), while the highest

\[3\] Of course this convention is for convenience only and bears no relation to the actual gender of subjects in GH’s experiments.

\[4\] For simplicity, we will write, in slight abuse of notation, \( V_i(s_i; \alpha, \delta, \pi) \) instead of \( V_i(s_i; \nu(\alpha, \delta, \pi)) \).
level of pessimism occurs when \( \alpha = 0 \). If \( 0 < \alpha < 1 \), the individual is neither purely ambiguity-averse nor purely ambiguity-loving, since (s)he responds to ambiguity partly in an optimistic way by over-weighting good outcomes and partly in a pessimistic way by over-weighting bad outcomes.

A possible interpretation is that the optimism parameter, \( \alpha \), is a personal characteristic of the decision maker like his/her risk preferences. In contrast, the degree of ambiguity, \( \delta \), may depend on the situation. In particular, one would expect there to be more ambiguity when players interact for the first time. Growing familiarity with the game and the behaviour of opponents is likely to reduce ambiguity.

CEG also show that these preferences may be represented in the multiple priors form:\(^5\)

\[
V_i (s_i; \alpha, \delta, \pi) = \alpha \max_{p \in \mathcal{P}} E_p u_i (s_i, s_{-i}) + (1 - \alpha) \min_{p \in \mathcal{P}} E_p u_i (s_i, s_{-i}),
\]

where \( \mathcal{P} := \{ p \in \Delta (S_{-i}) : p \geq (1 - \delta) \pi \} \). For the case of one opponent with three pure strategies, Figure 1 shows the set of probability distributions \( \mathcal{P}(\delta, \pi) \).

The multiple priors representation in equation (1) can be interpreted as follows. When an individual perceives a situation as ambiguous (s)he considers more than one probability distribution to be possible. He/she reacts to ambiguity partly in an optimistic way by using the most favourable possible probability and partly in a pessimistic way by using the least favourable distribution.

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\(^5\) Gilboa and Schmeidler (1989) axiomatized the multiple priors model, which represents ambiguous beliefs by sets of probability distributions. Multiple priors and non additive beliefs produce related models of ambiguity. However they are not, in general, identical.
Fig. 1 The set $\mathcal{P}(\delta, \pi)$
2.2 Evidence on Individual Decisions

This section presents a brief summary of the experimental evidence on individual decisions under uncertainty. First we discuss risks with known probabilities then we consider how these results are modified in the presence of ambiguity.

For risks with known probabilities, expected utility theory implies that individuals will maximize a weighted sum of the utilities, where the weights are equal to the probability of the relevant outcomes. However if individuals do not have expected utility preferences it is possible that these decision weights will differ from the probabilities. Tversky and Wakker (1995) study the relationship between decision weights and attitudes towards risk and characterize the possibility and certainty effects. A majority of individuals appear to behave cautiously when there is ambiguity. Following Wakker (2001), who relates such behaviour to a generalized version of the Allais paradox, we shall refer to such cautious behaviour as pessimism. This article surveys the relevant experimental literature, which shows, that a minority of individuals respond to ambiguity in the opposite way, i.e. they display optimism.

Experiments on decision-making with known probabilities have shown that individuals tend to overweight both high and low probability events. As a result the decision weights assigned to events are an inverse S-shaped function compared to the given probability distribution, (see for instance, Gonzalez and Wu (1999) and Abdellaoui (2000)). This can be explained by insensitivity of perception in the middle of the range. For instance, the change from a probability of 0.55 to 0.60 is not perceived as great as the change from 0 to 0.05.

If probabilities are not known, a similar phenomena has been found (see, Kilka and Weber (2001)). Individuals overweight highly unlikely events and underweight likely events. (In this case the likelihood of events is subjective.) This produces a pattern of decision weights like that illustrated in Figure 2. The curved line represents the decision weights of a typical experimental subject and the 45° line represents SEU beliefs for comparison. This diagram is based on observations that subjects are willing to take courses of action, which yield high outcomes in unlikely events but refuse to accept even a small chance of bad outcomes. The more unfamiliar the source of uncertainty is the lower is the elevation of the curve, i.e. the curve shifts downwards in less familiar situations. This can be interpreted as an effect of ambiguity.

Kilka and Weber (2001) report an experimental study of choices in financial markets, which was able to distinguish beliefs from decision weights. They found that decision weights deduced from actual choices were markedly non-additive. Moreover the weighting scheme of a neo-additive capacity provides a simple version of an inverse-S shaped function relating beliefs, p, to decision weights, w(p).

\[
w(p) := \begin{cases} 
1 & \text{for } p = 1, \\
\delta \alpha + (1 - \delta) \cdot p & \text{for } 0 < p < 1, \\
0 & \text{for } p = 0.
\end{cases}
\]
This weighting scheme is illustrated in Figure 3. It can be seen as a piecewise linear approximation to that in Figure 2.

Kilka and Weber (2001) used their data to estimate degrees of optimism and ambiguity separately. In terms of our notation, they report the following values:

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\alpha \delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>0.5</td>
<td>0.52</td>
<td>0.26</td>
</tr>
<tr>
<td>Max.</td>
<td>0.62</td>
<td>0.61</td>
<td>0.34</td>
</tr>
<tr>
<td>Min.</td>
<td>0.4</td>
<td>0.41</td>
<td>0.18</td>
</tr>
</tbody>
</table>

The values of both optimism $\alpha$ and ambiguity $\delta$ vary around 0.5 with deviations of 0.1.
In this paper we shall provide an explanation for evidence on behaviour in games by assuming that individuals have CEU preferences with parameters compatible with the experimental data from Kilka and Weber (2001). Their evidence suggests that we should consider $0.4 \leq \alpha \leq 0.62$ and $0.41 \leq \delta \leq 0.61$. We say that if $\alpha$ and $\delta$ satisfies these inequalities that they lie in the KW-range.\(^6\)

2.3 Equilibrium under Ambiguity

We shall use a solution concept based on Dow and Werlang (1994).\(^7\) Formally, we assume that each player maximizes his/her expected payoff with respect to a non-additive belief. In equilibrium, beliefs have to be reasonable in the sense that each player “believes” that the opponents play best responses. To model this we require that the support of any given player’s beliefs contain only best responses of the other players. Denote by $R_i(\nu_{-i}) = \arg \max \{ V_i(s_i, \nu_{-i}) | s_i \in S_i \}$ the best response correspondence of player $i$, given beliefs represented by the capacity $\nu_{-i}$ on $S_{-i}$.

Most theories of ambiguity are formulated for single person decisions. To study ambiguity in games it is necessary to extend them to allow for the interactions between different decision-makers. In the absence of ambiguity, each player is assumed to choose a strategy which maximizes his/her expected payoff with respect to beliefs which are compatible with the strategies of their opponents. Most equilibrium notions rest on some degree of consistency between actual behaviour and beliefs, since players are likely to adapt their beliefs if they observe behaviour which contradicts them. In the presence of ambiguity, perfect consistency is unlikely since there do not exist non-additive randomizing devices, which prevents us from constructing strategies corresponding precisely to ambiguous beliefs. We consider games where each player believes that the strategy choice of his/her opponents is possibly ambiguous.\(^8\) An equilibrium is a situation where players behave optimally relative to their beliefs.

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\(^6\) In the present paper we shall assume that $\alpha$ and $\delta$ are distributed independently. We believe independence to be a reasonable approximation. In any event, it is not possible to estimate the joint distribution from the data in Kilka and Weber (2001).


