Abstract
This paper studies n-player games where players’ beliefs about their opponents’ behaviour are modelled as non-additive probabilities. The concept of an "equilibrium under uncertainty" which is introduced in this paper extends the equilibrium notion of Dow and Werlang (1994) to n-player games in strategic form. Existence of such an equilibrium is demonstrated under usual conditions. For low degrees of ambiguity, equilibria under uncertainty approximate Nash equilibria. At the other extreme, with a low degree of confidence, maximin equilibria appear. Finally, robustness against a lack of confidence may be viewed as a refinement for Nash equilibria.

JEL Classification: C72, D81

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1. Introduction

Experimental evidence indicates that preferences over uncertain acts cannot be represented by expected utility functionals. In particular, Ellsberg (1961) and others have pointed out that decision-makers distinguish between situations under risk, where objective probabilities are known, and uncertainty, where this is not the case, in a way that cannot be represented by an additive probability distribution. Recently, Schmeidler (1989) and Gilboa (1987) have axiomatised a preference functional which does not imply that beliefs are represented by additive probabilities.

Von Neumann and Morgenstern (1944) were among the first to axiomatise a decision theory for lotteries with known probabilities. This was necessary to describe the payoffs of players who decide to choose a pure strategy randomly. Since then, this has been the dominant paradigm for the analysis of games. In a Nash equilibrium, beliefs about the behaviour of opponents were supposed to coincide with the actual behaviour of these players. Interestingly, von Neumann and Morgenstern (1944) did not propose this equilibrium concept. They worked with a decision theory of complete ignorance assuming that players choose maximin strategies. According to this behavioural assumption agents consider the worst outcome for all strategies available and choose the strategy which yields the best among these worst outcomes. Von Neumann and Morgenstern (1944) could prove however that the value concept built on maximin behaviour coincides with the equilibrium concept for the class of two-player zero-sum games.

The Nash equilibrium concept can be viewed as combining the assumption of optimising behaviour of players for given beliefs about the opponents’ behaviour with the consistency requirement that all players’ beliefs be correct. If beliefs cannot be represented by additive probability distributions, then it is impossible to define consistency of beliefs by identifying mixed strategies actually played with the beliefs players hold about their opponents. With these new approaches to choice under uncertainty, new concepts for how to guarantee some degree of consistency between equilibrium behaviour and equilibrium beliefs about opponents’ behaviour have to be found. Equilibrium concepts under uncertainty can therefore no longer be as tight as they were in traditional Nash equilibrium.

With few exceptions, no attempt has been made so far to investigate the implications of a decision theory without additive probabilities for game theory. In the context of two-player games, Dow and Werlang (1991, 1994) were the first to study a solution concept that allowed for non-additive beliefs of players about the strategy choice of their opponents. They assume that players choose pure strategies and that equilibrium beliefs about these pure strategy choices are concentrated on best responses of the opponents. The precise notion of what it means that beliefs are concentrated on best responses becomes crucial in context of this equilibrium concept. For two-player games, Dow and Werlang (1994) show existence of an equilibrium under uncertainty. Marinacci (1996) proposes an alternative equilibrium concept and proves existence. Haller (1997) investigates equilibria under uncertainty in the context of zero-sum games. Hendon, Jacobsen, Sloth and Tranæs (1993a) study a similar equilibrium concept for the case of belief functions.

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1 There is a large literature on the distinction between uncertainty and risk which cannot be reviewed here. Ellsberg (1961) contains a discussion of the most important contributions to this literature.

2 Groes et al. (1998), however, define the support of beliefs differently.
In two recent papers, Klibanoff (1996) and Lo (1996) have developed an alternative approach to equilibrium under uncertainty. Their approach is based on a related decision theory known as maximin expected utility (MMEU) which was introduced by Gilboa and Schmeidler (1989)\(^3\). In many respects, these equilibrium notions are similar to the one proposed by Dow and Werlang (1994). Since players’ beliefs are represented by sets of additive probability distributions these papers allow players to choose mixed strategies. Expected payoffs can be formed by combining probabilities from mixed strategies that are actually chosen with the additive probabilities in the sets representing beliefs. The tightness of the equilibrium set depends again on the consistency notion. Klibanoff (1996) requires the mixed strategy played in equilibrium to be contained in the set of additive probability distributions representing equilibrium beliefs, while in Lo (1996) all additive probabilities in the equilibrium set of beliefs have to be best responses. Both papers define equilibrium in terms of mixed strategies. A consequence of their choice of strategy space and the way they compute expected payoffs is a strict preference of players for randomisation in situations where decision makers are indifferent about pure strategies with non-comonotonic payoffs. This suggests that where players have access to randomising devices they are likely to use them. Combining actions and beliefs in this way raises some difficult issues since the evaluation of actions, i.e., mixed strategies, may depend on the order in which uncertainty, represented by a set of additive probabilities, is resolved (Sarin and Wakker, 1992). If a player consults a randomising device before playing, does he conceive his exposure to uncertainty as being less than if he had played the same strategy without consulting a randomising device? Eichberger and Kelsey (1996a) show that this distinction depends crucially on the way uncertainty is modelled. In the Ascombe-Aumann axiomatisation of Schmeidler (1989), the preference for randomisation is an immediate consequence of the axiom of comonotonic independence applied to the lottery-outcome space. In contrast, for axiomatisations based on the Savage notion of an act, as in Gilboa (1987) or Sarin and Wakker (1992), no such implication follows.

This paper will extend the approach of Dow and Werlang (1994) to n-player games. Such an extension is essential in order to apply the new approach to economic problems as in Eichberger and Kelsey (1997, 1999). We show how the theory of non-additive beliefs allows us to reconcile the maximin approach with the Nash equilibrium concept for general games. The notions of a degree of confidence and a degree of ambiguity can be used to relate the two approaches. If the degree of confidence in a belief is low players will behave like maximin players whereas they will act as expected utility maximisers for a low degree of ambiguity. It can be shown that, in general, equilibrium behaviour with non-additive beliefs will be as predicted by the Nash equilibrium concept if the degree of ambiguity is low. For low levels of confidence, however, equilibrium behaviour under uncertainty will be maximin behaviour. A further result is the observation that a low degree of confidence will rule out play of dominated strategies in equilibrium. This provides the possibility of a refinement of Nash equilibrium that is weaker than perfectness but still eliminates dominated strategy play in many games.

Ellsberg (1961) suggested a preference representation also built on the notion of a degree

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3 Compare Kelsey (1994) for a simplified exposition of this theory.
of confidence. Though it has a similar interpretation, his representation is distinct from the one advanced in this paper. In particular, while he argues for his representation purely on the grounds of plausibility, the representation in this paper has an axiomatic foundation in the work of Gilboa (1987) and Schmeidler (1989). Eichberger and Kelsey (1996b) provide an axiomatisation of the Ellsberg functional and show that the Choquet integral of simple capacities forms a special case.

The paper is organised as follows. The next section presents notation for games where players’ beliefs are represented by non-additive probabilities. It introduces the concept of the degree of confidence and the degree of ambiguity. Section 3 discusses different notions of equilibrium in this context, provides a definition of an equilibrium under uncertainty and proves existence of equilibrium for any given degree of confidence or ambiguity. Section 4 relates this equilibrium concept to the familiar notions of an equilibrium in maximin strategies and Nash equilibrium. Section 5 shows how robustness against lack of confidence may be used as a refinement of Nash equilibrium and section 6 applies the concept to an economic example.

2. Games and Beliefs

Consider a game \( \Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I}) \) with a finite player set \( I \) and, for each player \( i \in I \), with a finite pure strategy set \( S_i \) and a payoff function \( p_i(s_i, s_{-i}) \) depending on the strategy combination \( (s_i, s_{-i}) \) played. The notation \( s_{-i} := (s_1, \ldots s_{i-1}, s_{i+1}, \ldots s_I) \) indicates a strategy combination for all players except player \( i \). It is convenient to denote by \( S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_I \) the set of strategy combinations which players other than \( i \) could choose.

In contrast to standard game theory, beliefs of players about opponents’ behaviour are represented by non-additive probabilities (or capacities). A capacity assigns non-additive weights to subsets of opponents’ strategy combinations. Formally, capacities are defined as follows.

**Definition 2.1** A capacity on \( S_{-i} \) is a real-valued function \( \nu \) on the subsets of \( S_{-i} \) which satisfies the following properties:

a) \( A \subseteq B \Rightarrow \nu(A) \leq \nu(B) \);  
b) \( \nu(\emptyset) = 0, \nu(S_{-i}) = 1 \).

The capacity is called convex if \( \nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B) \) holds.

Throughout this paper, we will restrict attention to convex capacities. In contrast to an additive probability, it is not required that \( \nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B) \). For a convex capacity \( \nu(A) + \nu(S_{-i} \setminus A) < 1 \) may hold, implying that not all probability mass is allocated to a set and its complement. A deviation of the sum of the capacity values of an event and its complement from one indicates that the decision-maker does not attribute full mass to her belief about an event and its complement. One can therefore view this sum as a measure of the confidence that the decision-maker puts into her beliefs regarding this event. Typically, a capacity \( \nu \) will be characterized by a vector of
confidence parameters\(^4\) \(\{\nu(A) + \nu(S_{-i} \setminus A) \mid A \subset S_{-i}\}\). It is therefore useful to define the following two measures.

**Definition 2.2** The degree of confidence of belief \(\nu\) is

\[
\gamma(\nu) := \max\{\nu(A) + \nu(S_{-i} \setminus A) \mid A \subset S_{-i}\}.
\]

If the degree of confidence of a capacity is zero, players will be completely uncertain in regard to all events. On the other hand, from a degree of confidence of one, one cannot conclude that the player’s beliefs can be represented by an additive probability distribution.

Conversely, one may call the greatest deviation of \(\nu(A) + \nu(S_{-i} \setminus A)\) from one the degree of ambiguity.

**Definition 2.3** The degree of ambiguity of belief \(\nu\) is

\[
\lambda(\nu) := 1 - \min\{\nu(A) + \nu(S_{-i} \setminus A) \mid A \subset S_{-i}\}.
\]

For a degree of ambiguity of zero, a convex capacity \(\nu\) must be an additive probability distribution. The following example shows that capacities put far less restrictions on beliefs than additive probabilities.

**Example 2.1** Consider the case of one opponent with pure strategy set \(S_{-i} = \{s_1, s_2, s_3\}\).

Beliefs of player \(i\) are represented by the following convex capacity:

\[
\begin{align*}
\nu(\{s_1, s_2, s_3\}) &= 1, \quad \nu(\{s_1, s_2\}) = 1, \quad \nu(\{s_1, s_3\}) = 0, \\
\nu(\{s_2, s_3\}) &= 0, \quad \nu(\{s_1\}) = 0, \quad i = 1, 2, 3, \quad \nu(\emptyset) = 0.
\end{align*}
\]

A player with such a belief is extremely confident that his opponent will play either \(s_1\) or \(s_2\), but is extremely ambiguous in regard to any other possible behaviour. This belief combines a full degree of confidence and a full degree of ambiguity, i.e. \(\lambda(\nu) = \gamma(\nu) = 1\).

