

Ambiguity and the Centipede Game: Strategic Uncertainty in Multi-Stage Games*

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Abstract

We propose a solution concept for a class of extensive form games with ambiguity. Specifically we consider multi-stage games. Players have CEU preferences. The associated ambiguous beliefs are revised by Generalized Bayesian Updating. We assume individuals take account of possible changes in their preferences by using consistent planning. We show that if there is ambiguity in the centipede game it is possible to sustain ‘cooperation’ for many periods as part of a consistent-planning equilibrium under ambiguity. In a non-cooperative bargaining game we show that ambiguity may be a cause of delay in bargaining.

Keywords: optimism, neo-additive capacity, dynamic consistency, consistent planning, centipede game, multi-stage game.

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1 Introduction

Ellsberg (1961) argues that there are types of uncertainty for which it is difficult or impossible to assign subjective probabilities. He describes such uncertainty as *ambiguous*. It is not implausible that an individual might perceive ambiguity when the resolution of uncertainty depends on the behaviour of other people. This suggests that many interactions in economics and game theory may involve ambiguity. Indeed ambiguity may play an important role in the social sciences more generally.

Traditionally, game theory assumed that one can deduce the behaviour of one's opponents from knowledge of the pay-offs (games of complete information) or from knowing a probability distribution over their possible pay-off parameters (games of incomplete information). Nash equilibrium (or Bayes-Nash equilibrium) may be viewed as a requirement that beliefs about the opponents' behaviour coincide with their actual behaviour in pure or mixed strategies.

Game theory usually assumes that players hold beliefs, which are both unambiguous and correct, about their opponents' behaviour. This stands in stark contrast to recent studies of choice under uncertainty where agents may face ambiguity about the outcome distribution. In the present paper we model such difficulties in predicting behaviour by assuming that players perceive their opponents actions as ambiguous. There is a recent literature on ambiguity in games. However most of it studies games in strategic form.¹ In this paper we extend these approaches to games in extensive form.² In a first step, we will focus on games with perfect information.

1.1 Backward Induction and Alternatives

The standard analysis of sequential two-player games with complete and perfect information uses backward induction or subgame perfection in order to rule out equilibria which are based on “incredible” threats or promises. In sequential two-player games with perfect information, this principle successfully narrows down the set of equilibria and leads to precise predictions. Sequential bargaining, Rubinstein (1982), repeated prisoner's dilemma, chain store paradox, Selten (1978) and the centipede game, Rosenthal (1981) provide well-known examples.

Experimental evidence, however, suggests that in all these cases the unique backward induction

¹See Dow and Werlang (1994), Eichberger and Kelsey (2000) and Riedel and Sass (2014) for games of complete information and Azrieli and Teper (2011), Kajii and Ui (2005) and Grant, Meneghel, and Tourky (2016) for games of incomplete information.

²There is a small earlier literature on extensive form games with ambiguity, see for instance, Bose and Renou (2014), Eichberger and Kelsey (1999), Eichberger and Kelsey (2004), Kellner and LeQuement (2015), Lo (1999), or Mouraviev, Riedel, and Sass (2017),

equilibrium is a poor predictor of behaviour.³ It appears as if pay-offs received off the “narrow” equilibrium path do influence behaviour, even if a step by step analysis shows that it is not optimal to deviate from it at any stage, see Greiner (2016). This suggests that we should reconsider the logic of backward induction. We propose a concept of Consistent Planning Equilibrium (in Beliefs) Under Ambiguity (CP-EUA), which extends the notion of strategic ambiguity to sequential two-player games with perfect information. Despite ambiguity, players remain consistent with regard to their own strategic plans of action. With this notion of equilibrium, we reconsider some of the well-known games mentioned above in order to see whether ambiguity about the opponent’s strategy brings game-theoretic predictions closer to observed behaviour. CP-EUA suggests a general principle for analysing extensive form games without embedding them into an appropriately structured game of incomplete information.

The notion of an Equilibrium under Ambiguity (EUA) for strategic games in Eichberger and Kelsey (2014) rests on the assumption that players take their knowledge about the opponents’ incentives reflected in their pay-offs seriously but not as beyond doubt. Although they predict their opponents’ behaviour based on their knowledge about the opponents’ incentives, they do not have full confidence in these predictions. There may be little ambiguity if the interaction takes place in a known context with familiar players or it may be large in unfamiliar situations where the opponents are strangers. In contrast to standard Nash equilibrium theory, in an EUA the cardinal pay-offs of a player’s own strategies may matter if they are particularly high (optimistic case) or particularly low (pessimistic case). Hence, there will be a trade off between relying on the prediction about the opponents’ behaviour and the salience of one’s own strategy in terms of the outcome.