\(^8\) There are other possible modelling choices, for instance, one could consider the case where there is ambiguity about the opponents’ type.
Definition 1 A pair of capacities \( \nu^* = \langle \nu_1^*, \nu_2^* \rangle \) is an *Equilibrium Under Ambiguity* (EUA) if

\[
\emptyset \neq \text{supp } \nu_1^* \subseteq R_1(\nu_2^*) \quad \text{and} \quad \emptyset \neq \text{supp } \nu_2^* \subseteq R_1(\nu_1^*). \tag{9}
\]

If \( s_i^* \in \text{supp } \nu_i^* \) for \( i = 1, 2 \), we say that \( s^* = \langle s_1^*, s_2^* \rangle \) is an *equilibrium strategy profile*. If \( \text{supp } \nu_i^* \) contains a single strategy profile for \( i = 1, 2 \) we say that it is a *pure* equilibrium, otherwise we say that it is *mixed*.\(^10\)

A mixed equilibrium, where the support contains multiple strategy profiles, should be interpreted as an equilibrium in beliefs rather than randomizations.

2.4 Support of Ambiguous Beliefs

There is more than one way to extend the notion of support from probability distributions to capacities. This definition determines how tight the relationship between beliefs and actual behaviour is. Definition 2 requires the strategies in the support of a player’s equilibrium belief be best responses. However it is ambiguous whether the opponent plays best responses. As result, in addition, the best and worst possible plays by one’s opponent are taken into account when evaluating a strategy. Decision-relevant strategies outside the support can be interpreted as events a player views as unlikely but which, due to ambiguity cannot be completely ruled out.

Several solution concepts for games with strategic ambiguity have been suggested, (see for instance Marinacci (2000) and Lo (1996)). The main difference between these solution concepts is that they use different support notions. Thus the definition of support deserves careful consideration.

Definition 2 We define the *support* of the neo-additive capacity \( \nu(\cdot|\alpha, \delta, \pi) \) by \( \text{supp } \nu = \text{supp } \pi \).

As explained above a neo-additive capacity is intended to represent a situation where the decision-maker’s belief is represented by the additive probability distribution \( \pi \) but (s)he is not fully confident in this belief. Given this it is plausible that the support of \( \nu \) should coincide with that of \( \pi \). Eichberger and Kelsey (2009) show that, for a neo-additive capacity \( \nu(\cdot|\alpha, \delta, \pi) \),

\[
\text{supp } \nu = \bigcap_{p \in \mathcal{P}} \text{supp } p,
\]

where \( \mathcal{P} \) is the set of probability distributions defined in equation (1).\(^11\) Much of the existing literature on ambiguity in games has explicitly or implicitly


\(^10\) Our aim is to modify Nash equilibrium by allowing for the possibility that players may view their opponent’s behaviour as ambiguous. If beliefs were additive, then in a 2-player game, Definition 1 would coincide with Nash equilibrium. In this sense we have modified Nash equilibrium to allow for ambiguity.

\(^11\) This definition is a special case of the inner support notion in Ryan (1997).
restricted attention to the case of pessimistic players. In the present paper, an important part of our explanation of behaviour in experimental games relies to a large extent on optimistic responses to ambiguity. It is, therefore, necessary to reconsider the support notions put forth in the previous literature. For a more detailed discussion of the relation of our proposal to earlier support notions see Eichberger and Kelsey (2009).

3 EXPERIMENTAL GAMES

Goeree and Holt (2001) present evidence that NE is a good predictor in some games but not others. In particular they consider five simultaneous move games, in which there is evidence in favour of NE. However, in each case, a seemingly irrelevant parameter change produces the opposite result. In this section we argue that much of this evidence can be explained by ambiguity. For expository reasons we shall discuss the experiments in a different order to GH. To avoid undue repetition, we shall discuss the first game in detail and discuss the remaining cases more briefly.

3.1 The Kreps Game

The impact of ambiguity is illustrated by the Kreps game, which is an asymmetric coordination game with a safe strategy for Player 2, \( NN \). The normal form of the game is described in the following matrix:

<table>
<thead>
<tr>
<th>Game A</th>
<th>Player 2</th>
<th>Player 1</th>
<th>L (26%)</th>
<th>M (8%)</th>
<th>NN (68%)</th>
<th>R (0%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (68%)</td>
<td>200, 50</td>
<td>0, 45</td>
<td>10, 30</td>
<td>20, -250</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B (32%)</td>
<td>0, -250</td>
<td>10, -100</td>
<td>30, 30</td>
<td>50, 40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The numbers in brackets denote the number of subjects playing the respective strategies in GH’s experiment. The only two Nash equilibria in pure strategies are \( (T, L) \) and \( (B, R) \). There is also a mixed strategy equilibrium, in which Player 2 chooses \( M \) and \( L \) each with positive probability. The only strategy not played in any NE is \( NN \). In stark contrast, over two thirds of subjects chose \( NN \). Interestingly, this game shows another behavioural feature not mentioned in GH. Given the strong incentive of Player 2 to choose \( NN \), Player 1 could be expected to play the best reply \( B \). This is, however, not the case for the majority of subjects in GH’s experiment.

We claim that these results can be explained by ambiguity and are compatible with the estimated values of \( \alpha \) and \( \delta \). For player 2, strategy \( NN \) gives a
certain pay-off of 30, even with ambiguity. Pessimistic responses to ambiguity can motivate him to choose NN, since all the other strategies can potentially yield a much lower pay-off. Suppose that he has an ambiguous belief that player 1 will play $s_1$, where $s_1$ can either take the values $T$ or $B$; then the Choquet expected utility of his other strategies is given by:\textsuperscript{13}

$$V_2(L) \leq \delta \cdot (\alpha \cdot 50 - (1 - \alpha) \cdot 250) + (1 - \delta) \cdot \max \{u_2(T, L), u_2(B, L)\} \leq 50 - \delta \cdot (1 - \alpha) \cdot 300 \leq 30;$$

$$V_2(M) \leq \delta (\alpha \cdot 45 - (1 - \alpha) \cdot 100) + (1 - \delta) \cdot \max \{u_2(T, M), u_2(B, M)\} = 45 - \delta (1 - \alpha) \cdot 145 \leq 30;$$

$$V_2(R) \leq \delta \cdot (\alpha \cdot 40 - (1 - \alpha) \cdot 250) + (1 - \delta) \cdot \max \{u_2(T, R), u_2(B, R)\} = 40 - \delta \cdot (1 - \alpha) \cdot 290 \leq 30;$$

provided $\delta(1 - \alpha) \geq \max\{\frac{20 - 30}{300}, \frac{45 - 30}{145}, \frac{40 - 30}{250}\} = \frac{3}{25} = 0.12$. If $\alpha$ and $\delta$ lie in the KW-range then $0.16 \leq \delta (1 - \alpha) \leq 0.37$. Hence in EUA, Player 2 will choose NN.

The observed behaviour of more than two thirds of Player 1’s choosing $T$, can be obtained as an equilibrium under ambiguity, $\pi_1^*(T) = \pi_2^*(NN) = 1$, but never as a NE. Assuming 2 is believed to play NN, the CEU value of pay-offs for Player 1 are:

$$V_1(s_1, NN) = \delta \alpha \cdot \max_{s_2 \in S_2} u_1(s_1, s_2) + \delta (1 - \alpha) \cdot \min_{s_2 \in S_2} u_1(s_1, s_2) + (1 - \delta)u_1(s_1, NN)$$

$$= \left\{ \begin{array}{ll}
\delta \cdot \alpha \cdot 200 + (1 - \delta) \cdot 10 & \text{for } s_1 = T, \\
\delta \cdot \alpha \cdot 50 + (1 - \delta) \cdot 30 & \text{for } s_1 = B.
\end{array} \right.$$ 

Thus $T$, is preferred to $B$ if and only if $150\delta \alpha - (1 - \delta)20 \geq 0 \iff \frac{\delta \alpha}{1 - \delta} \geq \frac{2}{15} = 0.13$. This inequality is satisfied for all $\alpha, \delta$ in the KW-range. Hence, with the experimentally observed parameter values for $\alpha$ and $\delta$, there is a unique equilibrium under ambiguity with equilibrium strategies, $(T, NN)$.