Of special interest is therefore the case where \(\nu(A) + \nu(S_{-i} \setminus A)\) are constant across events. A constant degree of confidence, or equivalently a constant degree of ambiguity, means that the decision-maker has the same degree of confidence in her beliefs with respect every event \(A\). In this case, \(\max\{\nu(A) + \nu(S_{-i} \setminus A) \mid A \subset S_{-i}\} = \min\{\nu(A) + \nu(S_{-i} \setminus A) \mid A \subset S_{-i}\}\) and, therefore, \(\lambda(\nu) = 1 - \gamma(\nu)\). In this case, beliefs represented by an additive capacity can be interpreted as beliefs held with full confidence or no ambiguity, i.e. \(\gamma(\nu) = 1\) and \(\lambda(\nu) = 0\). Capacities with a constant degree of confidence form a special class of non-additive beliefs which is often useful in applications for comparative static analysis (compare, e.g., Dow and Werlang 1992, Marinacci 1996).

Given beliefs of player \(i\) about the opponents’ behaviour represented by a capacity \(\nu_i\), one can apply the concept of the Choquet integral to determine the expected value of the payoff obtained from pure strategies\(^5\). Since the set of strategy combinations of the

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\(^4\) Marinacci (1996) classifies capacities by the vector of these confidence parameters \(\{\nu(A) + \nu(S_{-i} \setminus A) \mid A \subset S_{-i}\}\).

\(^5\) Schmeidler (1989), Gilboa (1987), and Sarin and Wäcker (1992) provide a system of axioms for the representation of beliefs by capacities and preferences over uncertain acts by a Choquet integral of utilities over these capacities.
opponents $S_{-i}$ is finite, one can order a player’s payoffs from a strategy $s_i$:
\[ p_i^1(s_i) > p_i^2(s_i) > \ldots > p_i^{r_i}(s_i) > p_i^{r_i+1}(s_i). \]

Denote by
\[ T^k(s_i) = \{ s_{-i} \in S_{-i} \mid p_i(s_i, s_{-i}) \geq p_i^k(s_i) \} \]
the set of strategy combinations of the opponents that yield a payoff at least as high as $p_i^k(s_i)$. By convention, let $T^0(s_i) = \emptyset$. The Choquet integral can now be formally defined.

**Definition 2.4** The Choquet integral of the payoff function $p_i(s_i, s_{-i})$ with respect to the capacity $\nu_i$ on $S_{-i}$ is
\[ P_i(s_i, \nu_i) := \sum_{k=1}^r p_i^k(s_i) \cdot [\nu_i(T^k(s_i)) - \nu_i(T^{k-1}(s_i))]. \]

**Remark 2.1** It is an important feature of the Choquet integral that payoffs of events that have measure zero may still influence the expected payoff\(^6\). A player may believe that the opponents will not choose a particular strategy combination and still give some weight to the payoff that would arise in this case. It is this feature of the Choquet integral that allows us to model players that are only to some degree confident about their beliefs regarding the other players’ behaviour.

For game-theoretic applications, by far the most critical aspect of general capacities is the lack of restrictions that they impose on beliefs. There seems to be no obvious restriction that one may want to impose in general on beliefs, once one deviates from their representation by additive probability distributions. From this perspective, it appears desirable to have a special case of a capacity which imposes as little non-additivity as possible, yet so that one can still accommodate the known behavioural inconsistencies\(^7\). In the context of games in strategic form, where no prior objective knowledge is assumed, simple capacities\(^8\) are the natural specialisation of this approach.

**Definition 2.5** A capacity $\nu$ is called simple if there exists an additive probability distribution $\pi$ on $S_{-i}$ and a real number $\gamma \in [0, 1]$ such that for all events $E \subset S_{-i}$, $\nu(E) = \gamma \cdot \pi(E)$.

A simple capacity can be thought of as a contraction of an additive probability distribution. This can be interpreted as lack of confidence of the decision-maker in regard to this probability assessment. The parameter $\gamma$ measures the degree of confidence that the

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\(^6\) Hence, sets that have a capacity value of zero are in general not Savage null sets. On the other hand, Savage null sets have always capacity zero.

\(^7\) In Eichberger and Kelsey (1996b), we introduce the concept of an E-capacity with this property and provide an axiomatisation for it. Interestingly, E-capacities have a Choquet integral which corresponds to a representation of preferences suggested by Ellsberg (1961) without axiomatic foundation.

\(^8\) The terminology simple capacity is used in most of the literature on capacities and non-additive beliefs. Capacities are structurally identical to games with transferable utility. In this context simple games refers to zero-one-valued games.
agent has in the probabilistic assessment given by the additive probability distribution $\pi$, i.e., for simple capacities,

$$\gamma(\nu) = \gamma \quad \text{and} \quad \lambda(\nu) = 1 - \gamma$$

holds\(^9\). The smaller $\gamma$, the degree of confidence, the more uncertain is the agent about the probability distribution. Uncertainty measured by $\gamma$ can be distinguished from the likelihood of a particular strategy combination $s_{-i}$ represented by $\pi(s_{-i})$.

In Eichberger and Kelsey (1996b) we show that the Choquet integral of a simple capacity has the following form:

$$P_i(s_i, \gamma_i \cdot \pi_i) = \gamma_i \cdot \left[ \sum_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}) \cdot \pi_i(s_{-i}) \right] + (1 - \gamma_i) \cdot \min_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}).$$

For simple capacities the Choquet integral is a convex combination between the expected value of $p_i(s_i, s_{-i})$ given the additive measure $\pi_i$ and the worst outcome of the strategy $s_i$ with weight $\gamma_i \in [0, 1]$. The full weight of the non-additivity falls on the worst outcome. If $\gamma_i$ is close to zero, uncertainty dominates and the value of integral is close to the worst outcome. Thus, a decision-maker who chooses an action to maximise the Choquet integral with respect to a simple capacity will,

- for $\gamma_i = 1$, maximise expected utility with probability distribution $\pi_i$, and
- for $\gamma_i = 0$, choose a maximin action.

With beliefs characterised by a simple capacity and a degree of confidence $\gamma_i \in (0, 1)$, a player shows a decision-making behaviour which lies between expected utility maximisation and the extremely uncertainty-averse maximin behaviour.

### 3. Equilibrium under Uncertainty

Consider a game where each player $i \in I$ has beliefs about the opponents’ behaviour, represented by a capacity $\nu_i$ on $S_{-i}$. The expected payoff from a strategy $s_i$ is determined by the Choquet integral given in Definition 2.4. Players are assumed to choose a best response given their beliefs about their opponents’ behaviour. If an equilibrium is supposed to represent a lasting and stable situation, then the beliefs of players cannot be unrelated to their actually observed behaviour.

In a game-theoretic context, beliefs have traditionally been represented by additive probability distributions. Beliefs of players can therefore be identified with mixed strategies actually played. Nash equilibrium requires that beliefs are rationalized by actual behaviour. This can be formulated as the requirement that any pure strategy that a player believes another player may choose, i.e., any pure strategy in the support of the equilibrium mixed strategy of that player, is a best response for this player given his or her beliefs.

For capacities, a support can be defined either as the smallest set of opponents’ strate-

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\(^9\) A simple capacity has a constant degree of confidence and a constant degree of ambiguity, respectively. A capacity with a constant degree of confidence or ambiguity need however not be simple.
gies with measure one or the smallest set of opponents’ strategies with a complement of measure zero\(^{10}\). We will use the latter notion which can be formally stated as follows:

**Definition 3.1** The support of the capacity \( \nu \) is a set \( E \subseteq S_{-i} \) such that \( \nu (S_{-i} \setminus E) = 0 \) and \( S_{-i} \setminus E \subseteq F \) implies \( \nu (F) > 0 \).

Example 2.1 shows that the same capacity \( \nu_i \) may have more than one support. In this example \( \{s_1\} \) and \( \{s_2\} \) are each a support of the capacity \( \nu_i \). Uniqueness of the support is however a desirable property of a capacity because it gives a clear meaning to the characterisation of the support as "the set of strategies that a player considers possible". This recommends restrictions on capacities that guarantee a unique support.

At this point a caveat is in order. In the context of non-additive beliefs, the notion of a support becomes itself ambiguous, even if beliefs have a unique support. If a player’s beliefs are represented by a capacity and if preferences are modelled by a Choquet integral, then believing that an opponent will not use a particular strategy does not imply that this strategy will have no impact on the player’s behaviour. It is well-known that a set of strategies which a capacity assigns a weight of zero is not necessarily Savage-null. \( Lo (1995) \) calls strategies which influence a decision but are not in the support of a player’s beliefs "infinitely less likely". Appealing to a hierarchy of beliefs may be problematical in the context of games. In games, beliefs are at least partially determined in equilibrium. Hence, one cannot take the set of acts in the sense of \( Savage (1954) \) as exogenous. Even if a player does not seriously consider the possibility of the opponent using a particular strategy, it does not appear unreasonable to assume that this strategy may influence the behaviour of this player, e.g., if the consequences of this strategy were particularly grave or if the player has some doubts about the correctness of the equilibrium prediction.

In equilibrium, it is necessary to relate players’ beliefs to their actual behaviour. When players’ beliefs are represented by additive probabilities, expectations can be rational in the sense that players choose (mixed) strategies which are best responses to the actually played (mixed) strategies of the opponents. In this case, beliefs and actual behaviour need not be distinguished\(^{11}\). This consistency requirement of Nash equilibrium is equivalent to the condition that pure strategies played with positive probability, i.e., those in the support of the equilibrium mixed strategy, form best responses of the respective player. In other words, no player expects another player to choose a pure strategy that is not a best response of this player.

The following definition of an equilibrium under uncertainty is also built on the notion that players expect other players to use only best responses. It generalises the equilibrium concept for two-player games in \( Dow and Werlang (1994) \) to games with an arbitrary finite number of players and pure strategies. It assumes maximising behaviour of all players given their beliefs and consistency of these beliefs in the sense that no player expects other players to choose actions that are not best responses given their beliefs.

Denote by \( R_i (\nu_i) = \arg \max \{ P_i (s_i, \nu_i) \mid s_i \in S_i \} \) the best response correspondence of

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\(^{10}\) Compare \( Dow and Werlang (1991) \) and \( Haller (1997) \) for a discussion and analysis of these two notions of a support. \( Ryan (1997) \) contains an extensive discussion of various support concepts for capacities.