In dynamic games, where a strategy involves a sequence of moves, the observed history may induce a reconsideration of previously planned actions. As a result one needs to consider issues of dynamic consistency and also whether equilibria rely upon incredible threats or promises. The logic of backward induction forbids a player to consider any move of the opponent which is not optimal, no matter how severe the consequences of such a deviation may be. This argument is weaker in the presence of ambiguity. In contrast in our equilibrium, players maintain sophistication by having correct beliefs about their own future moves.

These considerations suggest that ambiguity makes it harder to resolve dynamic consistency problems. However there are also advantages to studying ambiguous beliefs. We shall update beliefs by

³For the bargaining game Güth, Schmittberger, and Schwarze (1982) provided an early experimental study and for the centipede game McKelvey and Palfrey (1992) find evidence of deviations from Nash predictions.

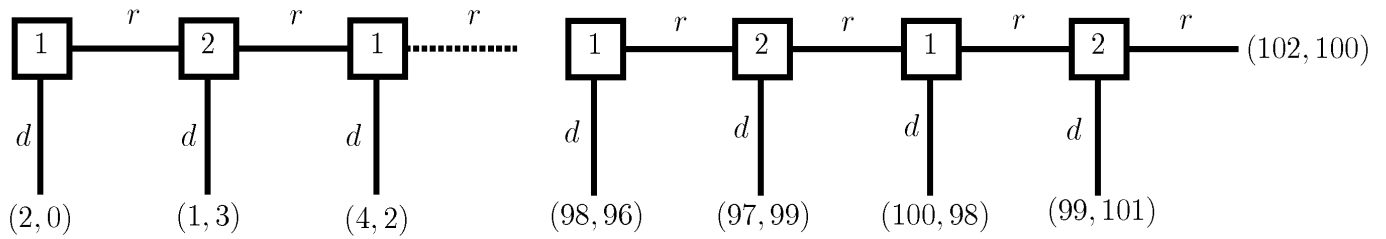


Figure 1: The Centipede Game

the commonly used Generalized Bayesian Updating rule (henceforth GBU). It has the advantage that it is usually defined both on and off the equilibrium path. This contrasts with standard solution concepts, such as Nash equilibrium or subgame perfection, where beliefs off the equilibrium path are somewhat arbitrary, since Bayes' rule is not defined at such events.

We do not assume that players are ambiguity-averse but also allow for optimistic responses to ambiguity. Ambiguity-aversion is different to risk aversion since it is possible for an individual to be both ambiguity-loving and ambiguity-averse at the same time. This would imply that the decision-maker over-weights both high and low pay-offs compared to an individual with SEU preferences. As a result middle ranking outcomes are under-weighted. We find that cooperation in the centipede game requires moderate rather than extreme ambiguity-loving. This is compatible with experimental data on ambiguity-attitudes, for a survey see Trautmann and de Kuilen (2015).

We show that a game of complete and perfect information need not have a pure strategy equilibrium. Thus a well-known property of Nash equilibrium need not apply when there is ambiguity.

1.2 Ambiguity in the Centipede Game

The centipede game is illustrated in figure 1. It has been a long-standing puzzle in game theory. Intuition suggests that there are substantial opportunities for the players to cooperate. However standard solution concepts imply that cooperation is not possible. In this game there are two players who move alternately. At each move a player has the option of giving a benefit to his/her opponent at a small cost to himself/herself. Alternatively (s)he can stop the game at no cost. The centipede game could represent a strategic situation which resembles the case of two countries disarming or dismantling tariff barriers in stages.

Conventional game theory makes a clear prediction. Nash equilibrium and iterated dominance both imply that the only equilibrium in the centipede game is where the first player to move stops the

game by playing down, d . This is despite the fact that both players could potentially make large gains if the game continues until close to the end. However intuition suggests that it is more likely that players will cooperate, at least for a while, thereby increasing both pay-offs. This is confirmed by the experimental evidence, see McKelvey and Palfrey (1992). Thus it is desirable to find an alternative theory which is more in accord with what we observe. It is plausible that playing right, r , may be due to optimistic preferences. Players are choosing a high but uncertain pay-off in preference to a low but safe pay-off. Such behaviour can be predicted by models of ambiguity.

One reason why ambiguity may be present in the centipede game, is that many rounds of deletion of dominated strategies are needed to produce the standard prediction. A player may be uncertain as to whether his/her opponent performs some or all of them.

Our conclusions are that with ambiguity-averse preferences the only equilibrium is one without cooperation. Ambiguity aversion increases the attraction of playing down and receiving a certain pay-off. However if players are ambiguity-loving they may be tempted to cooperate by the high pay-offs towards the end of the game.