In Game A, $\frac{3}{4}$ of the subjects play the strategy NN, which is not part of any NE. This is clear evidence of a deviation from NE. A possible explanation of this is Loss Aversion, see Kahneman et al (1991). This refers to the fact that individuals avoid taking actions which potentially could result in losing money. Loss aversion is in addition to conventional risk aversion, which can arise even when all pay-offs are positive. Note that NN is the only strategy for player 2 which does not potentially lead to a negative pay-off. Hence Loss Aversion could potentially be a reason why 68% of subjects choose NN in Game A.

Game B, is similar to Game A except that 300 has been added to all pay-offs. This change does not affect the set of Nash equilibria nor does it affect the set of EUA.

\textsuperscript{13} For convenience we are suppressing the arguments $\alpha, \delta$ and $\pi$.  

No strategy in Game B can yield losses. If significantly less players had played NN in Game B than in Game A, this would have been evidence that play of NN in the Game A was due to loss aversion. In fact adding the constant had little effect on the behaviour of player 2. This suggests that ambiguity-aversion is a more likely explanation for playing NN.

The “Treasure” treatment of the Kreps game is as follows.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L (24%)</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>500, 350</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>300, 50</td>
</tr>
</tbody>
</table>

The pay-offs from the strategy combination \( B;R \) have been substantially increased. This does not affect the set of Nash equilibria. It does vastly increase the number of players using the Nash equilibrium strategies \( B \) and \( R \).

Consider first player 1. Suppose that she believes that player 2 will play \( R \), then her (Choquet) expected utility from her two strategies are \( V_1 (T) = \delta \alpha 200 - \delta (1 - \alpha) 250 - (1 - \delta) 250 \) and \( V_1 (B) = \delta \alpha 350 + (1 - \delta) 350 \). It can be seen that \( B \) will be preferred to \( T \) for any \( \alpha \) and \( \delta, 0 \leq \alpha \leq 1, 0 \leq \delta \leq 1 \).

Now suppose that player 2 believes that 1 will play \( B \). Then his Choquet expected utility from his strategies is:

\[
\begin{align*}
V_2 (L) &= \delta \alpha 50 - \delta (1 - \alpha) 250 - (1 - \delta) 250; \\
V_2 (M) &= \delta \alpha 45 - \delta (1 - \alpha) 100 - (1 - \delta) 100; \\
V_2 (NN) &= 30; \\
V_2 (R) &= \delta \alpha 400 - \delta (1 - \alpha) 250 + (1 - \delta) 400.
\end{align*}
\]

First note that \( R \) will be preferred to \( L \) and \( M \) for any \( \alpha, \delta \). For \( R \) to be preferred to \( NN \) we require: \( \delta \alpha 40 - \delta (1 - \alpha) 25 + (1 - \delta) 40 \geq 3 \Leftrightarrow 0.57 = \frac{22}{37} \geq \delta (1 - \alpha) \). All values of \( \alpha, \delta \) within the KW-range are compatible with this inequality and hence \( (B, R) \) is the unique equilibrium under ambiguity of Game C for parameter values in this range.

### 3.2 The Traveller’s Dilemma

In the Traveller’s Dilemma, each player makes a claim \( n_i \) for a payment between 180 and 300 cents, i.e., \( n_i \in S := \{ 180, 181, 182, \ldots, 298, 299, 300 \}. \)

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14 Assuming that 0 is the reference point from which subjects measure gains and losses.

15 The story which motivates the game can be found in Basu (1994), where this game was introduced.
Given two claims \((n_1, n_2)\), both players obtain the minimum \(\min\{n_1, n_2\}\), but, if the claims are not equal, the player with the higher claim pays \(R > 1\) to the other, yielding the payoff function:

\[
u_i(n_1, n_2) = \min\{n_1, n_2\} + R \cdot \text{sign}(n_j - n_i),
\]

with \(i, j \in \{1, 2\}\) and \(i \neq j\).

It is easy to see that for \(R > 1\) each player has an incentive to undercut the opponent’s claim by one unit. Figure 4 shows the best-reply of Player 1. Hence, for any \(R > 1\), claiming the minimum amount, \((n_1^*, n_2^*) = (180, 180)\), is the unique NE. In fact, \(n_i^* = 180\) is the only rationalizable strategy for each player, since it is the only strategy which cannot be undercut by the opponent.

GH show, however, that the experimental results, depend on \(R\). For large \(R\), players claim 180, or close to this amount, as predicted by the NE. For \(R = 180\), almost 80 percent of the subjects chose \(n_i \leq 185\). In contrast to the NE predictions, for small \(R\), players made claims close to 300, i.e., for \(R = 5\), almost 80 percent of the players chose \(n \geq 295\).

The evidence can be explained by ambiguity as follows. In the Traveller’s Dilemma, pay-offs are high if players coordinate on a high claim. As a result there are two strategies which are potentially best responses to an action by one’s opponent. Either one can undercut by one unit or alternatively one can choose 299, which may yield the highest coordination gain and maintains at
least the chance to avoid the penalty. For $R = 180$ however, the penalty for being the highest bidder is extreme, wiping out any possible gain from coordination. Hence, even a small amount of pessimism in response to ambiguity will deter players from making a high claim and the only possible equilibrium is where both claim $180$.

![Fig. 5 Best reply of Player 1 with ambiguity](image)

In contrast, for $R = 5$, the penalty is so low that a little ambiguity and optimism, $\frac{\alpha \delta}{1-\delta} \geq 0.1$, suffices to make it worthwhile to claim $299$. Figure 5 illustrates the equilibrium best-reply correspondence. There is a mixed equilibrium with two best responses $299$ and $\tilde{n}$. For example, if $\alpha = 0.4$ and $\delta = 0.6$, the ambiguous beliefs that one’s opponent would choose $[\tilde{n}] = 285$ would be $0.18$ and the belief for $299$ would equal $0.82$. The observation, that 80 percent of subjects chose a claim higher than $295$, seems to be not obviously incompatible with EUA for these values of $\alpha$ and $\delta$. This analysis is summarised in the following proposition.

**Proposition 1** Suppose players’ beliefs are given by a neo-additive capacity with parameters $\alpha, \delta$ in the KW-range.

1. For $R = 180$, in the unique symmetric EUA both players have beliefs with support $n_{-i} = 180$. For each player $i$, $n_i = 180$ is the unique best response.
2. For $R = 5$, the unique symmetric EUA is a mixed equilibrium in which each player has two best responses, $[\pi]$ and $299$ where

$$\pi(\alpha, \delta, R) := \left(300 - \frac{1-\delta}{\alpha\delta}(2R - 1)\right).$$

The table below gives the values of $[\bar{n}]$ for $R = 5$ and parameter values in the KW-range.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\alpha$</th>
<th>0.4</th>
<th>0.5</th>
<th>0.62</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.41</td>
<td></td>
<td>268</td>
<td>274</td>
<td>280</td>
</tr>
<tr>
<td>0.52</td>
<td></td>
<td>279</td>
<td>283</td>
<td>286</td>
</tr>
<tr>
<td>0.61</td>
<td></td>
<td>285</td>
<td>288</td>
<td>291</td>
</tr>
</tbody>
</table>

3.3 Matching Pennies

In experiments on the Matching Pennies game, GH discover that subjects tend to conform with NE predictions if the game is symmetric, but deviate systematically if the pay-offs are asymmetric. They study the following two versions of matching pennies.\(^{17}\) The ratios to the left of the strategies indicate the unique NE mixed strategies and the bold numbers to the right of the strategies show the percentage of subjects choosing the respective strategy in the experiments.