\(^{11}\) An exception is \( Crawford (1990) \) who considers beliefs as additive probability distributions over mixed strategy spaces.
player $i$ given beliefs $\nu_i$.

**Definition 3.2** A belief system $(\nu_1^*, \ldots, \nu_T^*)$ is an equilibrium under uncertainty if for all $i \in I$ there exists $\text{supp } \nu_i^*$ such that

$$\text{supp } \nu_i^* \subseteq \bigtimes_{j \in I \setminus \{i\}} R_j(\nu_j^*).$$

Though similar in spirit to the Nash equilibrium concept, the consistency requirement here is much weaker. In particular, since capacities are defined on $S_\omega$, players may believe that their opponents do not choose their strategies independently. Furthermore, there is no requirement that two players have to hold consistent beliefs in regard to the other players’ behaviour. The following examples will highlight these differences to the standard Nash equilibrium concept. In section 4, we will investigate the relationship between an equilibrium under uncertainty and other equilibrium concepts.

Equilibria under uncertainty are equilibria in beliefs. In general, they will not specify exactly which strategy will actually be chosen. Equilibrium beliefs determine precisely which strategy will be actually played if and only if the support of the belief for each player consists of a single strategy. Otherwise, any strategy combination contained in the Cartesian product of the supports of the two players may be played in equilibrium. The following example illustrates this point.

**Example 3.1** Consider the well-known matching pennies game.

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H$</td>
<td>$T$</td>
</tr>
<tr>
<td>$H$</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>$T$</td>
<td>0, 1</td>
<td>1, 0</td>
</tr>
</tbody>
</table>

The only equilibrium $(\nu_1^*, \nu_2^*)$ is $\nu_1^*(H) = \nu_2^*(T) = \alpha_1$ and $\nu_2^*(H) = \nu_1^*(T) = \alpha_2$ for some positive $\alpha_1$ and $\alpha_2$. Obviously, $\text{supp } \nu_i^* = \{H, T\}$ in this case. The expected payoffs of player 1 are $P_1(H, \nu_1^*) = 1 \cdot \nu_1^*(H) + 0 \cdot (1 - \nu_1^*(H)) = \alpha_1$ and $P_1(T, \nu_1^*) = 1 \cdot \nu_1^*(T) + 0 \cdot (1 - \nu_1^*(T)) = \alpha_1$, respectively. Hence, $R_1(\nu_1^*) = \{H, T\}$. Similarly, one obtains $P_2(H, \nu_2^*) = P_2(T, \nu_2^*) = \alpha_2$ and $R_2(\nu_2^*) = \{H, T\}$. The equilibrium concept does not specify which strategy each player will choose in this equilibrium. Since players are indifferent between their strategies, they may actually play either of them.

A special feature of two-player games is the obvious fact that an equilibrium under uncertainty is a Nash equilibrium if beliefs are additive.

If there are more than two players in the game, this is no longer true. The following example illustrates that beliefs of different players need not be consistent.

**Example 3.2** Consider the three-player game given in the matrix below.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$l$</td>
<td>$r$</td>
<td>$l$</td>
<td>$r$</td>
</tr>
<tr>
<td>$T$</td>
<td>1, 0, 1</td>
<td>0, 1, 0</td>
<td>0, 1, 1</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>$B$</td>
<td>0, 0, 0</td>
<td>0, 0, 0</td>
<td>1, 0, 0</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td>$L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The following beliefs \((\nu_1^*, \nu_2^*, \nu_3^*)\) are an equilibrium under uncertainty:

\[
\begin{align*}
\nu_1^*(E) &= \begin{cases} 
\alpha_1 & \text{if } (l, L) \in E, \quad \alpha_1 > 0, \\
0 & \text{otherwise}
\end{cases}, \\
\nu_2^*(E) &= \begin{cases} 
\alpha_2 & \text{if } (T, R) \in E, \quad \alpha_2 > 0, \\
0 & \text{otherwise}
\end{cases}, \\
\nu_3^*(E) &= \begin{cases} 
\alpha_3 & \text{if } (T, l) \in E, \quad \alpha_3 > 0, \\
0 & \text{otherwise}
\end{cases}.
\end{align*}
\]

Clearly, \(\text{supp } \nu_1^* = \{l, L\}, \text{supp } \nu_2^* = \{T, R\}, \text{and supp } \nu_3^* = \{T, l\}\). For player 1, one computes the following expected payoffs:

\[
\begin{align*}
P_1(T, \nu_1^*) &= 1 \cdot \nu_1^* \left( \{(l, L)\} \right) + 0 \cdot (1 - \nu_1^* \left( \{(l, L)\} \right)) = \alpha_1, \\
P_1(B, \nu_1^*) &= 1 \cdot \nu_1^* \left( \{(l, R)\} \right) + 0 \cdot (1 - \nu_1^* \left( \{(l, R)\} \right)) = 0.
\end{align*}
\]

Hence, \(R_1(\nu_1^*) = \{T\}\). Similarly, one easily confirms that \(R_2(\nu_2^*) = \{l\}\) and \(R_3(\nu_3^*) = \{L, R\}\) are best responses for player 2 and 3, respectively.

In this game player 1 believes that player 3 will choose the left matrix, \(L\), while player 2 believes that player 3 will choose \(R\). Both \(L\) and \(R\) are best responses of player 3.

The beliefs of player 1 and 2 in regard to player 3’s behaviour are obviously not mutually consistent. Yet, no player expects another player to behave non-optimally. A slight modification of the previous example illustrates that players may assume that their opponents correlate their actions.

**Example 3.3** Consider the following three-player game.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th>Player 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T)</td>
<td>(l)</td>
<td>0, 0, 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0, 0, 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(B)</td>
<td>0, 0, 0</td>
<td></td>
</tr>
<tr>
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</table>

The beliefs \((\nu_1^*, \nu_2^*, \nu_3^*)\),

\[
\begin{align*}
\nu_1^*(E) &= \begin{cases} 
\alpha_1 & \text{if } (l, R) \in E, \quad \alpha_1 > 0, \\
0 & \text{otherwise}
\end{cases}, \\
\nu_2^*(E) &= \begin{cases} 
\alpha_2 & \text{if } (T, R) \in E, \quad \alpha_2 > 0, \\
0 & \text{otherwise}
\end{cases}, \\
\nu_3^*(E) &= \begin{cases} 
\alpha_3 & \text{if } (T, l) \in E \text{ and } (B, r) \in E, \quad \alpha_3 = \beta_3 + \gamma_3, \\
\beta_3 & \text{if } (T, l) \in E \text{ and } (B, r) \notin E, \quad \beta_3 > 0, \gamma_3 > 0, \\
\gamma_3 & \text{if } (T, l) \notin E \text{ and } (B, r) \in E, \\
0 & \text{otherwise}
\end{cases}.
\end{align*}
\]

form an equilibrium under uncertainty. Clearly, \(\text{supp } \nu_1^* = \{(l, R)\}, \text{supp } \nu_2^* = \{(T, R)\}, \text{and supp } \nu_3^* = \{(T, l), (B, r)\}\). For player 1, one computes the following expected payoffs:

\[
\begin{align*}
P_1(T, \nu_1^*) &= P_1(B, \nu_1^*) = 1 \cdot \nu_1^* \left( \{(l, R)\} \right) + 0 \cdot (1 - \nu_1^* \left( \{(l, R)\} \right)) = \alpha_1.
\end{align*}
\]

Hence, \(R_1(\nu_1^*) = \{(T, B)\}\). Similarly, one easily confirms that \(R_2(\nu_2^*) = \{(l, r)\}\) and \(R_3(\nu_3^*) = \{R\}\) are best responses for player 2 and 3, respectively. In this game player 3
believes that player 1 and player 2 will coordinate their actions and choose either \((T, l)\) or \((B, r)\).

### 3.1 2x2 Matrix Games

Many properties of equilibrium concepts can be exemplified in games with two players where each player has just two strategies. This class of games has been studied extensively. 2x2 matrix games can be classified in terms of their best response structure\(^{12}\). Generically there are only three constellations:

1. at least one player has a dominant strategy and there is a unique Nash equilibrium,
2. no player has a dominant strategy and there is a unique Nash equilibrium,
3. no player has a dominant strategy and there are three Nash equilibria.

These three cases will be considered in turn.

Consider two players with pure strategy sets \(S_1 = \{s_1, s_2\}\) and \(S_2 = \{t_1, t_2\}\). The notation for the payoffs is given in the matrix below.

\[
\begin{array}{c|cc}
\text{Player 2} & t_1 & t_2 \\
\hline
s_1 & a_{11}, b_{11} & a_{12}, b_{12} \\
\hline
s_2 & a_{21}, b_{21} & a_{22}, b_{22}
\end{array}
\]

If each player has just two strategies, then the capacity representing beliefs can be described by two numbers:

\(\nu_1 : (q_{t1}, q_{t2})\), \(q_{t1} \geq 0, q_{t2} \geq 0, q_{t1} + q_{t2} \leq 1\),

\(\nu_2 : (q_{s1}, q_{s2})\), \(q_{s1} \geq 0, q_{s2} \geq 0, q_{s1} + q_{s2} \leq 1\).

Notice that these probabilities need not sum to one. The support of these capacities is simply

\[
\text{supp } \nu_1 = \left\{ \begin{array}{l}
\{t_1\}, \{t_2\} \quad \text{for } q_{t1} = 0, q_{t2} = 0 \\
\{t_1\} \quad \text{for } q_{t1} > 0, q_{t2} = 0 \\
\{t_2\} \quad \text{for } q_{t1} = 0, q_{t2} > 0 \\
\{t_1, t_2\} \quad \text{for } q_{t1} > 0, q_{t2} > 0
\end{array} \right.
\]

and

\[
\text{supp } \nu_2 = \left\{ \begin{array}{l}
\{s_1\}, \{s_2\} \quad \text{for } q_{s1} = 0, q_{s2} = 0 \\
\{s_1\} \quad \text{for } q_{s1} > 0, q_{s2} = 0 \\
\{s_2\} \quad \text{for } q_{s1} = 0, q_{s2} > 0 \\
\{s_1, s_2\} \quad \text{for } q_{s1} > 0, q_{s2} > 0
\end{array} \right.
\]

Given beliefs \(\nu_1\) and \(\nu_2\), it is straightforward to compute the following Choquet integrals:

\[
P_1 \left(s_i, \nu_1 \right) = \left\{ \begin{array}{ll}
a_{i1} \cdot q_{t1} + a_{i2} \cdot (1 - q_{t1}) & \text{for } a_{i1} > a_{i2} \\
a_{i1} & \text{for } a_{i1} = a_{i2} \\
a_{i2} \cdot q_{t2} + a_{i1} \cdot (1 - q_{t2}) & \text{for } a_{i1} < a_{i2}
\end{array} \right.
\]

\(^{12}\) Eichberger, Haller, Milne (1993) provide a complete classification of 2x2 matrix games.