1.3 Bargaining

As a second application we consider non-cooperative bargaining. Sub-game perfection suggests that agreement will be instantaneous and outcomes will be efficient. However these predictions do not seem to be supported in many of the situations which bargaining theory is intended to represent. Negotiation between unions and employers often take substantial periods of time and involve wasteful actions such as strikes. Similarly international negotiations can be lengthy and may yield somewhat imperfect outcomes. It is not implausible that ambiguity-loving behaviour could play a role in explaining this. Parties to a bargain initially choose ambitious positions in the hope of achieving large gains. If these expectations are not realised they later shift to make more reasonable demands.

Organization of the paper We first describe in section 2, how we model ambiguity and the rule we use for updating as well as our approach to dynamic choice. We then introduce in section 3 the class of games we shall be studying along with the attendant notation, as well as explaining how we incorporate the model of ambiguity developed in the previous section. In section 4 we present our solution concept. We demonstrate existence and show that games of complete and perfect information may not have pure equilibria. This is applied to the centipede game in section 5 and to bargaining

in section 6. The related literature is discussed in section 7 and section 8 concludes. The appendix contains proofs of those results not proved in the text.

2 Framework and Definitions

In this section we describe how we model ambiguity, updating and dynamic choice.

2.1 Ambiguous Beliefs and Expectations

For a typical two-player game let $i \in \{1, 2\}$, denote a generic player. We shall adopt the convention of referring to Player 1 (respectively, Player 2) by female (respectively, male) pronouns. Let S_i and S_{-i} denote respectively the finite strategy set of i and that of his/her opponent. We denote the payoff to Player i from choosing his/her strategy s_i in S_i , when his/her opponent has chosen s_{-i} in S_{-i} by $u_i(s_i, s_{-i})$. Following Schmeidler (1989) we shall model ambiguous beliefs with capacities on S_{-i} , which are defined as follows.

Definition 2.1 A capacity on S_{-i} is a real-valued function ν on the subsets of S_{-i} such that $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$ and $\nu(\emptyset) = 0$, $\nu(S_{-i}) = 1$.

The ‘expected’ payoff obtained from a given act, with respect to the capacity ν can be found using the Choquet integral, defined below.

Definition 2.2 The Choquet integral of $u_i(s_i, s_{-i})$ with respect to capacity ν on S_{-i} is:

$$V_i(s_i) = \int u_i(s_i, s_{-i}) d\nu = u_i(s_i, s_{-i}^1) \nu(s_{-i}^1) + \sum_{r=2}^R u_i(s_i, s_{-i}^r) [\nu(s_{-i}^1, \dots, s_{-i}^r) - \nu(s_{-i}^1, \dots, s_{-i}^{r-1})],$$

where $R = |S_{-i}|$ and the strategy profiles in S_{-i} are numbered so that $u_i(s_i, s_{-i}^1) \geq u_i(s_i, s_{-i}^2) \geq \dots \geq u_i(s_i, s_{-i}^R)$.

Preferences represented by a Choquet integral with respect to a capacity are referred to as Choquet Expected Utility (henceforth CEU). Chateauneuf, Eichberger, and Grant (2007) have axiomatised a special case of CEU, that we shall refer to as *non extreme outcome (neo)-expected utility* preferences. In this model a decision-maker maximises the expected value of utility with respect to a *neo-additive* capacity, ν . These capacities are characterized by two parameters $\alpha, \delta \in [0, 1]$ and a probability distribution π_i on S_{-i} . A neo-additive capacity is defined by setting: $\nu(A|\alpha, \delta, \pi_i) = 1$, for $A = S_{-i}$;

$\nu(A|\alpha, \delta, \pi_i) = \delta(1 - \alpha) + (1 - \delta)\pi_i(A)$, for $\emptyset \subsetneq A \subsetneq S_{-i}$; and $\nu(A|\alpha, \delta, \pi_i) = 0$ for $A = \emptyset$. Fix the two parameters α and δ . For any probability distribution π_i on S_{-i} , it is straightforward to show that the Choquet integral of $u_i(s_i, s_{-i})$ with respect to the neo-additive capacity $\nu(\cdot|\alpha, \delta, \pi_i)$ on S_{-i} can be expressed as the neo-expected payoff function given by:

$$V_i(s_i|\nu(\cdot|\alpha, \delta, \pi_i)) = (1 - \delta) \cdot \mathbf{E}_{\pi_i} u_i(s_i, s_{-i}) + \delta \left[\alpha \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1 - \alpha) \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \right], \quad (1)$$

where $\mathbf{E}_{\pi_i} u_i(s_i, s_{-i})$, denotes a conventional expectation taken with respect to the probability distribution π_i .⁴

Thus, given the two parameters α and δ and the probability distribution π_i , the player's preferences over her strategies in S_i can be represented by a weighted average of the mean payoff, the lowest payoff and the highest payoff.