**Game D**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 $T$ (48%)</td>
<td>80, 40</td>
</tr>
<tr>
<td>0.5 $B$ (52%)</td>
<td>40, 80</td>
</tr>
</tbody>
</table>

**Game E**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 $T$ (96%)</td>
<td>320, 40</td>
</tr>
<tr>
<td>0.5 $B$ (4%)</td>
<td>40, 80</td>
</tr>
</tbody>
</table>

The games differ only in the payoff of Player 1 for $(T, L)$, which is indicated by a bold-face number. In mixed-strategy NE, the probabilities, which a given player uses to randomize, are chosen to make his/her opponent indifferent between all of his/her equilibrium strategies. Thus NE predicts that a change

\(^{16}\) As usual, $[y]$ denotes the largest integer smaller than $y$.

\(^{17}\) GH also consider a third version of matching pennies. The impact of ambiguity on this game can be analysed in a similar way.
in player 1’s payoff will leave her own behaviour unchanged, while causing a change in the behaviour of player 2.

Actual play reveals a quite different pattern. While the relative frequency of strategy choices in Game D corresponds to the NE prediction, they deviate dramatically from the predictions in the asymmetric game. In Game E, Player 1’s choose almost exclusively strategy T, the payoff of which has been increased. As a result they make their behaviour predictable, which is exploited by the subjects in the role of Player 2.\textsuperscript{15} It is surprising that Player 1 does not appear to foresee this shift in the behaviour of her opponent. It appears as if Player 2 understands the change in Player 1’s incentives better than she does herself. One interpretation of Player 1’s behaviour may be a shift in decision weights to extremely attractive albeit unlikely events.

Ambiguity makes little difference to game D. Symmetry implies that the only equilibrium is where each player believes that his/her opponent is equally likely to use either strategy. In game E, optimistic responses to ambiguity cause player 1 to overweight unlikely events which yield the high payoff 320. This causes her to choose strategy T almost exclusively. From Player 2’s point of view, the two strategies are symmetric. However 1 has a bias in favour of T. Hence R is a best response for 2. Thus for the parameter values we find in Kilka and Weber’s experiments, there is an equilibrium with ambiguity where the equilibrium strategy combination is \((T, R)\). (For details see the appendix.)

There is no NE which describes such behaviour. There is a unique NE, where Player 2 plays \(\pi(L) = \frac{1}{8}\) and Player 1 plays \(\pi(T) = \frac{1}{2}\). Such randomizations are incompatible with the observed choices.

This discussion is summarised in the following proposition, which is proved in the appendix.

**Proposition 2** Suppose players’ beliefs are given by a neo-additive capacity with parameters \(\alpha, \delta\) in the KW-range.

1. In game D there is a unique EUA where each player is believed to be equally likely to use either strategy i.e. \(\nu_1(T) = \nu_1(B)\) and \(\nu_2(L) = \nu_2(R)\).
2. In game E there is a unique EUA in which T is the best response of player 1 and R is the best response of player 2.

### 3.4 A Coordination Game with a Secure Option

GH study two coordination games, which have been modified by giving Player 2 an extra secure option.

\textsuperscript{15} GH’s experiments were one shot. In contrast, Ochs (1995) reports experiments on repeated games with a unique mixed strategy equilibrium. The results in that case are different as he finds that behaviour adapts over time. However he does not, in general, find convergence to Nash equilibrium behaviour. We have focused on GH’s experiments as we believe their design is more suitable to study the effects of ambiguity.
Both of these games have two NEs, \((L, L)\) and \((H, H)\). The data suggests that in Game \(F\) subjects are nearly all playing the \((H, H)\) equilibrium. In game \(G\) the majority of the subjects are still playing the \((H, H)\) equilibrium, while a significant minority have switched to other strategies.\(^{19}\)

For this experiment we are not able to make a precise prediction since both games have multiple equilibria with and without ambiguity. However we shall show that the results agree in a qualitative sense with EUA. The following proposition shows that while in Game \(F\), \((H, H)\) is an equilibrium for all parameters values in the KW-range, in Game \(G\) this equilibrium only exists for some parameters in this range. Thus in moving from Game \(F\) to Game \(G\) one would expect some subjects to switch from \(H\) to another strategy. It is not possible to predict how many subjects will switch without more details of the distribution of \(\alpha\) and \(\delta\). Unless the distributions of the parameters are concentrated at the upper ends of their ranges, only a minority of subjects would switch. Thus the theoretical prediction agrees with the experimental results at least qualitatively.

**Proposition 3** For the coordination game with a secure option:

1. In game \(F\) both \((H, H)\) and \((L, L)\) are EUA for all parameters in the KW-range;
2. In game \(G\),
   
   (a) \((H, H)\) is an EUA for \(\alpha, \delta\) in the KW-range provided \(0.82 \geq \frac{\delta\alpha}{1-\delta}\);
   (b) \((L, L)\) is an EUA for all parameters in the KW-range.

Whether the \((H, H)\) equilibrium exists in game \(G\) depends on the parameter \(\eta = \frac{\delta\alpha}{1-\delta}\). KW’s evidence shows that \(0.28 \leq \eta \leq 0.97\). For illustrative purposes assume that \(\eta\) is uniformly distributed on this range. Then for \(21\%\) of subjects, the \((H, H)\) equilibrium will not exist in Game \(G\). This is qualitatively similar to the experimental results, where \(36\%\) play \(L\) in Game \(G\). We have two possible explanations for the discrepancy between \(21\%\) and \(36\%\). Firstly it could be that \(\eta\) is not uniformly distributed. Secondly subjects may perceive co-ordination games to be more ambiguous than KW’s single person decisions, (i.e. \(\delta\) is larger than in the KW experiment). This is not implausible since the presence of multiple equilibria means that even if one believes one’s

\(^{19}\) GH do not report whether player 2 switches to \(L\) or \(S\).
opponent will play an equilibrium strategy it is not clear which one. Moreover there is no natural way to assign probabilities to the different equilibria. If in addition it is not clear whether one’s opponents will play an equilibrium strategy, ambiguity will be increased still further.

3.5 A Minimum-Effort Coordination Game

The minimum effort coordination game was introduced by Huyck et al (1990), (it is also know as the weakest link model of public goods, see Cornes and Sandler (1996)). In this model, provision of the public good is equal to the *minimum* of the individual contributions. This is in contrast to a standard model in which provision of the public good depends on the *sum* of the individual contributions. An example would be a number of military commanders defending a frontier, where the enemy will choose the most favourable point at which point to attack.