11
nation games and games of the type of the battle of the sexes. The following example
strategy and there exist three Nash equilibria. This group of games comprises coordi-
Case 3
thor is however no requirement that players have to play each strategy with a certain prob-
Nash equilibrium behaviour, since players are indifferent between their strategies. There
best responses. Behaviour in an equilibrium under uncertainty is in this case similar to
consist of the full strategy set. Otherwise, players would have unique but incompatible
Case 2
and the other not. Whether an equilibrium under uncertainty corresponding to the unique
rationality, and form an equilibrium under uncertainty.
Example 3.4 Consider the following game.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_1$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>1, 2</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Clearly, $P_1(s_1, \nu_1) = 1 > 0 = P_1(s_2, \nu_1)$ and therefore $R_1(\nu_1) = \{s_1\}$ for all $\nu_1$. For
player 2, however, one computes $P_2(t_1, \nu_2) = 2 \cdot q_{s_1}$ and $P_2(t_2, \nu_2) = 1$. Hence, for
$q_{s_1} < \frac{1}{2}$, $R_2(\nu_2) = \{t_2\}$. The beliefs $(\nu_1^*, \nu_2^*)$ determined by $(q_{s_1}, q_{s_2}) = (\alpha, 0)$, $\alpha < \frac{1}{2}$,
and $(q_{t_1}, q_{t_2}) = (0, \beta)$, $\beta > 0$ have supports $\text{supp} \nu_1^* = \{t_2\}$ and $\text{supp} \nu_2^* = \{s_1\}$ respectively, and form an equilibrium under uncertainty.
Behaviour in this equilibrium under uncertainty does not correspond to a Nash equilib-
rium, but makes sense if player 2 is not completely confident that player 1 will rationally
play the dominant strategy. After all, by choosing $t_2$ player 2 can guarantee herself a
riskless payoff of 1.

Case 2. If no player has a strictly dominant strategy and there exists a unique equilib-
rium, then the game is similar to matching pennies discussed in Example 3.1. In this
case, both players must be indifferent between their two strategies, i.e., the support has to
consist of the full strategy set. Otherwise, players would have unique but incompatible
best responses. Behaviour in an equilibrium under uncertainty is in this case similar to
Nash equilibrium behaviour, since players are indifferent between their strategies. There
is however no requirement that players have to play each strategy with a certain probabil-
Case 3. The most interesting case of games arises when no player has a strictly dominant
strategy and there exist three Nash equilibria. This group of games comprises coordi-
ation games and games of the type of the battle of the sexes. The following example
illustrates the range of equilibrium behaviour.

\[
P_2(t_i, \nu_2) = \begin{cases} 
   b_{i1} \cdot q_{s_1} + b_{2i} \cdot (1 - q_{s_1}) & \text{for } b_{i1} > b_{2i} \\
   b_{i1} & \text{for } b_{i1} = b_{2i} \\
   b_{2i} \cdot q_{s_2} + b_{i2} \cdot (1 - q_{s_2}) & \text{for } b_{i1} < b_{2i} 
\end{cases}
\]

Case 1. If a player has a strictly dominant strategy, e.g., \(a_1 > a_{21}\) and \(a_{12} > a_{22}\),
then the best-response correspondence of this player will consist of this strategy only,
\(R_1(\nu_1) = \{s_1\}\) for all \(\nu_1\). If both players have a strictly dominant strategy, then equilib-
rium beliefs must give positive weight to these strategies and zero weight to their other
strategy. Thus, an equilibrium under uncertainty corresponds to the unique Nash equilib-
rium in this case.

More interesting is the case where one player, say player 1, has a strictly dominant strategy
and the other not. Whether an equilibrium under uncertainty corresponding to the unique
Nash equilibrium occurs in this case will depend on the belief of player 2 about whether
player 1 will use the dominant strategy, i.e. whether player 1 behaves rationally, as the
following example shows.
Example 3.5  Consider the following version of the battle of the sexes.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>s1</th>
<th>s2</th>
</tr>
</thead>
<tbody>
<tr>
<td>t1</td>
<td>1,2</td>
<td>1,1</td>
</tr>
<tr>
<td>t2</td>
<td>0,0</td>
<td>2,1</td>
</tr>
</tbody>
</table>

The Nash equilibria in pure strategies are \((s_1, t_1)\) and \((s_2, t_2)\). The mixed strategy Nash equilibrium is \(((\frac{1}{2}s_1, \frac{1}{2}s_2), (\frac{1}{2}t_1, \frac{1}{2}t_2))\). One computes easily that there are exactly four types of equilibrium under uncertainty:

(A) \((\nu^+_1, \nu^+_2)\) : \[
q_{s1} > \frac{1}{2}, \quad q_{s2} = 0, \quad \text{supp } \nu^+_1 = \{t_1\}, \quad \text{supp } \nu^+_2 = \{s_1\}, \quad R_1(\nu^+_1) = \{s_1\}, \quad R_2(\nu^+_2) = \{t_1\}
\]

(B) \((\nu^+_1, \nu^+_2)\) : \[
q_{s1} = 0, \quad q_{s2} > 0, \quad \text{supp } \nu^+_1 = \{t_1\}, \quad \text{supp } \nu^+_2 = \{s_1\}, \quad R_1(\nu^+_1) = \{s_2\}, \quad R_2(\nu^+_2) = \{t_2\}
\]

(C) \((\nu^+_1, \nu^+_2)\) : \[
q_{s1} = 0, \quad q_{s2} > 0, \quad \text{supp } \nu^+_1 = \{t_1, t_2\}, \quad \text{supp } \nu^+_2 = \{s_1, s_2\}, \quad R_1(\nu^+_1) = \{s_1, s_2\}, \quad R_2(\nu^+_2) = \{t_1, t_2\}
\]

(D) \((\nu^+_1, \nu^+_2)\) : \[
q_{s1} \in [0, \frac{1}{2}], \quad q_{s2} = 0, \quad \text{supp } \nu^+_1 = \{t_2\}, \quad \text{supp } \nu^+_2 = \{s_1\}, \quad R_1(\nu^+_1) = \{s_1\}, \quad R_2(\nu^+_2) = \{t_2\}
\]

In the equilibria under uncertainty of type A and B players behave as in the pure-strategy Nash equilibria. Behaviour in an equilibrium of type C is similar to mixed-strategy Nash equilibrium behaviour. Players are indifferent between their strategies and any strategy combination may be played.

More interesting is behaviour in an equilibrium under uncertainty of type D which does not correspond to a Nash equilibrium. Here players are uncertain as to which equilibrium will be played. Strategies \(s_1\) and \(t_2\) yield players a certain payoff of one, insuring them against strategic risk.

The analysis of 2x2 matrix games makes it clear that behaviour which is incompatible with the consistency requirements of a Nash equilibrium may be accommodated in an equilibrium under uncertainty. Moreover, as these examples show, such behaviour may be quite sensible if players are concerned about strategic uncertainty.

3.2 Related Equilibrium Concepts

The concept of an equilibrium under uncertainty introduced in this section generalises the equilibrium concept proposed by Dow and Werlang (1991,1994) for two-player games if beliefs may be non-additive. Also in the two-player context, Haller (1997) and Marinacci (1996) propose a modification of the Dow-Werlang concept which is mainly distinguished by a different notion of support for a capacity. They suggest as support of a
capacity the set of all strategies that have positive measure as a single element set, i.e. 
\[
\text{supp } \nu_i = \{s_{-i} \in S_{-i} | \nu_i(s_{-i}) > 0\}. 
\]
This concept of support has the disadvantage of an empty support whenever all single strategy combinations have measure zero as, e.g., in Example 2.1. This is a case where the support concept of Definition 3.1 leads to a non-unique support. In game-theoretic applications, one often puts restrictions on the capacities which make these two concepts equivalent. For simple capacities, for instance, the two notions of support are the same. In many game-theoretic applications, this difference in the definition of a support is therefore of no consequence.

All of these papers restrict analysis to two-player games. To our knowledge, this is the first paper that extends the Dow-Werlang approach to games with more than two players. Of course it is only in this context that the problems of correlated beliefs and possibly inconsistent beliefs may arise (compare Examples 3.2 and 3.3). A related, but different approach which models players with non-additive beliefs uses the multiple-prior concept axiomatised by Gilboa and Schmeidler (1989)\(^{13}\). Klibanoff (1996) and Lo (1996) provide equilibrium concepts for n-player games in which players’ beliefs are represented by multiple probability distributions.

Klibanoff (1996) and Lo (1996) show in the context of a multiple prior model where players evaluate expected payoffs from compound probabilities, i.e., probability distributions over mixed strategies, that their model induces a probability distribution over payoffs conditional on any ambiguous event. In both papers, players can choose mixed strategies. The expected value of a mixed strategy is evaluated by forming compound payoffs conditional on any ambiguous event. In both papers, players evaluate expected payoffs from compound probabilities, i.e., probability distributions over mixed strategies, that their model induces a probability distribution over payoffs conditional on any ambiguous event. In both papers, players can choose mixed strategies. The expected value of a mixed strategy is evaluated by forming compound payoffs conditional on any ambiguous event. In both papers, players evaluate expected payoffs from compound probabilities, i.e., probability distributions.

For the case of two player-games, one can illustrate the approach of Klibanoff and Lo as follows. Denote by \(M_i\) the set of mixed strategies of player \(i \in I\). Uncertainty is modelled by a set of mixed strategies of player \(j\), \(B_j \subseteq M_j\), that player \(i\) considers possible. Complete uncertainty is represented by \(B_j = M_j\) and complete certainty about the opponent’s mixed strategy by \(B_j = \{m_j\}\) for some \(m_j \in M_j\). Expected payoffs from a mixed strategy \(m_i\) given beliefs \(B_j\) are defined as

\[
\tilde{P}_i(m_i, B_j) := \min_{m_j \in B_j} \sum_{s_i \in S_i} \sum_{s_j \in S_j} p_i(s_i, s_j) \cdot m_i(s_i) \cdot m_j(s_j).
\]

Notice that the minimum is taken over the average payoff given a mixed strategy of player \(i\). Since the minimum of averages is usually bigger than the average of the minima it is not surprising that mixed strategies are preferred for pure strategies with non-comonotonic payoffs\(^{14}\).

Klibanoff (1996) requires for an equilibrium that the best responses of players are contained in the set of their opponent’s beliefs, i.e. \(\arg\max_{m_i \in M_i} \tilde{P}_i(m_i, B_j) \subseteq B_i\). Lo (1996) requires that, in equilibrium, all beliefs are best responses, i.e., \(\arg\max_{m_i \in M_i} \tilde{P}_i(m_i, B_j) = B_i\).