One can interpret π_i as the decision-maker's ("probabilistic") belief or "theory" about how her opponent is playing. However, she perceives there to be some degree of ambiguity associated with her theory. Her confidence in this theory is modelled by the weight $(1 - \delta)$ given to the expected pay-off $\mathbf{E}_{\pi_i} u_i(s_i, s_{-i})$. Correspondingly, the highest (respectively, lowest) possible degree of ambiguity corresponds to $\delta = 1$, (respectively, $\delta = 0$). Her attitude toward such ambiguity is measured by α . Purely ambiguity-loving preferences are given by $\alpha = 0$, while the highest level of ambiguity-aversion occurs when $\alpha = 1$. If $0 < \alpha < 1$, the individual has a mixed attitude toward ambiguity, since she responds to ambiguity partly in a pessimistic way by over-weighting bad outcomes and partly in an optimistic way by over-weighting good outcomes.

2.2 The Support of a Capacity

In this section we define the support of a capacity, which represents the strategies that a given player believes his/her opponent will play. It is not possible for us to use existing definitions of the support unmodified since many of them have implicitly assumed ambiguity-aversion or equivalently that the capacity is convex. Two of the more prominent definitions of support from the literature are the *Marinacci (M) support*, and the *inner (I) support*. Marinacci (2000) defines the support of a capacity ν to be the set of strategies with positive capacity, $\text{supp}_M \nu = \{s_{-i} \in S_{-i} : \nu(s_{-i}) > 0\}$. Ryan (2002) defines the inner support, supp_I , as follows.

⁴To simplify notation we shall suppress the arguments and write $V_i(s_i|\nu)$ for $V_i(s_i|\nu(\cdot|\alpha, \delta, \pi_i))$ when the meaning is clear from the context.

Definition 2.3 If μ is a convex capacity on S_{-i} , we define the inner support of μ , $\text{supp}_I \mu$, by

$$\text{supp}_I \mu = \bigcap_{p \in \mathcal{C}(\mu)} \text{supp } p, \quad (2)$$

where the core, $\mathcal{C}(\nu)$, of a capacity ν is defined by,

$$\mathcal{C}(\nu) = \{p \in \Delta(S_{-i}); \forall A \subseteq S_{-i}, p(A) \geq \nu(A)\}.$$

However neither of these support notions are suitable for capacities which are not convex. For example, consider the neo-additive capacity $\nu = \delta(1 - \alpha) + (1 - \delta)\pi$. This capacity assigns positive values to all strategies $s_{-i} \in S_{-i}$, provided $\alpha < 1, \delta > 0$. Thus $\text{supp}_M \nu = \text{supp}_I \nu = S_{-i}$. Hence, neither of these concepts allow us to make a distinction between those strategies which a given player believes are possible for his opponent and others. Whenever there is even a small amount of ambiguity-loving, neo-additive capacities assign positive capacity to all strategies. However, this does not mean the player “believes” in these strategies. Looking at strategies with positive capacity confounds belief and ambiguity-attitude, since optimism increases the capacity values assigned to all strategies. To overcome these problems we propose the following definition.

Definition 2.4 If ν is a capacity on S_{-i} , define

$$\mathcal{B}(\nu) = \{s_{-i} \in S_{-i} : \forall A \subsetneq S_{-i}, s_{-i} \notin A; \nu(A \cup s_{-i}) > \nu(A)\}.$$

The set $\mathcal{B}(\nu)$ consists of those strategies of i 's opponent which always get positive weight in the Choquet integral, no matter which of i 's strategies is being evaluated. To see, this, recall that the Choquet expected utility of a given strategy, s_i , is a weighted sum of utilities. The decision-weight assigned to strategy \tilde{s}_{-i} is $\nu(\{s_{-i} : u_i(s_i, s_{-i}) \succ u_i(s_i, \tilde{s}_{-i})\} \cup \{\tilde{s}_{-i}\}) - \nu(\{s_{-i} : u_i(s_i, s_{-i}) \succ u_i(s_i, \tilde{s}_{-i})\})$. These weights depend on the way in which the strategy s_i ranks the strategies in S_{-i} . Since there are $R!$ ways the elements of S_{-i} can be ranked, in general there are $R!$ decision weights used in evaluating the Choquet integral with respect to a given capacity.⁵ If $\forall A \subsetneq S_{-i}, \tilde{s}_{-i} \notin A, \nu(A \cup \tilde{s}_{-i}) > \nu(A)$ then the decision-weight on the strategy \tilde{s}_{-i} is positive no matter how \tilde{s}_{-i} is ranked by strategy s_i . The set of strategies S_{-i} can be partitioned into three sets, those strategies which are given positive weight by all of the decision weights, those strategies given positive weight by some sets of decision weights