In the game, two players have to choose effort levels from the set \( E = \{110, \ldots, 170\} \) at a marginal cost of \( c < 1 \) yielding pay-offs

\[
    u_i(e_1, e_2) := \min\{e_1, e_2\} - c \cdot e_i,
\]

for \( i = 1, 2 \). GH ran this experiment with the marginal cost parameters 0.1 and 0.9 and observed the following distributions of play:\(^{20}\)

<table>
<thead>
<tr>
<th>Game H</th>
<th>( e = 115 )</th>
<th>( e = 125 )</th>
<th>( e = 135 )</th>
<th>( e = 145 )</th>
<th>( e = 155 )</th>
<th>( e = 165 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e = 0.1 )</td>
<td>0.1</td>
<td>0.02</td>
<td>0.1</td>
<td>0.1</td>
<td>0.08</td>
<td>0.6</td>
</tr>
<tr>
<td>( e = 0.9 )</td>
<td>0.5</td>
<td>0.18</td>
<td>0.05</td>
<td>0.07</td>
<td>0.05</td>
<td>0.15</td>
</tr>
</tbody>
</table>

The observations show a clear concentration of play on high effort levels in the case of low costs, \( c = 0.1 \), and on the low ones for high costs, \( c = 0.9 \). Coordinating on any of the six possible effort levels is a NE for either possible value of costs i.e., the set of NE’s is \( \{(e_1^*, e_2^*) \in E^2 \mid e_1^* = e_2^*\} \). Thus NE is unable to explain why an increase in the cost parameter changes behaviour. Since the experiments were one-shot games and there were many possible equilibria, coordination is not very likely.

We shall argue that such observations can be explained by ambiguity. In this game, the best outcome is that your opponent plays the highest possible strategy. Suppose there is an equilibrium with ambiguity in which both players coordinate on an effort level other than the highest. If Player 1 increases her effort by one unit, the perceived marginal benefit is at least \( \delta \alpha \), which is the weight on the highest outcome. The marginal cost of increasing effort is \( \alpha \). Thus if \( \delta \alpha > \alpha \) it is always in her interest to increase her contribution. Player

\(^{20}\) Note that GH have grouped the data for ten successive integers.
2 will think similarly. Hence if $\delta \alpha > c$ the only possible equilibrium is where both players make the highest contribution.

Suppose there is an equilibrium with ambiguity in which both players use an effort level other than the lowest (i.e. 110). If Player 1 decreases her contribution by one unit the perceived marginal reduction in benefit is at most $\delta \alpha + (1 - \delta)$, (this is the sum of the decision weight on the worst outcome and the equilibrium outcome.) The marginal cost saving is $c$. Thus if $c > \delta \alpha + (1 - \delta)$ it is worth decreasing effort, which implies that the only possible equilibrium is where both players coordinate on the lowest effort level.

If $\alpha$ and $\delta$ lie in the KW-range we have $0.38 \geq \delta \alpha \geq 0.16$, which implies $\delta \cdot \alpha - c > 0$ for $c = 0.1$. Similarly Kilka and Weber’s experiment implies $0.85 \geq \delta \alpha + (1 - \delta) \geq 0.64$ and hence $c > |\delta \cdot \alpha + (1 - \delta)|$ for $c = 0.9$. For marginal costs of 0.1, equilibrium with ambiguity predicts that players would try to coordinate on the highest effort level, while for $c = 0.9$ they should coordinate on the lowest effort level. The observed behaviour seems to correspond reasonably well with this prediction.

4 COMPETING EXPLANATIONS

For economic theory, the notion of a Nash equilibrium is attractive. If the pay-offs of players are common knowledge, NE identifies strategy combinations which leave no incentives for individual players to change their behaviour as long as the opponents’ follow their equilibrium strategy. Thus, it combines two key ideas:

(i) Players choose the best strategy given their beliefs about the behaviour of the opponents;

(ii) Beliefs about the behaviour of the opponents are correct.

In order to infer their opponents’ behaviour in one-shot games, however, players need to know the incentives of these opponents. Whether the pay-offs of a player reflect the incentives of this player correctly is however questionable. Attitudes towards risk and ambiguity may matter, just as social attitudes like preferences for fairness, altruism or spite may affect behaviour.

The regularities of behaviour recorded in GH’s experiments challenge the idea that the interaction of subjects in a one-shot game can be appropriately described by Nash-equilibrium. Yet, as the treasure treatments show, NE does seem to work well for some payoff constellations.

There are several responses to the challenge posed by these experimental results. One may take into account additional aspects of preferences, one may relax the assumption that players maximize an objective function, or one may give up the idea of consistency between beliefs and actual behaviour. Examples of the first approach include fairness-based payoff transformations in the
spirit of Fehr and Schmidt (1999) and Rabin (1993) and the EUA approach suggested in the present paper. The *Quantal Response Equilibrium (QRE)* proposed by McKelvey and Palfrey (1995) provides an example for the second type of response. QRE assumes that players play mixed strategies which assign probabilities to the pure strategies based on the relative expected payoff obtained with them.\(^{21}\) McKelvey and Palfrey (1995) show that QRE can be viewed as a Bayesian equilibrium of a game of imperfect information with payoff uncertainty.\(^{22}\) In contrast to models which explicitly assume a bias in the payoff evaluation, QRE does not suggest a reason for the particular error structure assumed. The third group of responses is represented by the *Cognitive Hierarchy Model (CHM)* of Camerer et al (2004). Building on the level-k optimality model proposed by Stahl and Wilson (1995) this approach abandons the notion of equilibrium. It assumes a distribution of player types characterised by different levels of rationality. These different levels of rationality remain unrelated to the actual behaviour of players.

For the asymmetric Matching-Pennies game, Boylan and Grant (2006) consider fairness-based payoff transformations in the spirit of Fehr and Schmidt (1999) and Rabin (1993). They show, however, that such preferences do not predict the observed behaviour. In contrast, they find that the QRE can explain the experimental data of Goeree and Holt (2001). To the best of our knowledge there exists no study comparing these approaches. It appears however likely that all of them will explain the observed data better than NE since they all add more parameters. EUA adds parameters for ambiguity and ambiguity-attitude, QRE adds a sensitivity parameter and CHM allows for arbitrary levels of rationality. If these parameters can be adjusted freely, it is not surprising that a superior fit to the data can be achieved.

A better test requires constraints on these new parameters. In our opinion, this is the main advantage of the EUA. Ambiguity and ambiguity-attitudes have been studied in large numbers of non-interactive decision-making experiments. From these studies we can obtain additional constraints for the parameters \(\alpha\) and \(\delta\). As we have shown in this paper, estimates of these parameters from the range obtained in many experiments seem to suffice to explain the observed behaviour in games. Ideally, one would want to assess these parameters from the participants of the game-theoretic experiments. For instance, one could run pretests in which subjects’ choices in Ellsberg-type problems reveal their ambiguity attitude, before they interact with others. Neither QRE nor CHM offer such possibilities because the parameters of “sensitivity” and “level of rationality” lack an interpretation outside the given game. They just measure the deviation from full rationality embodied in the Nash equilibrium concept.

\(^{21}\) Since the expected payoffs from a player’s pure strategy depend on the mixed strategies chosen by the opponents, QRE has to solve a fixed point problem in order to obtain consistency among the players’ mixed strategies.

\(^{22}\) For further discussion of the properties of QRE see Haile et al (2008) and Goeree and Holt (2005).
All these approaches are more flexible in describing actual behaviour than NE and therefore will provide a better fit for experimental data. We believe that EUA may be given a more attractive interpretation. Ambiguity-attitude, optimism and pessimism, can be viewed as personal characteristics, which we may take as given, like a player’s preferences. Ambiguity, on the other hand, is more situation-dependent. In particular, one would predict that a one-shot game is, by nature, more ambiguous than later rounds of a repeated game. Hence, there are testable hypotheses regarding EUA, which have no counterpart in QRE and CHM. Moreover, there is substantial evidence on individual behaviour, which allows one to predict the attitude of a decision-maker towards ambiguity. Such evidence can help to restrict behaviour in EUA, which makes the theory more powerful. It is a particular strength of EUA that it can explain the behaviour in many different games with the same set of parameters. The hypothesis that players view their opponents’ actions as ambiguous, produces some quite precise comparative static predictions, which could in principle be tested experimentally.