The following example will illustrate the difference between the equilibrium concepts of

\(^{13}\) It is well-known that the two approaches to the representation of beliefs coincide for convex capacities. Gilboa and Schmeidler (1994) investigates the relationship between these two concepts.

\(^{14}\) This is the point where it becomes crucial for the results of the theory in which way beliefs, represented by a set of additive probabilities, and mixed strategies are combined.
Klibanoff (1996) and Lo (1996) and the approach of this paper.

Example 3.6 The following slightly modified version of a coordination game, discussed in Klibanoff (1996) on page 15, may serve to distinguish the various equilibrium concepts.

\[
\begin{array}{c|cc}
\text{Player 1} & L & R \\
\hline
U & 2, 2 & 0, \frac{1}{2} \\
D & \frac{1}{2}, 0 & 1, 1 \\
\end{array}
\]

This game has three Nash equilibria \((U, L), (D, R),\) and the mixed strategy equilibrium \((\frac{3}{5} U, \frac{2}{5} D), (\frac{2}{5} L, \frac{3}{5} R).\) Since each player has only two strategies, mixed strategies can be identified with a single number \(\alpha \in [0, 1]\) by setting \(m_1(U) = \alpha, m_1(D) = 1 - \alpha,\) and similarly for player 2. The following table shows all equilibria of the Klibanoff concept for this game.

<table>
<thead>
<tr>
<th>beliefs ((i = 1, 2))</th>
<th>mixed strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 \in B_i \subseteq [0, \frac{2}{5}])</td>
<td>((0U, 1D), (0L, 1R))</td>
</tr>
<tr>
<td>(1 \in B_i \subseteq (\frac{2}{5}, 1])</td>
<td>((1U, 0D), (1L, 0R))</td>
</tr>
<tr>
<td>(\frac{1}{2} \in B_i, B_i \cap (\frac{1}{2}, 1) \neq \emptyset)</td>
<td>((\frac{1}{2}U, \frac{1}{2}D), (\frac{1}{2}L, \frac{1}{2}R))</td>
</tr>
</tbody>
</table>

Apart from the Nash equilibria, the mixed strategy combination \((\frac{1}{5} U, \frac{4}{5} D), (\frac{4}{5} L, \frac{1}{5} R))\) is the only other equilibrium behaviour in the Klibanoff concept. Notice that these mixed strategies do not coincide with the Nash equilibrium in mixed strategies. They guarantee the player however an expected payoff of \(\frac{3}{5}\).

In contrast, Lo’s concept has three equilibrium belief systems corresponding to the set of Nash equilibria:

\[
\begin{align*}
B_1 &= B_2 = \{1\}, \\
B_1 &= B_2 = \{0\}, \\
B_1 &= B_2 = \{\frac{1}{2}\}.
\end{align*}
\]

For equilibria under uncertainty, the set of equilibria will depend on the degree of confidence of the players. If the degree of confidence is high, in this example \(\gamma(\nu_1), \gamma(\nu_2) \geq \frac{1}{4},\) then all Nash equilibria are equilibria under uncertainty:

<table>
<thead>
<tr>
<th>equilibrium beliefs for (\gamma(\nu_2) = \gamma \geq \frac{1}{4})</th>
<th>corresponding equilibrium behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu_1 = (\gamma, 0), \nu_2 = (\gamma, 0))</td>
<td>((U, L)),</td>
</tr>
<tr>
<td>(\nu_1 = (0, \gamma), \nu_2 = (0, \gamma))</td>
<td>((D, R)),</td>
</tr>
<tr>
<td>(\nu_1 = \left(\frac{1+\gamma}{2}, \frac{1-\gamma}{2}\right), \nu_2 = \left(\frac{1+\gamma}{2}, \frac{1-\gamma}{2}\right))</td>
<td>(((U, D), (L, R))).</td>
</tr>
</tbody>
</table>

This prediction appears to be reasonable, because a high degree of confidence can be interpreted as a strong belief that the Nash equilibrium prediction about the opponent’s behaviour is correct.

On the other hand, with \(\gamma(\nu_2) < \frac{1}{4},\) only \((D, R)\) will be belief in an equilibrium under uncertainty. To see this, note that \(P_1(U, \nu_1) = 2 \cdot \nu_1(L)\) and \(P_1(D, \nu_1) = 1 \cdot \nu_1(R) + 0.5 \cdot (1 - \nu_1(R)).\) Hence, \(P_1(U, \nu_1) \leq P_1(D, \nu_1)\) if \(4 \cdot \nu_1(L) \leq (1 + \nu_1(R)).\) From \(\nu_1(R) \leq \gamma(\nu_1) < \frac{1}{4},\) we can conclude that \(D\) is the only best response for player 1. By symmetry of the game, \((\nu_1, \nu_2)\) with \(\frac{1}{4} > \nu_1(R) \geq \nu_1(L) = 0\) and \(\frac{1}{4} > \nu_2(D) \geq \nu_2(U) = 0\) is the
only equilibrium in beliefs for such low degrees of confidence. The behaviour in such an equilibrium under uncertainty with a low degree of confidence appears not unreasonable for uncertainty averse players. After all, players can guarantee themselves payoffs between $\frac{1}{2}$ and 1 by playing D or R, respectively. By playing U or L they may gain 2, but they risk also to obtain nothing if the opponent chooses the second strategy for whatever reason. Good reasons could be uncertainty about the equilibrium selection or uncertainty about the opponent's perception of the game. It is worth noting that in this example $(U, L)$ is the Pareto-optimal and the risk-dominant Nash equilibrium. We consider it a strength of the equilibrium-under-uncertainty concept that it parametrises the equilibrium correspondence with the degree of confidence that player have regarding the Nash equilibrium prediction. Thus, with high confidence, Nash equilibrium predictions are appropriate, while with low degrees of confidence Nash equilibrium is not necessarily a good prediction of behaviour.

Example 3.6 is typical. Lo (1996) imposes more stringent consistency conditions on beliefs in equilibrium than Klibanoff (1996). Equilibrium behaviour in Lo’s model is therefore close to Nash equilibrium behaviour. In particular, in two-player games, it coincides always with Nash equilibrium. A main difference in the predicted behaviour between the capacity-based approach advocated in this paper and the multiple-prior approach of Klibanoff is the strict preference for mixed strategies which is illustrated in Example 3.6.

As argued above, in an equilibrium under uncertainty, beliefs can no longer be interpreted as equilibrium mixed strategies because beliefs may be non-additive capacities and a player may believe that her opponents’ strategy choices are correlated. Extending the concept to mixed strategy sets raises the difficult problem of how to combine non-additive beliefs, represented by capacities, with the objective probabilities of different pure strategies that arise from the random device of a mixed strategy. In this sense, Lo and Klibanoff’s approach falls into the Anscombe-Aumann framework. Not surprisingly from this perspective, they find a preference of players for mixed strategies. The results in Eichberger and Kelsey (1996a) suggest that the Anscombe-Aumann approach, though equivalent under additivity, has behavioural implications under uncertainty that are incompatible with the Savage approach. In particular, there is not a general preference for randomisation in a Savage framework.

For these reasons, we do not consider the equilibrium concepts of Klibanoff (1996) and Lo (1996) and the one proposed by Dow and Werlang (1994) and in this paper as equivalent. It is therefore worthwhile to investigate both equilibrium concepts. In our opinion, a final verdict of the appropriateness will depend as much on successful economic applications as on experimental evidence.

3.3 Existence of Equilibrium under Uncertainty

Without further constraints on beliefs existence of an equilibrium under uncertainty is no problem as the following lemmata show.

Lemma 3.1 Every mixed-strategy Nash equilibrium is an equilibrium under uncertainty.
Proposition 3.1  

Particularly of interest are capacities with a constant uncertainty level for each level of uncertainty. 

Proof. Consider a Nash-equilibrium mixed strategy combination \( (m_1^*, ..., m_i^*) \). For all \( s_{-i} \in S_{-i} \), define \( \nu_i^*(\{s_{-i}\}) = \prod_{j \neq i} m_j^*(s_j) \). Thus defined, \( \nu_i^* \) is an additive capacity. These beliefs \( (\nu_1^*, ..., \nu_i^*) \) form an equilibrium under uncertainty. 

Since, in finite games, a Nash equilibrium in mixed strategies always exists, existence of an equilibrium under uncertainty is also guaranteed. This is the case of no ambiguity \( \lambda(\nu_i^*) = 0 \) and full confidence \( \gamma(\nu_i^*) = 1 \) for all players \( i \in I \) which is analysed extensively in traditional game theory. 

Lemma 3.2  

Every pure-strategy maximin strategy combination is an equilibrium under uncertainty. 

Proof. The pure strategy combination \( (\hat{s}_1, ..., \hat{s}_I) \) is a maximin strategy combination if for all \( i \in I \)

\[
\hat{s}_i \in \arg \max_{s_{-i} \in S_{-i}} \min_{s_i \in S_i} p_i(s_i, s_{-i})
\]

holds. Define \( \hat{\nu}_i(E) = 0 \), for all \( E \subset S_{-i} \). In this case, the Choquet integral is

\[
P_i(s_i, \hat{\nu}_i) = \min\{p_i(s_i, s_{-i})|s_{-i} \in S_{-i}\}.
\]

Furthermore, every strategy combination \( s_{-i} \) forms a support of \( \hat{\nu}_i \), i.e., for all \( s_{-i} \in S_{-i} \), \( \text{supp}(\hat{\nu}_i) = \{s_{-i}\} \).

Hence,

\[
\text{supp}(\hat{\nu}_i) = \{s_{-i}\} \subseteq \times_{j \neq i} \arg \max_{s_j \in S_j} P_j(s_j, \hat{\nu}_j) = \times_{j \neq i} R_j(\hat{\nu}_j).
\]

\((\hat{\nu}_1, ..., \hat{\nu}_I)\) is therefore an equilibrium under uncertainty. 

Pure-strategy maximin strategy combinations exist in all finite games. This lemma therefore guarantees existence of an equilibrium under uncertainty for the case of complete ambiguity \( \lambda(\nu_i^*) = 1 \) and no confidence \( \gamma(\nu_i^*) = 0 \) for all players. This case has also been studied in traditional game theory, in particular in the context of zero-sum games.

More interesting are the intermediate cases of some ambiguity, \( 0 < \lambda(\nu_i^*) < 1 \), and incomplete confidence, \( 0 < \gamma(\nu_i^*) < 1 \), for at least some players. The more important question concerns therefore the existence of an equilibrium under uncertainty in these cases.

Particularly of interest are capacities with a constant uncertainty level \( \nu_i(E) + \nu_i(S_{-i} \setminus E) = \gamma_i \). Restricting capacities to such a class raises the question of whether there is an equilibrium under uncertainty for each level of \( \gamma_i \). If this is the case, one can derive comparative static results by varying these parameters and comparing the equilibrium properties.