⁵Recall that $R = |S_{-i}|$, is the number of strategies available to i 's opponent.

but not others and those strategies which are given weight zero by all decision weights.⁶

Sarin and Wakker (1998) argue that the decision-maker's beliefs may be deduced from the decision weights in the Choquet integral. With this interpretation, $\mathcal{B}(\nu)$ is the set of strategies in which the decision-maker 'believes' in the strong sense that they always get positive weight. Thus $\mathcal{B}(\nu)$ can be viewed as an extension of the inner support concept to capacities which are not necessarily convex, since it is the intersection of the supports of the decision-weights. Similarly the strategies which always get zero weight are those which the decision-maker believes to be impossible. The remaining strategies can be interpreted as those which the decision-maker believes to be ambiguous. The weight they get in the Choquet integral may or may not be positive depending on the context. Note that for a convex capacity $\mathcal{B}(\nu)$ coincides with the inner support.

The example below shows that $\mathcal{B}(\nu)$ yields an intuitive result when it is applied to a neo-additive capacity.

Example 2.1 *Let ν be a neo-additive capacity on S_{-i} where $\nu(A) = \delta(1 - \alpha) + (1 - \delta)\pi(A)$, then $\mathcal{B}(\nu) = \text{supp } \pi = \{s_{-i} \in S_{-i} : \pi(s_{-i}) > 0\}$ provided $0 \leq \alpha < 1$.*

Proof. If $s \notin A \subseteq S$, $\nu(A \cup s) - \nu(A) = [\delta(1 - \alpha) + (1 - \delta)\pi(A \cup s)] - [\delta(1 - \alpha) + (1 - \delta)\pi(A)] = [(1 - \delta)\pi(A) + (1 - \delta)\pi(s)] - [(1 - \delta)\pi(A)] = (1 - \delta)\pi(s)$. Thus $\nu(A \cup s) > \nu(A) \Leftrightarrow \pi(s) > 0$. ■

Recall that we usually interpret π as an ambiguous belief. Thus it is natural that the support of the capacity ν is the support in the usual sense of the additive probability π . For further discussion of the various support notions and the relation between them see Eichberger and Kelsey (2014) and Ryan (2002).

2.3 Updating Ambiguous Beliefs

CEU is a theory of decision-making at one point in time. To use it in extensive form games we need to extend it to multiple time periods. We do this by employing Generalized Bayesian Updating (henceforth GBU) to revise beliefs. One problem which we face is that the resulting preferences may not be dynamically consistent. We respond to this by assuming that individuals take account of future preferences by using *consistent planning*, defined below. The GBU rule has been axiomatized in Eichberger, Grant, and Kelsey (2007) and Horie (2013). It is defined as follows.

Definition 2.5 *Let ν be a capacity on S_{-i} and let $E \subseteq S_{-i}$. The Generalized Bayesian Update*

⁶The third set of profiles is a null set.

(henceforth GBU) of ν conditional on E is given by:

$$\nu^E(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)},$$

where $E^c = S_{-i} \setminus E$ denotes the complement of E .

The GBU rule coincides with Bayesian updating when beliefs are additive. For a neo-additive belief $\nu(\cdot|\alpha, \delta, \pi_i)$ the GBU conditional on E is given by

$$\nu^E(A|\alpha, \delta, \pi_i) = \begin{cases} 0 & \text{if } A \cap E = \emptyset, \\ \delta^E(1 - \alpha) + (1 - \delta^E)\pi^E & \text{if } \emptyset \subsetneq A \cap E \subsetneq E, \\ 1 & \text{if } A \cap E = E, \end{cases}$$

where $\delta^E = \frac{\delta}{\delta + (1 - \delta)\pi(E)}$, and $\pi^E(A) = \frac{\pi(A \cap E)}{\pi(E)}$. Notice that for a neo-additive belief with $\delta > 0$, the GBU update is well-defined even if $\pi(E) = 0$ (that is, E is a zero-probability event according to the individual's 'theory'). In this case the updated parameter $\delta^E = 1$, which implies the updated capacity is a Hurwicz capacity that assigns the weight 1 to every event that is a superset of E , and $(1 - \alpha)$ to every event that is a non-empty strict subset of E .