For economic analysis, it is desirable to have a model with parameters which one can interpret behaviourally and which include the “full rationality” of a Nash equilibrium as a special case. While Nash equilibrium can be obtained as a limiting case in all these models, a behavioural interpretation can be given only in the context of the fairness approach of Fehr and Schmidt (1999) and the EUA. Since the parameters of QRE and CHM are harder to interpret, it appears also more difficult to use these concepts in economic analysis.

Schmutzler (2008) also studies some of the GH treasures. He shows that if a game has strategic complementarity and increasing differences then the comparative statics are predictable. This result is used to explain the comparative static results in 7 of the ten treasures. However he does not attempt to predict point outcomes in any single experiment.

5 CONCLUSION

In this paper we have shown that many of GH’s treasures of game theory can be explained as responses to ambiguity. We have shown that four of the five Treasures can be explained by the hypothesis that subjects have CEU preferences with parameters in the KW-range. In the coordination game with a secure option there are multiple equilibria with and without ambiguity. Thus it is not possible to make a point prediction. However the shift in behaviour between Games F and G appears, at least in qualitative terms to be compatible with our model of ambiguity in games.

We have only analysed those treasures based on normal form games. The other experiments concern dynamic games some of which also have incomplete information. To study the impact of ambiguity in these cases it would be necessary to develop new solution concepts for such games. This is beyond the scope of the present paper. Nevertheless we believe that explanations based on
ambiguity could be found for many of these games as well. For instance, the
treasure from GH entitled ‘Should you believe a threat that is not credible’ is
similar to the model of frivolous lawsuits in Eichberger and Kelsey (2004).

The preferences we use have the effect of over weighting the best and worst
outcomes. It is likely that ambiguity-aversion would cause other bad outcomes
to be over weighted. Similarly optimism might have the effect that a number
of good outcomes are over-weighted rather than just the best outcome. While
this objection may have some merits in general, the games studied in this paper
typically have salient best and worst outcomes. It does not seem implausible
that subjects should focus on these outcomes. In Eichberger and Kelsey (2009)
we show that much of our analysis can be extended to the more general case
where a number of good and bad outcomes are over-weighted.

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A APPENDIX

The appendices provide details of the model and proofs of the claims made in the corresponding section of the text.

A.1 The Traveller’s Dilemma

Consider Player 1. Suppose she holds beliefs which are represented by a neo-additive capacity:\[\nu(A; \alpha, \delta, \pi) = \alpha \delta + (1 - \delta) \pi(A).\] Given these beliefs, Player 1’s Choquet expected utility from the choice of \(n_1\) is

\[
V_1(n_1; \alpha, \delta, \pi) = \delta \left( \alpha \max_{n_2 \in S} u_1(n_1, n_2) + (1 - \alpha) \min_{n_2 \in S} u_1(n_1, n_2) \right) \\
+ (1 - \delta) \cdot \sum_{n_2 \in S} u_1(n_1, n_2) \cdot \pi(n_2).
\]

**Lemma 1** Suppose Player 1’s beliefs are given by a neo-additive capacity \(\nu(\cdot|\alpha, \delta, \pi^{n_2}).\) If \(\alpha\) and \(\delta\) lie in the KW-range then the best-reply correspondence is

\[
R_1(\nu(\cdot|\alpha, \delta, n_2)) = \begin{cases} 
299 \quad &\text{for} \quad n_2 < \pi(\alpha, \delta, R), \\
\pi(\alpha, \delta, R) - 1 &\text{otherwise},
\end{cases}
\]

with

\[
\pi(\alpha, \delta, R) := 300 - \frac{1 - \delta}{\alpha \delta} (2R - 1).
\]

**Proof** First note since \(R > 1, n_1 = 299\) weakly dominates \(n_1 = 300,\) moreover, for \(\alpha, \delta > 0, n_1 = 300\) is strictly dominated. To see this note that the highest pay-off for \(n_1 = 299\) is greater than that for \(n_1 = 300,\) and the highest payoff gets positive weight in the Choquet integral. (Given our assumptions on \(\alpha\) and \(\delta.\) Thus we may eliminate the possibility that either player plays strategy 300.

Consider \(n_2 = 180.\) The CEU of a pure strategy \(n_1\) can be computed as,

\[
V_1(n_1; \alpha, \delta, \pi^{180}) := \int u_1(n_1, \cdot) d\nu(\cdot|\alpha, \delta, \pi^{180}) = \delta \left( \alpha \max_{n_2 \in S} u_1(n_1, n_2) + (1 - \alpha) \min_{n_2 \in S} u_1(n_1, n_2) \right) + (1 - \delta) \cdot u_1(n_1, 180)
\]

\[
= \begin{cases} 
\delta \left( \alpha (180 + R) + (1 - \alpha)180 \right) + (1 - \delta) \cdot 180 \quad &\text{for} \quad n_1 = 180, \\
\delta \left( \alpha (n_1 + R) + (1 - \alpha) (180 - R) \right) + (1 - \delta) \cdot (180 - R) \quad &\text{for} \quad 180 < n_1 < 300.
\end{cases}
\]

Consider next \(n_2 \in (180, 300).\) The CEU of a pure strategy combination \((n_1, n_2)\) is,

\[
V_1(n_1; \alpha, \delta, \pi^{n_2}) := \int u_1(n_1, \cdot) d\nu(\cdot|\alpha, \delta, \pi^{n_2})
\]

\footnote{Here \(\pi^{n_2}\) denotes the Dirac measure with support \(n_2.\)}
\[
\delta [\alpha (180 + R) + (1 - \alpha)180] + (1 - \delta) \cdot [180 + R] \quad \text{for } n_1 = 180;
\]
\[
\delta [\alpha (n_1 + R) + (1 - \alpha) (180 - R) + (1 - \delta) \cdot [n_1 + R] \quad \text{for } 180 < n_1 < n_2;
\]
\[
\delta [\alpha (n_1 + R) + (1 - \alpha) (180 - R) + (1 - \delta) \cdot n_1 \quad \text{for } n_1 = n_2;
\]
\[
\delta [\alpha (n_1 + R) + (1 - \alpha) (180 - R) + (1 - \delta) \cdot [n_2 - R] \quad \text{for } n_2 < n_1 < 300.
\]

(ii) For \( n_2 = 180 \), \( V_1(n_1; \alpha, \delta, \pi^{180}) \) is strictly increasing in \( n_1 \) in the range \( 299 > n_1 > 180 \). Hence,
\[
V_1(299; \alpha, \delta, \pi^{180}) - V_1(180; \alpha, \delta, \pi^{180}) = a\delta (299 - 180) - (1 - a)\delta R = 119a\delta - (1 - a)\delta R.
\]
Thus
\[
R_1(\nu(a; \alpha, \delta, 180)) = \begin{cases} 299 & \text{for } a\delta > \frac{R}{R_{180}}, \\ 180 & \text{otherwise.}
\end{cases}
\]

Notice, for \( R = 5 \), \( \frac{R}{R_{180}} = \frac{5}{299} \approx 0.041 < 0.4 \cdot 0.41 \). Hence, \( R_1(\nu(a; \alpha, \delta, 180)) = 299 \) for \( R = 180 \), we have \( \frac{R}{R_{180}} = \frac{180}{299} = 0.6 \) 0.61 - 0.62 and \( R_1(\nu(a; \alpha, \delta, 180)) = 180 \).