Proposition 3.1  

For any vector of confidence parameters \( \gamma := (\gamma_1, ..., \gamma_I) \) there exists a equilibrium under uncertainty. 

Proof. Simple capacities \( \nu_i(E) = \gamma_i \cdot \pi_i(E) \) satisfy the condition \( \nu_i(E) + \nu_i(S_{-i} \setminus E) = \gamma_i \) for all \( E \subset S_{-i} \) because \( \pi_i \) is an additive measure. The proof will proceed in two steps. First, it will be shown that simple capacities have a unique support. Then it will
be demonstrated that an equilibrium in simple capacities exists for any given vector of confidence parameters $\gamma$.

Lemma 3.3 For a simple capacity with $\gamma \neq 0$ the support is unique and consists of all states with positive probability.

Proof. Let $\nu_i$ be a simple capacity on $S_{-i}$. Since $\nu_i$ is simple there exists an additive probability $\pi_i$ on $S_{-i}$ and $\gamma_i \in [0, 1]$ such that for all $E \subset S_{-i}, \nu_i(E) = \gamma_i \cdot \pi_i(E)$. Let $B \subset S_{-i}$ denote the set of all states with positive measure $\pi_i$. By definition $\pi_i(S_{-i} \setminus B) = 0$, hence $\nu_i(S_{-i} \setminus B) = 0$. Let $C$ be an event such that $S_{-i} \setminus B \subset C$. Then there exists $s_{-i} \in C \setminus (S_{-i} \setminus B) \subset B$. By monotonicity $\nu_i(C) \geq \nu_i(\{s_{-i}\}) = \gamma_i \cdot \pi_i(\{s_{-i}\}) > 0$.

This demonstrates that $B$ is the support of $\nu_i$.

To show uniqueness, suppose there is another support $B' \neq B$. Then there must be $s_{-i} \in B'$ with $\pi_i(s_{-i}) = 0$. Consider the set $C = (S_{-i} \setminus B') \cup \{s_{-i}\}$. Clearly, $S_{-i} \setminus B' \subset C$, but $\nu_i(C) = \gamma_i \cdot \pi_i(C) = 0$. This contradicts the assumption that $B'$ is a support.

Denote by $A(\gamma) := \{i \in I | \gamma_i > 0\}$ the set of players with $\gamma_i > 0$. For $A(\gamma) = \emptyset$, Lemma 3.2 shows existence of an equilibrium under uncertainty.

Let $m_i(s_i) := \min \{p_i(s_i, s_{-i}) | s_{-i} \in S_{-i}\}$ be the minimum payoff a player obtains from playing strategy $s_i$. Define new payoff functions

$$\Psi_i(s_i, s_{-i}) := \begin{cases} p_i(s_i, s_{-i}) + [(1 - \gamma_i) / \gamma_i] \cdot m_i(s_i) & \text{for } i \in A(\gamma) \\ m_i(s_i) & \text{otherwise} \end{cases}$$

The new game $\Gamma' = (I, (S_i)_{i \in I}, (\Psi_i)_{i \in I})$ is well-defined and has a Nash equilibrium $\pi^* = (\pi_1^*, \ldots, \pi_I^*)$ in mixed strategies. Thus, for all $i \in I$ and all $s_i \in S_i$ with $\pi_i^*(s_i) > 0$,

$$\sum_{s_{-i} \in S_{-i}} \Psi_i(s_i, s_{-i}) \cdot \Pi_i^*(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \Psi_i(s_i, s_{-i}) \cdot \Pi_i^*(s_{-i})$$

for all $s_{-i} \in S_{-i}$ must hold, where

$$\Pi_i^*(s_{-i}) := \pi_1^*(s_1) \cdot \pi_2^*(s_2) \cdot \ldots \cdot \pi_i^*(s_i) \cdot \pi_{i+1}^*(s_{i+1}) \cdot \ldots \cdot \pi_I^*(s_I)$$

denotes the probability that $s_{-i}$ is played.

Let $\nu_i^*(s_{-i}) := \gamma_i \cdot \Pi_i^*(s_{-i})$ for all $s_{-i} \in S_{-i}$ denote the belief of player $i \in I$. It will be shown that $\nu^* = (\nu_1^*, \ldots, \nu_I^*)$ is an equilibrium under uncertainty for the original game with a vector of confidence parameters $\gamma$.

For player $i \in A(\gamma)$, consider $s_{-i}' \in \text{supp } \nu_i^*$. By Lemma 3.3,

$$\nu_i^*(s_i') = \gamma_i \cdot \Pi_i^*(s_i') = \gamma_i \cdot [\pi_1^*(s_1') \cdot \pi_2^*(s_2') \cdot \ldots \cdot \pi_i^*(s_i') \cdot \pi_{i+1}^*(s_{i+1}') \cdot \ldots \cdot \pi_I^*(s_I')] > 0.$$ 

Hence, for $j \neq i$, $\pi_j^*(s_j') > 0$ and

$$s_j' \in \text{arg max}_j \{ \sum_{s_{-j} \in S_{-j}} \Psi_j(s_j, s_{-j}) \cdot \Pi_j^*(s_{-j}) | s_j \in S_j \}$$

by the definition of a Nash equilibrium.
Since the capacity $\nu_\gamma$ is simple, it follows from (*) that
\[
P_j (s'_j, \nu_\gamma^j) = \gamma_j \sum_{s_{-j} \in S_{-j}} p_j (s'_j, s_{-j}) \cdot \Pi_j^\ast (s_{-j}) + (1 - \gamma_j) \cdot m_j (s'_j)
\]
\[
= \gamma_j \sum_{s_{-j} \in S_{-j}} \left[ \psi_j (s'_j, s_{-j}) \cdot \Pi_j^\ast (s_{-j}) \right]
\]
\[
= \gamma_j \sum_{s_{-j} \in S_{-j}} \psi_j (s_j, s_{-j}) \cdot \Pi_j^\ast (s_{-j})
\]
\[
\geq \gamma_j \sum_{s_{-j} \in S_{-j}} \Psi_j (s_j, s_{-j}) \cdot \Pi_j^\ast (s_{-j})
\]
\[
= P_j (s_j, \nu_\gamma^j) \quad \text{for all } s_j \in S_j.
\]
If $i \notin A(\gamma)$, then any strategy combination of the opponents can form the support of the capacity $\nu_\gamma^i$. Thus, one can choose any pure strategy combination of best responses $s_{-i} \in \text{supp} \Pi_j^\ast$. This proves that $(\nu_\gamma^1, ..., \nu_\gamma^i)$ is an equilibrium under uncertainty.

4. Nash Equilibrium and Maximin Strategies

This section relates equilibria under uncertainty to Nash equilibria on the one hand and maximin strategies on the other. It is a well-known fact that, in zero-sum games, every Nash equilibrium strategy combination is a maximin strategy combination. This equivalence of prudent and equilibrium behaviour does however not carry over to general games. The concept of an equilibrium under uncertainty makes it possible to relate these two traditional solution concepts of a game (compare, e.g., Moulin (1986)). Indeed, the less confidence players have in their beliefs about opponents’ behaviour the more likely the pure strategies chosen in an equilibrium under uncertainty will be maximin strategies. On the other hand, at least in two-player games, if players are confident about their beliefs, then they will play as in a Nash equilibrium. For games, with more than two players, however, this simple relationship does not necessarily hold. This section will investigate to which degree equilibrium behaviour under uncertainty resembles the behaviour predicted by the maximin and Nash equilibrium concept.

First, low degrees of confidence, i.e. $\gamma(\nu_i)$ close to zero, are considered. Recall that $m_i (s_i) := \min \{ p_i (s_i, s_{-i}) \mid s_{-i} \in S_{-i} \}$ denotes the worst payoff player $i$ may obtain from playing strategy $s_i$. An equilibrium under uncertainty $\nu^\ast = (\nu^\ast_1, ..., \nu^\ast_i)$ induces maximin play if, for all $i \in I$,
\[
R_i (\nu_i^\ast) \subseteq \arg \max \{ m_i (s_i) \mid s_i \in S_i \}.
\]
The following result shows that, for low degrees of confidence, equilibria under uncertainty are maximin strategy combinations.

**Proposition 4.1** For every game, there exists $\varepsilon > 0$ such that every equilibrium under uncertainty $(\nu_\gamma^1, ..., \nu_\gamma^i)$ with $\gamma(\nu_i^\ast) \in [0, \varepsilon]$ for all $i \in I$ induces maximin play.

**Proof.** Let $M_i := \arg \max \{ m_i (s_i) \mid s_i \in S_i \}$ be the set of player $i$’s maximin strategies. Given an equilibrium under uncertainty $(\nu_\gamma^1, ..., \nu_\gamma^i)$, consider any strategy $s_i \in R_i (\nu_i^\ast)$ such that $s_i \notin M_i$. If no such strategy exists then the proposition is true.
Suppose there exists \( s_i \) such that \( m_i(s_i') > m_i(s_i) \) and \( P_i(s_i, \nu_i^\ast) \geq P_i(s_i', \nu_i^\ast) \) for all \( s_i' \in M_i \).

Let \( \bar{\pi}_i := \max \{ p_i(s_i, s_{-i}) | s_i \in S_i, s_{-i} \in S_{-i} \} \) be the highest possible payoff of player \( i \). Then,
\[
\varepsilon \cdot \bar{\pi}_i + m_i(s) \cdot [1 - \varepsilon] \geq \gamma(\nu_i^\ast) \cdot \bar{\pi}_i + m_i(s) \cdot [1 - \gamma(\nu_i^\ast)] \geq P_i(s_i, \nu_i^\ast)
\]
and
\[
P_i(s_i', \nu_i^\ast) \geq m_i(s_i').
\]

\( m_i(s_i') > m_i(s_i) \) implies that there is a positive
\[
\varepsilon_i := \frac{m_i(s_i') - m_i(s_i)}{\bar{\pi}_i - m_i(s_i)} > 0
\]
such that \( P_i(s_i', \nu_i^\ast) > P_i(s_i, \nu_i^\ast) \) for all beliefs \( \nu_i^\ast \) with \( \gamma_i(\nu_i^\ast) \leq \varepsilon_i \). For \( \varepsilon < \varepsilon_i \), \( s_i \in R_i(\nu_i^\ast) \) implies \( s_i \in M_i \). Thus, \( \varepsilon = \min \{ \varepsilon_i | i \in I \} \) provides an upper bound on the degree of confidence such that \( R_i(\nu_i^\ast) \subseteq M_i \) for all \( i \in I \).

Proposition 4.1 follows from the continuity of the Choquet integral \( P_i(\cdot, \nu_i) \) in beliefs \( \nu_i \) and the fact that \( \text{supp} \nu_i = R_i(\nu_i) = M_i \) for \( \gamma(\nu_i) = 0 \) holds. The best response correspondence \( R_i(\nu_i) \) is upper semi-continuous which implies our result \( \text{supp} \nu_i = R_i(\nu_i) = M_i \) for capacities with a degree of confidence in the neighbourhood of zero.