The following result states that a capacity is neo-additive, if and only if both it and its GBU update admit a multiple priors representation with the same α and the updated set of beliefs is the prior by prior Bayesian update of the initial set of probabilities.⁷

Proposition 2.1 *The capacity ν is neo-additive for some parameters α and δ and some probability π , if and only if both the ex-ante and the updated preferences respectively admit multiple priors representations of the form:*

$$\int u_i(s_i, s_{-i}) d\nu = \alpha \cdot \min_{q \in \mathcal{P}} \mathbf{E}_q u_i(s_i, s_{-i}) + (1 - \alpha) \cdot \max_{q \in \mathcal{P}} \mathbf{E}_q u_i(s_i, s_{-i}),$$

$$\int u_i(s_i, s_{-i}) d\nu^E = \alpha \cdot \min_{q \in \mathcal{P}^E} \mathbf{E}_q u_i(s_i, s_{-i}) + (1 - \alpha) \cdot \max_{q \in \mathcal{P}^E} \mathbf{E}_q u_i(s_i, s_{-i}),$$

where $\mathcal{P} := \{p \in \Delta(S_{-i}) : p \geq (1 - \delta)\pi\}$, $\mathcal{P}^E := \{p \in \Delta(E) : p \geq (1 - \delta^E)\pi^E\}$, $\delta^E = \frac{\delta}{\delta + (1 - \delta)\pi(E)}$, and $\pi^E(A) = \frac{\pi(A \cap E)}{\pi(E)}$.

⁷A proof can be found in Eichberger, Grant, and Kelsey (2012).

We view this as a particularly attractive and intuitive result since the ambiguity-attitude, α , can be interpreted as a characteristic of the individual which is *not updated*. In contrast, the set of priors is related to the environment and one would expect it to be revised on the receipt of new information.⁸

2.4 Consistent Planning

As we have already foreshadowed, the combination of CEU preferences and GBU updating is not, in general, dynamically consistent. Perceived ambiguity is usually greater after updating. Thus for an ambiguity-averse individual, constant acts will become more attractive after updating. Hence if an individual is ambiguity-averse, in the future (s)he may wish to take an option which gives a certain payoff, even if it was not in his/her original plan to do so. Following Strotz (1955), Siniscalchi (2011) argues against *commitment to a strategy* in a sequential decision problem in favour of *consistent planning*. This means that a player takes into account any changes in his/her own preference arising from updating at future nodes. As a result, players will take a sequence of moves which is consistent with backward induction. In general it will differ from the choice a player would make at the first move with commitment.⁹ With consistent planning, however, dynamic consistency is no longer an issue.¹⁰ The dynamic consistency issues and consistent planning are illustrated by the following example of individual choice in the presence of sequential resolution of uncertainty.

Example 2.2 *Consider the following setting of sequential resolution of uncertainty. There are three time periods, $t = 0, 1, 2$. In period 0 the decision-maker decides whether or not to accept a bet b_W which pays 1 in the event W (Win) and 0 in the complementary event L (Lose). The alternative is to choose an act \bar{b} which yields a certain pay-off of $x, 0 < x < 1$. At time $t = 1$ she receives a signal which is either good G or bad B . A good (resp. bad) signal increases (resp. decreases) the likelihood of winning. If at time 0 she chose to bet and the signal is good she now has the option of switching to a certain payment b_G . In effect selling her bet. To summarise at time $t = 0$ the individual can choose among the following three ‘strategies’:*

\bar{b} accept a non-state contingent (that is, guaranteed) pay-off of x ; or,

⁸There are two alternative rules for updating ambiguous beliefs, the Dempster-Shafer (pessimistic updating) rule and the Optimistic updating rule, Gilboa and Schmeidler (1993). However neither of these will leave ambiguity-attitude, α , unchanged after updating. The updated α is always 1 (respectively, 0) for the Dempster-Shafer (respectively, Optimistic updating) rule. See Eichberger, Grant, and Kelsey (2010). For this reason we prefer the GBU rule.

⁹From this perspective, commitment devices should be explicitly modelled. If a commitment device exists, e.g., handing over the execution of a plan to a referee or writing an enforceable contract, then no future choice will be required.

¹⁰Bose and Renou (2014) and Karni and Safra (1989) use versions of consistent planning in games.

b_G accept the bet but switch to a certain payment if the signal at $t = 1$ is good or

b_W accept the bet and retain it in period 1.

Suppose that the individual is a neo-additive expected payoff maximizer with capacity ν . Her ‘(probabilistic) belief’ about the data generating process can be summarized by the following three probabilities: $\pi(G) = p$, $\pi(W|G) = q$ and $\pi(W|B) = 0$, where $\max\{p, q\} < 1$ and $\min\{p, q\} > 0$. Her ‘lack of confidence’ in her belief is given by the parameter $\delta \in (0, 1)$, and her attitude toward ambiguity is given by the parameter α which we assume lies in the interval $(1 - q, 1)$.