(iii) Consider now \( n_2 \in (180, 300) \). For \( n_1 \in (180, n_2) \cup (n_2, 300) \) the CEU value is strictly increasing in \( n_1 \). Hence, only \( n_1 = 180, n_1 = n_2 - 1, n_1 = n_2, \) or \( n_1 = 299 \) can be best responses to \( n_2 \).

(a) Comparing \( n_1 = n_2 - 1 \) and \( n_1 = n_2 \), we observe that:
\[
V_1(n_2; \alpha, \delta, \pi^{n_2}) - V_1(n_2 - 1; \alpha, \delta, \pi^{n_2}) = \{\delta [\alpha (n_2 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot n_2\}
\]
\[
- \{\delta [\alpha (n_2 - 1 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot (n_2 - 1 + R)\}
\]
\[
= \delta \alpha + (1 - \delta) \cdot (1 - R) = \delta \alpha + (R - 1) - (R - 1) < 0
\]
holds, for \( R > 1 + \alpha \). For \( R = 5 \) and \( R = 180 \), this condition is satisfied. Hence, \( n_1 = n_2 \) cannot be a best reply.

(b1) Suppose \( 181 \leq n_2 \leq 298 \), then we observe that
\[
V_1(299; \alpha, \delta, \pi^{n_2}) - V_1(180; \alpha, \delta, \pi^{n_2}) = \{\delta [\alpha (299 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot [n_2 - R]\}
\]
\[
- \{\delta [\alpha (180 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot [180 + R]\}
\]
\[
= 119\delta \alpha + (1 - \delta) \cdot (n_2 - 180 - 2R) \geq \frac{\delta \alpha}{1 - \delta} 119.
\]

By the assumption \( 0.28 < \frac{a\delta}{1 - \delta} < 0.97 \), we obtain for \( R = 5, 180 > 180 + 2 \cdot 0.28 - 0.28 < 119 \geq 119 + 2 \cdot 2 R - \frac{\delta \alpha}{1 - \delta} \). Hence, for \( R = 5, V_1(299; \alpha, \delta, \pi^{n_2}) > V_1(180; \alpha, \delta, \pi^{n_2}) \) for all \( n_2 > 180 \).

Moreover, for \( R = 180 \), we find that \( 180 + 2 R - \frac{\delta \alpha}{1 - \delta} 119 > 180 + 2 \cdot 0.97 \cdot 119 > 300 \). Thus, \( V(180; \alpha, \delta, \pi^{n_2}) > V(299; \alpha, \delta, \pi^{n_2}) \) in this case.

(b2) For \( n_2 = 299 \), \( V_1(299; \alpha, \delta, \pi^{299}) - V_1(180; \alpha, \delta, \pi^{299}) \)
\[
= \{\delta [\alpha (299 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot 299\}
\]
\[
- \{\delta [\alpha (180 + R) + (1 - \alpha) (180 - R)] + (1 - \delta) \cdot [180 + R]\}
\]
\[
= 119\delta \alpha + (1 - \delta) \cdot (119 - R) \leq 0 \iff 119 \left(1 + \frac{a\delta}{1 - \delta}\right) \leq R.
\]
For \( R = 5 \), this condition is satisfied, hence, \( n_3 = 299 \) is the best reply to a belief concentrated on \( n_2 = 299 \). For \( R = 180 \), \( 119 \left(1 + \frac{a\delta}{1 - \delta}\right) \leq 119 (1 + 0.28) < 180 = R \). Hence, \( V_1(299; \alpha, \delta, \pi^{299}) < V_1(180; \alpha, \delta, \pi^{299}) \).
Comparing $n_1 = n_2 - 1$ with $n_1 = 299.$

For $n_2 = 299,$ we obtain
\[
V_1(299; \alpha, \delta, \pi^{299}) - V_1(299, \alpha, \delta, \pi^{299}) = \{\delta \alpha (298 + R) + (1 - \alpha) (180 - R)} + (1 - \delta) \cdot (298 + R) \}
- \{\delta \alpha (299 + R) + (1 - \alpha) (180 - R)} + (1 - \delta) \cdot (299) \}
= \delta \alpha (-1) + (1 - \delta) \cdot (R - 1) \leq 0 \quad \text{as} \quad R \geq 1 + \frac{\alpha \delta}{1 - \alpha}. \]
For $R \geq 2.$ Thus, $n_1 = 298$ is the best reply to 299.

For $n_2 < 299,$ we obtain
\[
V_1(n_2 - 1, \alpha, \delta, \pi^{n_2}) - V_1(299, \alpha, \delta, \pi^{n_2}) = \{\delta \alpha (n_2 - 1 + R) + (1 - \alpha) (180 - R)} + (1 - \delta) \cdot (n_2 - 1 + R) \}
- \{\delta \alpha (299 + R) + (1 - \alpha) (180 - R)} + (1 - \delta) \cdot (n_2 - R) \}
= \delta \alpha (n_2 - 300) + (1 - \delta) \cdot (2R - 1),
\]
thus
\[
V_1(n_2 - 1, \alpha, \delta, \pi^{n_2}) \geq V_1(299; \alpha, \delta, \pi^{n_2}) \quad \text{as} \quad n_2 \geq 300 - \frac{1 - \delta}{\alpha \delta} (2R - 1) =: \pi(\alpha, \delta, R).
\]
Hence,
\[
R_1(\nu(\alpha, \delta, n_2)) = \begin{cases} 299 & \text{for} \quad n_2 < \pi(\alpha, \delta, R), \\ n_2 - 1 & \text{otherwise}. \end{cases}
\]

For $R = 180,$ $\pi(\alpha, \delta, R) < 0.$ More precisely, for the parameter values of $\alpha$ and $\delta$ it ranges between $300 - \frac{\alpha \delta}{1 - \alpha} = -982$ and $300 - \frac{\alpha \delta}{1 - \alpha} = -70.$ Since $n_2 > 180,$ $n_1 = n_2 - 1$ is the best response.

For $R = 5,$ $\pi(\alpha, \delta, R)$ can range between $300 - \frac{9}{0.28} = 287$ and $300 - \frac{9}{0.97} = 290.$

The following proposition yields the symmetric equilibrium under ambiguity of this game. The notation $[x]$ denotes the smallest integer larger or equal to $x.$ For ease of notation, we will suppress the arguments of the function $\pi(\alpha, \delta, R)$ and will write $\pi$ for its value.

**Proposition 4** Suppose the conditions of Lemma 1 are satisfied.

1. For $R = 180,$ in the unique symmetric EUA both players have beliefs:
\[
\pi^*(180) = 1.
\]
In response, both players choose $n_1^* = n_2^* = 180.$

2. For $R = 5,$ in the unique symmetric EUA both players have beliefs:
\[
\pi^*(\pi) = \left(1 + \frac{\alpha \delta}{1 - \delta}\right) \frac{299 - \pi}{299 - \pi + R},
\]
\[
\pi^*(299) = \frac{R}{(299 - \pi + R)} \frac{\alpha \delta}{1 - \delta} \cdot (299 - \pi),
\]
where
\[
\pi(\alpha, \delta, R) := 300 - \frac{(1 - \delta)}{\alpha \delta} (2R - 1).
\]
In response, both players choose $n_1^*, n_2^* \in \{\pi, 299\}.$
Proof The equilibrium beliefs $\pi^*$ of an EUA must make players indifferent between claiming \([\pi]\) and 299. Clearly, all strategies which are not best responses will be played with probability zero. Hence, we can set $\pi^*(n) = 0$ for all $n \notin \{\pi, 299\}$. In an EUA the following equation must be satisfied:

$$V_1([\pi]; \alpha, \delta, \pi^*) - V_2(299; \alpha, \delta, \pi^*) = 0.$$  

Expanding, one has

$$\delta \{\alpha ([\pi] + R) + (1 - \alpha) (180 - R)\} + (1 - \delta) \cdot ([\pi] \cdot \pi^*([\pi]) + ([\pi] + R) \cdot \pi^*(299))$$

$$- \delta \{\alpha (299 + R) + (1 - \alpha) (180 - R)\} + (1 - \delta) \cdot ([\pi] - R) \cdot \pi^*([\pi]) + (299 - \pi^*(299)) = 0.$$  