At the other extreme, if the degree of ambiguity \( \lambda(\nu_i) \) converges to zero, one cannot draw a similar conclusion since \( \text{supp} \nu_i = R_i(\nu_i) \) cannot be guaranteed for \( \lambda(\nu_i) = 0 \). One can however prove the weaker result that, under certain conditions, a sequence of equilibria under uncertainty converges to a Nash equilibrium.

Example 3.4 suggests that, for low degrees of ambiguity, equilibrium play will resemble Nash equilibrium play. Indeed, for two-player games, Dow and Werlang (1994) prove this result. There are, however, complications if more than two players are considered. For games with three or more players,

(i) a player may believe that the opponents’ behaviour is correlated, and
(ii) two players’ beliefs about a third player’s behaviour may not coincide.

**Definition 4.1** Beliefs about the opponents’ behaviour \( \nu_i \) are independent if there exist beliefs \( \nu_i^\ast \) on \( S_j \) for all \( j \neq i \) such that for all \( E = \times_{j \neq i} E_j \),
\[
\nu_i(E) = \prod_{j \neq i} \nu_i^\ast(E_j).
\]

Definition 4.1 requires the capacity \( \nu_i \) to be a product of the capacities \( \nu_i^\ast \) only for events that are Cartesian products. For events, that cannot be decomposed in a Cartesian product there is no constraint implied beyond monotonicity and, eventually, convexity. There are several proposals in the literature for how to construct a product capacity\(^{15}\). Defini-
Definition 4.2  Independent beliefs of two players $\nu_i, \nu_j$ are consistent if for all $k \neq i, j$
\[ \nu^k_i = \nu^k_j. \]

If beliefs of all players are independent and consistent, then there exist beliefs $\nu^k$ on $S_k$ for all players $k \in I$ such that $\nu_i(E) = \prod_{j \neq i} \nu^j(E_j)$ for all $E = \times_{j \neq i} E_j$ holds.

Proposition 4.2  If the beliefs of all players are independent and consistent, then every sequence of equilibria under uncertainty $(\nu^0_1, \ldots, \nu^0_I)$ with $\lambda(\nu^0_i) \to 0$ for all $i \in I$ converges to a Nash equilibrium.

Proof. It is first proved that for $\lambda(\nu^0_i) \to 0$ consistent and independent beliefs must be additive product measures.

Lemma 4.1  Consider a sequence of independent beliefs $\nu^n_i \to \nu^0_i$ as $n \to \infty$ such that $\lim_{n \to \infty} \lambda(\nu^n_i) = 0$. Then, for all $j \neq i, \nu^m_i \to \nu^0_i$ such that, for all $E \subseteq S_i, \nu^0_i(E) = \sum_{s_i \in E} \nu^0_i(\{s_i\})$.

Proof. For any $E_j \subseteq S_j$, consider the Cartesian product $E = E_j \times (\times_{k \neq i} S_k)$. Clearly, $S_i \subseteq E = E_j \times (\times_{k \neq i} S_k)$. Since $\nu_i$ is independent, $\nu_i(E) + \nu_i(S_i \setminus E) = \nu_i^0(E_j) + \nu_i^0(E_j \setminus E)$ follows. Hence, $\lim_{n \to \infty} \lambda(\nu^n_i) = \lambda(\nu^0_i) = 0$ implies $\lambda(\nu^m_i) \to \lambda(\nu^0_i) = 0$ for all $j \neq i$. The capacity $\nu^0_i$ must therefore be also additive. Hence, for any $E_j \subseteq S_j, \nu^0_i(E_j) = \sum_{s_i \in E_j} \nu^0_i(\{s_i\})$.  

Consider now a sequence of equilibria under uncertainty $(\nu^n_1, \ldots, \nu^n_I)$ with $\lambda(\nu^n_i) \to 0$ for all $i \in I$. Since beliefs are independent and consistent, there is a sequence of beliefs $\nu^m_i$ such that $\nu_i^n(\times_{j \neq i} E_j) = \prod_{j \neq i} \nu^m_j(E_j), j = 1, \ldots, I$. Furthermore, $\text{supp} \nu^n_i \subseteq \times_{j \neq i} R_j(\nu^n_j)$ for all $n$ and all $i \in I$ implies $\text{supp} \nu^0_i \subseteq \times_{j \neq i} R_j(\nu^0_j)$ for all $i \in I$. Since $\nu^0_i$ is additive and independent, one has $\nu^0_i = \prod_{j \neq i} \nu^j_0$ and $\text{supp} \nu^0_i \subseteq R_j(\nu^0_j) = R_j(\prod_{k \neq j} \nu^k_0)$ for all $i \in I$. Hence, $(\nu^0_1, \ldots, \nu^0_I)$ is a Nash equilibrium.  

In two-player games, the consistency and independence condition of Proposition 4.2 are trivially satisfied. With appropriate notions of a product capacity and of consistency, the
intuition developed in two-player games (Example 3.4) that low degrees of ambiguity induce Nash equilibrium behaviour while low degrees of confidence lead to maximin behaviour carries over to general games. In this sense, equilibria under uncertainty represent a compromise between complete ambiguity and complete confidence in probabilistic beliefs.

One may wonder what kind of equilibrium would arise in the limit as the degree of ambiguity goes to zero, without the assumptions of independence of beliefs and consistency. It has been conjectured that a sort of correlated equilibrium would be the limit in this case. Though the beliefs of players may exhibit some kind of correlation there is no common belief or correlating device to coordinate the behaviour of the players. Example 3.2 shows that beliefs that different players hold about their common opponents need not be consistent. It is easy to check that this inconsistency will not vanish if the degree of ambiguity converges to zero\textsuperscript{16}. One can show, however, that the limit of a sequence of equilibria under uncertainty with limiting degree of ambiguity of zero equals a Bayesian Beliefs Equilibrium as introduced by Lo (1996).

5. Robustness Against Lack of Confidence

The previous section has demonstrated that for low degrees of ambiguity, i.e., for $\gamma(\nu_i) \leq 1 - \lambda(\nu_i)$ close to 0 for all $i \in I$, equilibrium behaviour under uncertainty is the same as in a Nash equilibrium, provided that beliefs of all players are independent and consistent. This raises the question whether all Nash equilibria have a nearby equilibrium under uncertainty with the same equilibrium play. A low degree of ambiguity implies a high degree of confidence, since

$$\gamma(\nu_i) \geq 1 - \lambda(\nu_i).$$

Hence, one may call a Nash equilibrium robust against lack of confidence if its equilibrium strategies occur also in an equilibrium under uncertainty with high degrees of confidence.

On the other hand, suppose a strategy is never played in an equilibrium under uncertainty, no matter how close to one the degree of confidence is. In this case, no Nash equilibrium using this strategy can have a nearby equilibrium under uncertainty in which this strategy is played. Thus, one can argue that a Nash equilibrium in which such a strategy is part of equilibrium play cannot be a representation of a stable situation, being sensitive to small degrees of strategic uncertainty of a player.

The following proposition shows that there are games with Nash equilibria whose equilibrium play occurs in no equilibrium under uncertainty.

**Proposition 5.1** Consider games such that, for all $i \in I$, $m_i(s_i) \neq m_i(s'_i)$ for all $s_i \neq s'_i$ holds. If beliefs of all players are represented by capacities with a degree of confidence $\gamma(\nu_i) < 1$, then an equilibrium under uncertainty does not use weakly dominated strategies.

**Proof.** Let $\nu^* = (\nu^*_1, ..., \nu^*_I)$ be an equilibrium under uncertainty. Suppose that some $s_i$ in

\textsuperscript{16} It suffices to consider the case where $\alpha_i$, $i = 1, 2, 3$, converges to 1.
the support of the equilibrium beliefs is weakly dominated by strategy \( \tilde{s}_i \), i.e., \( p_i(s_i, \tilde{s}_i) \geq p_i(s_i, s_{-i}) \) for all \( s_{-i} \in S_{-i} \) with strict inequality for some \( s_{-i} \). Then there exists \( \tilde{s}_{-i} \) such that \( p_i(s_i, \tilde{s}_{-i}) = m_i(s_i) \). Since \( m_i(s_i) \neq m_i(\tilde{s}_i) \) and \( \tilde{s}_i \) weakly dominates \( s_i \) one has \( m_i(\tilde{s}_i) > m_i(s_i) \) and \( p_i(\tilde{s}_i, \tilde{s}_{-i}) \geq m_i(\tilde{s}_i) > m_i(s_i) = p_i(s_i, \tilde{s}_{-i}) \). Consider the payoffs

\[
p_i^*(s_{-i}) = \begin{cases} p_i(s_i, s_{-i}) & s_{-i} \neq \tilde{s}_{-i} \\ p_i(s_i, \tilde{s}_{-i}) & s_{-i} = \tilde{s}_{-i} \end{cases}
\]

By construction, \( p_i(s_i, s_{-i}) \geq p_i^*(s_{-i}) \geq p_i(s_i, s_{-i}) \) for all \( s_{-i} \in S_{-i} \) and, by monotonicity of the Choquet integral, \( P_i^*(s_i, \nu_i^*) \geq P_i^*(\nu_i^*) \geq P_i(s_i, \nu_i^*) \), where \( P_i^*(\nu_i^*) \) denotes the Choquet integral of \( p_i^*(\cdot) \) with respect to \( \nu_i^* \).

By construction of \( p_i^*(\cdot) \),

\[
P_i^*(s_i, \nu_i^*) - P_i^*(\nu_i^*) = [1 - \nu_i^*(S_{-1} \setminus \tilde{s}_{-i})] \cdot [p_i(s_i, \tilde{s}_{-i}) - p_i(s_i, \tilde{s}_{-i})] \\
\geq [1 - \gamma(\nu_i^*)] \cdot [p_i(s_i, \tilde{s}_{-i}) - p_i(s_i, \tilde{s}_{-i})] > 0.
\]

Therefore, \( P_i^*(s_i, \nu_i^*) > P_i^*(\nu_i^*) \geq P_i(s_i, \nu_i^*) \) which contradicts the assumption that \( s_i \) is a best response to the belief \( \nu_i^* \).

Note that, in Proposition 5.1, a strategy is weakly undominated if there is no pure strategies that weakly dominates it. This is in contrast to the common use of this concept in standard game theory where mixed strategies are considered to be the “natural” extension of pure strategies. Clearly, pure strategies can be weakly dominated by mixed strategies. Such an extended notion of uncertainty is here questionable however, because this approach explicitly distinguishes between behaviour and beliefs. From this perspective, a game which allows players to use mixed strategies is a different game to one which does not allow for randomising devices.