The strategy b_G yields a constant pay-off of q if the signal realization is G , while b_W leads to a pay-off of 1 if the event W obtains and 0 otherwise. The state-contingent pay-offs associated with these three strategies are given in the following matrix.

| | | Events | | |
|------|-----------|--------|------------|------------|
| | | B | $G \cap L$ | $G \cap W$ |
| Bets | \bar{b} | x | x | x |
| | b_G | 0 | q | q |
| | b_W | 0 | 0 | 1 |

One-shot resolution If the DM is not allowed to revise her choice after learning the realization of the signal in period 1 (or she can commit not to revise her choice), then the choice between b_G and b_W is governed by her ex ante preferences which we take to be represented by the neo-expected pay-off function $V(\cdot)$ for which

$$V(b_G, \nu(\cdot|\alpha, \delta, \pi)) = (1 - \delta)pq + \delta(1 - \alpha)q, \quad V(b_W, \nu(\cdot|\alpha, \delta, \pi)) = (1 - \delta)pq + \delta(1 - \alpha).$$

Notice that $V(b_W, \nu) - V(b_G, \nu) = \delta(1 - \alpha)(1 - q) > 0$. Furthermore, if x is set so that $x = (1 - \delta)pq + \delta(1 - \alpha)\left(\frac{1+q}{2}\right)$, then we also have

$$V(\bar{b}, \nu(\cdot|\alpha, \delta, \pi)) = \frac{1}{2}V(b_G) + \frac{1}{2}V(b_W).$$

So for a one-shot resolution scenario, the DM will strictly prefer to choose b_W over both \bar{b} and b_G .

Sequential resolution Now consider the scenario in which the DM has the opportunity to revise her choice after she has learned the realization of the signal. Let ν^G (respectively, ν^B) denote the GBU

of ν conditional on G (respectively, B) realizing. If the signal realization is B then she is indifferent between the pair of bets b_W and b_G . However consider the case where she learns the signal realization is G . According to her *ex ante preferences*, she should stay with her choice of b_W . Her updated preference between the two bets b_W and b_G are characterized by the neo-expected pay-off function $V(\cdot, \nu^G)$ for which

$$V(b_G, \nu^G) = q, \quad V(b_W, \nu^G) = (1 - \delta^G)q + \delta^G(1 - \alpha),$$

where $\delta^G = \frac{\delta}{\delta + (1-\delta)^p}$.

Notice that $V(b_G, \nu^G) - V(b_W, \nu^G) = \frac{\delta}{\delta + (1-\delta)^p} \times (\alpha - [1 - q]) > 0$. Hence we have

$$V(b_G, \nu^G) > V(b_W, \nu^G) \text{ and } V(b_G, \nu^B) = V(b_W, \nu^B) (= 0),$$

$$\text{but } V(b_W, \nu) > V(b_G, \nu),$$

a violation of dynamic consistency (or what Skiadas (1997) calls “coherence”).

Naive Choice versus Consistent Planning If the DM is “naive” then in the sequential resolution setting, she does not choose \bar{b} in period 1, planning to go with b_W in the event the signal realization is G . However, given her updated preferences, she changes her plan of action and chooses the bet b_G instead, yielding her a now guaranteed payoff of q . On the other hand, a consistent planner, anticipating her future self would choose not to remain with the bet b_W after learning the realization of the signal was G , understands that her choice in the first period is really between \bar{b} and b_G . Hence she selects \bar{b} , since $V(\bar{b}, \nu) > V(b_G, \nu)$.

3 Multi-stage Games

We turn now to a formal description of the sequential strategic interaction between two decision-makers. This is done by way of multi-stage games that have a fixed finite number of time periods. In any given period the history of previous moves is known to both players. Within a time period simultaneous moves are allowed. We believe these games are sufficiently general to cover many important economic applications entailing strategic interactions. Below we describe our notation.

3.1 Description of the game.

There are 2 players, $i = 1, 2$ and T stages, $1 \leq t \leq T$. At each stage t , each player i simultaneously selects an action a_i^t .¹¹ Let $a^t = \langle a_1^t, a_2^t \rangle$ denote a profile of action choices by the players in stage t .

The game has a set \bar{H} of histories h which,

1. contains the empty sequence $h^0 = \langle \emptyset \rangle$ (no records);
2. for any non-empty sequence $h = \langle a^1, \dots, a^t \rangle \in \bar{H}$, all subsequences $\hat{h} = \langle a^1, \dots, a^{\hat{t}} \rangle$ with $\hat{t} < t$ are also contained in \bar{H} .