Recall $\pi^*(299) = 1 - \pi^*([\pi])$. Solving for $\pi^*([\pi])$, we obtain:

$$\pi^*([\pi]) = \left(1 + \frac{\alpha \delta}{1 - \delta}\right) \frac{299 - [\pi]}{299 - [\pi] + R}$$

$$\pi^*(299) = \frac{R}{299 - [\pi] + R} - \frac{\alpha \delta}{1 - \delta} \frac{299 - [\pi]}{299 - [\pi] + R}.$$  

A.2 Matching Pennies

Proof of Proposition 2 Let $x$ be the payoff of Player 1 at the strategy combination $(T, L)$. Consider first Player 1’s payoffs:

$$V_1(T) - V_1(B) = \delta [\alpha x + (1 - \alpha) 40] + (1 - \delta) [x \cdot \pi(L) + 40 \cdot \pi(R)]$$

$$- \delta [\alpha 80 + (1 - \alpha) 40] - (1 - \delta) [40 \cdot \pi(L) + 80 \cdot \pi(R)]$$

$$= \delta \alpha (x - 80) + (1 - \delta) [x \cdot \pi(L) - 40].$$  

Similarly, for Player 2 we obtain:

$$V_2(L) - V_2(R) = \delta [\alpha 80 + (1 - \alpha) 40] + (1 - \delta) [40 \cdot \pi(T) + 80 \cdot \pi(B)]$$

$$- \delta [\alpha 80 + (1 - \alpha) 40] - (1 - \delta) [80 \cdot \pi(T) + 40 \cdot \pi(B)]$$

$$= (1 - \delta) (80 \cdot \pi(B) - \pi(T)) = (1 - \delta) 80 \left(\frac{1}{2} - \pi(T)\right).$$  

In game $B$, where $x = 80$,

$$\pi^*(L) = \pi^*(R) = \frac{1}{2},$$

and $\pi^*(T) = \pi^*(B) = \frac{1}{2}$,

is the only EUA for any $\alpha, \delta, 0 < \alpha < 1, 0 < \delta < 1$.

When $x = 320$, $V_1(T) - V_1(B) = 240 \alpha a + (1 - \delta) 320 (\pi_1(L) - \frac{1}{2}) \geq 240 \alpha a - (1 - \delta) 40$.

Thus if

$$\frac{\delta \alpha}{1 - \delta} \geq \frac{1}{6},$$

(2)

$T$ will be preferred to $B$ for any beliefs about player 2’s behaviour (i.e. $T$ will be a best response whatever the value of $\pi_1(L)$). Provided $\alpha$ and $\delta$ are in the KW-range, $\frac{1}{6} \leq \alpha \delta \leq 0.97$, inequality (2) will hold.

In equilibrium, player 2’s beliefs must satisfy $\nu_2(B) = 0$, which implies:

$$V_2(L) = \delta \alpha 80 + \delta (1 - \alpha) 40 + (1 - \delta) 40$$

$$V_2(R) = \delta \alpha 80 + \delta (1 - \alpha) 40 + (1 - \delta) 80.$$  

Hence $L$ is a best response for player 2 for $0 < \alpha \leq 1, 0 < \delta < 1$.

Thus it follows that for parameters in the KW-range, Game E has a unique equilibrium with ambiguity in which 1 plays $T$ and 2 plays $R$.

\textsuperscript{24} For convenience we suppress the arguments $\alpha, \delta$ and $\pi$. 

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A.3 A Coordination Game with a Secure Option

Proof of Proposition 3

Conditions for \(\langle H, H \rangle\) to be an equilibrium in Game F

Suppose that player 2 ‘believes’ player 1 will play \(H\), i.e. \(H\) is the support of his beliefs. Then her (Choquet) expected utility is: \(V_2 (H) = \alpha 180 + (1 - \delta) 180\). \(V_2 (L) = \delta 90\) and \(V_2 (S) = 40\).

\[ V_2 (H) > V_2 (L) \text{ for all } \alpha, \delta, 0 < \alpha \leq 1, 0 < \delta < 1. \]

One can check this inequality holds for all \(\alpha, \delta\) in the KW range.

Suppose that player 1 ‘believes’ player 2 will play \(H\), then her (Choquet) expected utility is: \(V_1 (H) = \delta 180 + (1 - \delta) 180\) and \(V_1 (L) = \delta 90\). Hence \(V_1 (H) > V_1 (L)\) for all \(\alpha, \delta, 0 < \alpha \leq 1, 0 < \delta < 1\).

Conditions for \(\langle L, L \rangle\) to be an equilibrium in Game F

Suppose that player 2 ‘believes’ player 1 will play \(L\). Then his (Choquet) expected utility is: \(V_2 (H) = \delta 180\), \(V_2 (L) = \delta 90 + (1 - \delta) 90\) and \(V_2 (S) = 40\). Thus \(V_2 (L) > V_2 (S) \iff \delta 90 + (1 - \delta) 90 \geq 4\) and \(V_2 (L) > V_2 (H) \iff \delta \alpha + (1 - \delta) 90 \geq 2 \delta \alpha \iff 1 \geq \frac{1}{1 - \delta} \). One can check that both of these inequalities hold for all \(\alpha, \delta\) in the KW range.

Suppose that player 1 ‘believes’ player 2 will play \(L\). Then her (Choquet) expected utility is: \(V_1 (H) = \delta 180 + (1 - \delta) 180\) and \(V_1 (L) = \delta 90\). Hence \(V_1 (L) > V_1 (H)\) for all \(\alpha, \delta\) such that \(0 < \alpha \leq 1, 0 < \delta < 1\).

Conditions for \(\langle H, H \rangle\) to be an equilibrium in Game G

Suppose that player 1 ‘believes’ player 2 will play \(H\). Then her (Choquet) expected utility is: \(V_1 (H) = \delta 180\) and \(V_1 (L) = \delta 400\). For \(H\) to be a best response we require: \(\delta 90 + (1 - \delta) 90 \geq \delta 20\) or \(0.82 = \frac{2}{3} \geq \frac{1}{1 - \delta} \).

Suppose that player 2 ‘believes’ player 1 will play \(H\). Then \(H\) will be a best response for player 2 whenever the parameters are in the KW range. The reasoning for Game F applies here since the pay-offs of player 2 are the same in the two games.

Conditions for \(\langle L, L \rangle\) to be an equilibrium in Game G

Suppose that player 1 ‘believes’ player 2 will play \(L\). Then her (Choquet) expected utility is: \(V_1 (H) = \delta 180\) and \(V_1 (L) = \delta 400 + (1 - \delta) 90\). Hence \(V_1 (L) > V_1 (H)\) for all \(\alpha, \delta\) such that \(0 < \alpha \leq 1, 0 < \delta \leq 1\).

If player 2 ‘believes’ player 1 will play \(L\), then \(L\) is a best response for player 2 whenever the parameters are in the KW-range. As before the reasoning for Game F applies here. Thus \(\langle L, L \rangle\) is an equilibrium in Game G for all \(\alpha, \delta\) in the KW range.