The following examples illustrate the necessity of the condition \( m_i(s_i) \neq m_i(\tilde{s}_i) \) for this result.

**Example 5.1** Consider the following 2x2 matrix game.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Player 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( t_1 )</td>
<td>1, 1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( t_2 )</td>
<td>0, -1</td>
</tr>
</tbody>
</table>

It is easy to check that \( m_1(s_1) = 0 \neq m_1(s_2) = -1 \) and \( m_2(t_1) = 0 \neq m_2(t_2) = -1 \) holds. There are two Nash equilibria in pure strategies \( (s_1, t_1) \) and \( (s_2, t_2) \). The second Nash equilibrium \( (s_2, t_2) \) uses weakly dominated strategies. According to Proposition 5.1 there is no equilibrium under uncertainty with \( \gamma(\nu_i) < 1 \) that has \( (s_2, t_2) \) in its support.

To check this claim, consider player 1. The Choquet expected payoff of strategy 1 is \( P_1(s_1, \nu_1) = \nu_1(t_1) \) and for strategy 2, \( P_1(s_2, \nu_1) = \nu_1(t_2) - 1 \). For any degree of confidence \( \gamma(\nu_i) \) less than 1, it follows that

\[
P_1(s_1, \nu_1) = \nu_1(t_1) \geq 0 > \nu_1(t_2) - 1 = P_1(s_2, \nu_1).
\]

Hence, \( s_2 \) can never be a best response given the belief \( \nu_1 \). Since this condition is necessary for an equilibrium under uncertainty with support \( (s_2, t_2) \), no equilibrium under uncertainty can have such Nash equilibrium play.

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Example 5.2  Consider the following slight modification of the game in Example 5.1.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>t₁</th>
<th>t₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>s₁</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>s₂</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Notice that, compared to Example 5.1, neither did the set of pure Nash equilibria change nor the fact that $s_2$ and $t_2$ are dominated strategies. The condition of Proposition 5.1 is however no longer satisfied, since $m_1(s_1) = m_1(s_2) = 0$ and $m_2(t_1) = m_2(t_2) = 0$. From

$$P_1(s_1, \nu_1) = \nu_1(t_1) \geq 0 = P_1(s_2, \nu_1)$$

it follows that $s_2$ may be a best response if $\nu_1(t_1) = 0$ hold. Similarly, one can show that player 2 may find it optimal to choose $t_2$. Thus, in this case, there is an equilibrium under uncertainty compatible with the play of weakly dominated strategies.

Proposition 5.1 and Examples 5.1 and 5.2 show also that robustness against some lack of confidence does not coincide with refinements like perfectness or iterated deletion of strictly dominated strategies, i.e. rationalisability. The equilibrium $(s_2, t_2)$ in Example 5.2 uses weakly dominated strategies and is, therefore, not perfect. It is however robust against ambiguity. Furthermore, all strategy combinations in Example 5.1 are rationalisable but only $(s_1, t_1)$ and $(s_2, t_2)$ are equilibria under uncertainty. Equilibria under uncertainty are quite distinct from rationalisable strategy combinations. For example, games with the battle-of-the-sexes structure, e.g., Example 3.4, have rationalisable strategy combinations that will not be equilibria under uncertainty for any degree of ambiguity.

The necessary condition of Proposition 5.1, that there be no ties between the minimum payoffs achieved with different pure strategies of a player, is a generic property for games with finite pure strategy sets. This suggests that, at least for two-player games without ties, there may be a closer relationship between perfectness\(^{17}\) and robustness against ambiguity.

6. Increasing Uncertainty in Large Groups of Players

We conclude with an example illustrating how the notion of a degree of confidence may be used to explain observations in experimental studies that are difficult to explain with standard game-theoretic concepts. Games of the type presented in this example are discussed in more detail in Crawford (1995) and have been empirically tested by Van Huyck, Battalio, and Beil (1990).

Example 6.1  Consider $n$ communities that have to cooperate in the prevention of pollution. The quality of the environment $x$ depends on the effort contributed by each community $e_i$, $i = 1, ..., n$, according to the production function $x = \min \{e_i | i = 1, ..., n\}$. Thus, the minimal care taken determines the overall outcome. All communities have the same preferences over environmental quality and effort given by the utility function $u_i(x, e_i) = 2 \cdot x - e_i$. To simplify the exposition assume that effort levels can only take

\(^{17}\) Marinacci (1997) studies perfection in the context of ambiguity.
the values 1 and 2.

For the case of two communities the following payoff matrix arises from this scenario.

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Player 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>1</td>
<td>1, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

This game has two Nash equilibria in pure strategies, namely (2, 2) and (1, 1). It is straightforward to compute the following Choquet integral for community i:

\[ P_i(2, \nu_i) = 2 \cdot \nu_i(2) \text{ and } P_i(1, \nu_i) = 1. \]

Clearly, community i will choose effort level 2 if and only if \( \nu_i(2) \geq 0.5 \). For a degree of confidence \( \gamma(\nu_i) \geq 0.5 \), both Nash equilibria can be coordination under uncertainty. For a degree of confidence \( \gamma(\nu_i) < 0.5 \) however, there is a unique equilibrium under uncertainty where both players choose a low effort level.

Consider now the n-player case. Given the payoff function

\[ p_i(e_i, e_{-i}) = 2 \cdot \min \{ e_i | i = 1, \ldots, n \} - e_i, \]

one computes easily the Choquet integral of player i as

\[ P_i(2, \nu_i) = 2 \cdot \nu_i(2, \ldots, 2) \text{ and } P_i(1, \nu_i) = 1. \]

Assuming independent and consistent beliefs for all players, \( \nu_i(2, \ldots, 2) = \nu(2)^{n-1} \) follows for the Cartesian event where all other players choose the effort level 2. Note that, in this example, only this event matters for the Choquet integral. Hence, an equilibrium under uncertainty in which 2 is played requires that \( \nu_i(2, \ldots, 2) = \nu(2)^{n-1} \geq 0.5 \) holds. Let \( \gamma(\nu) \) be the degree of confidence in the belief that a particular opponent chooses the effort level 2. Since \( \gamma(\nu) \geq \nu(2) + \nu(1) \) by Definition 2.2, one can conclude that no equilibrium under uncertainty will have players contribute 2 units of effort if the degree of confidence is less than \( n^{-\sqrt{0.5}} \).

Thus, an increase in the number of participants reduces the scope of the good coordination equilibrium where players contribute 2 units of effort. In fact, for any degree of confidence \( \gamma(\nu) < 1 \), no matter how high, there is a number of players \( N \) large enough to make contributing 1 unit the only equilibrium under uncertainty, since \( \lim_{n \to \infty} n^{-\sqrt{0.5}} = 1. \) One can therefore conclude that the larger the number of players of this game the more likely is the equilibrium with the smallest possible contribution level.

The result of Example 6.1 that, in coordination games, the Nash equilibrium is more likely to be on the lowest level equilibrium if the number of players is large accords well with experimental results obtained by Van Huyck, Battalio, and Beil (1990). In their experimental study, they raise the question why this coordination to a Pareto-dominated
equilibrium is so consistently observed in experiments with a large number of players. It is of course well-known\(^\text{18}\) that the Pareto-dominant equilibrium becomes more and more risk-dominated as the number of participants rises. This observation alone, however, does not provide an explanation for such a particular equilibrium selection. This example suggests strategic uncertainty about the other players’ behaviour as a possible explanation. The economic situation described in Example 6.1 is a special case of a more general allocation problem for public goods when players are uncertainty averse. In *Eichberger and Kelsey (1999)* we study the public goods problem for more general production functions and preferences. Among other results, we can show that, for concave technologies, free-riding may decrease as players are more uncertain about the contributions of the other players.

7. Concluding Remarks

This paper has generalised *Dow and Werlang’s (1994)* equilibrium concept for games with two players whose beliefs are represented by non-additive probabilities to games with an arbitrary finite number of players. Relaxing the restrictions imposed on beliefs by the probabilistic nature of mixed strategies and the consistency of beliefs required in a Nash equilibrium increases the number of equilibria substantially. We propose two measures for the degree of uncertainty of players in their beliefs about the opponents’ behaviour. The degree of confidence measures the maximal trust players may have in their beliefs, while the degree of ambiguity measures maximal doubt of a player. It could be shown that for any degrees of confidence equilibria under uncertainty exist. With the help of these two concepts, it is possible to parametrise equilibria under uncertainty. It could be shown that, without further assumptions, equilibria under uncertainty will coincide with maximin strategies of the players if the degree of confidence of all players is low. On the other hand, for low degrees of ambiguity, equilibria under uncertainty are similar to Nash equilibria only if beliefs are independent and mutually consistent. Equilibria under uncertainty may provide a selection criterion for multiple Nash equilibria. It could be shown that robustness against small degrees of uncertainty provides a refinement of the Nash equilibrium concept that does not coincide with perfectness or robustness against iterated deletion of dominated strategies.

We conclude the paper with a brief discussion of some other game-theoretic equilibrium concepts where the assumption of expected-utility maximising agents is abandoned. *Crawford (1990)* introduces the notion of an equilibrium in beliefs. In *Crawford (1990)*, beliefs are probability distributions over mixed strategies, not capacities over pure strategies as in this paper. Nevertheless, there is a noteworthy relationship between the concept of an equilibrium in beliefs and an equilibrium under uncertainty. An equilibrium in beliefs requires a player’s belief to be concentrated on the set of best-reply mixed strategies of the other players given their beliefs. In this regard, equilibria in beliefs are similar to equilibria under uncertainty. In particular, strategies that are actually played need not

\(^\text{18}\) Compare, e.g., *Fudenberg and Tirole (1991)*, p. 20. *Crawford 1995* has shown that adaptive learning can explain why repeated play of this game in a given group of players may lead these players to coordinate on the risk-dominant equilibrium as observed in experiments.
coincide with the strategy that the opponent expects as long as it is part of the support of the opponent’s belief. Crawford (1990) introduces the concept of an equilibrium in beliefs for two-player games only. The issues of players’ believing that their opponents act independently and of consistency of beliefs do, therefore, not arise.

A second paper extending the Nash equilibrium concept to non-expected utility maximising players is Blume, Brandenburger, and Dekel (1991a, 1991b). They introduce the concept of a lexicographic Nash equilibrium which requires that players’ first-order beliefs about the opponents’ behaviour be concentrated on the best responses of the opponents given their belief system and that these first-order beliefs coincide with the equilibrium mixed strategy played. Lexicographic Nash equilibria are therefore Nash equilibria in first-order beliefs that have higher-order beliefs which allow players to assess the probability of strategy choices out of equilibrium. The main difference between the concept of an equilibrium under uncertainty and a lexicographic Nash equilibrium is the consistency requirement that first-order beliefs coincide with equilibrium mixed strategies.
References