The set of all histories at stage t are those sequences in \bar{H} of length $t - 1$, with the empty sequence h^0 being the only possible history at stage 1. Let H^{t-1} denote the set of possible histories at stage t with generic element $h^{t-1} = \langle a^1, \dots, a^{t-1} \rangle$.¹² Any history $\langle a^1, \dots, a^T \rangle \in \bar{H}$ of length T is a *terminal history* which we shall denote by z . We shall write $Z (= H^T)$ for the subset of \bar{H} that are terminal histories. Let $H = \bigcup_{t=1}^T H^{t-1}$ denote the set of all non-terminal histories and let $\theta = |H|$ denote the number of non-terminal histories.¹³ At stage \hat{t} , all players know the history of moves from stages $t = 1$ to $\hat{t} - 1$.

For each $h \in H$ the set $A^h = \{a \mid (h, a) \in \bar{H}\}$ is called the action set at h . We assume that A^h is a Cartesian product $A^h = A_1^h \times A_2^h$, where A_i^h denotes the set of actions available to player i after history h . The action set, A_i^h , may depend both on the history and the player. A pure strategy specifies a player's move after every possible history.

Definition 3.1 A (pure) strategy of a player $i = 1, 2$ is a function s_i which assigns to each history $h \in H$ an action $a_i \in A_i^h$.

Let S_i denote the strategy set of player i , $S = S_1 \times S_2$, the set of strategy profiles and $S_{-i} = S_j, j \neq i$, the set of strategies of i 's opponent. Following the usual convention, we will sometimes express the strategy profile $s \in S$ as (s_i, s_{-i}) , in order to emphasize that player i is choosing her strategy $s_i \in S_i$ given her opponent is choosing according to the strategy $s_{-i} \in S_{-i}$.

Fix a strategy profile $s = (s_1, s_2) \in S$. For each $t = 1, \dots, T$, let $s(h_s^{t-1})$ denote the unique action profile $a^t = (a_1^t, a_2^t)$ induced by this strategy profile s at stage t after history h^{t-1} .

¹¹It is without loss of generality to assume that each player moves in every time period. Games where one player does not move at a particular time, say \hat{t} , can be represented by assigning that player a singleton action set at time \hat{t} .

¹²Notice by definition, that $H^0 = \{h^0\}$.

¹³Notice that by construction $\bar{H} = H \cup Z$.

Thus for each strategy profile s , we have

$$\begin{array}{llll}
\cdot & h_s^0 = h^0 & \text{induces the action profile} & s(h^0) = a^1 \\
\cdot & h_s^1 = \langle a^1 \rangle & \text{induces the action profile} & s(\langle a^1 \rangle) = a^2 \\
\cdot & h_s^2 = \langle a^1, a^2 \rangle & \text{induces the action profile} & s(\langle a^1, a^2 \rangle) = a^3 \\
& \vdots & & \vdots \\
\cdot & h_s^{t-1} = \langle a^1, \dots, a^{t-1} \rangle & \text{induces the action profile} & s(\langle a^1, \dots, a^{t-1} \rangle) = a^t \\
& \vdots & & \vdots \\
\cdot & h_s^{T-1} = \langle a^1, \dots, a^{T-1} \rangle & \text{induces the action profile} & s(\langle a^1, \dots, a^{T-1} \rangle) = a^T \\
\cdot & h_s^T = \langle a^1, \dots, a^{T-1}, a^T \rangle & \text{is a terminal history in } Z. &
\end{array}$$

This gives rise to a collection of functions $\langle \zeta^t \rangle_{t=1}^T$, where for each $t = 1, \dots, T$, the function $\zeta^t : S \rightarrow H^t$ defined $\zeta^t(s) := \langle s(h_s^{\tau-1}) \rangle_{\tau=1}^t \in H^t$. The function ζ^t assigns strategies to histories of length t . It is, by construction, surjective since every history must arise from some combination of strategies. A pay-off function u_i for player i , assigns a real number to each terminal history $z \in Z$. With a slight abuse of notation, we shall write $u_i(s)$ for the convolution $u_i \circ \zeta^T(s)$. We now have all the elements to define a multi-stage game.

Definition 3.2 *A multi-stage game Γ is a triple $\langle \{1, 2\}, \bar{H}, u_i, i = 1, 2 \rangle$, where \bar{H} is the set of all histories, and $u_i, i = 1, 2$ characterizes the players' pay-offs.*

To study the impact of ambiguity we may wish to put restrictions on players' degree of ambiguity or their ambiguity attitude. This can be achieved by assigning each player, i , a set of admissible capacities, C_i , which is a subset of the set of capacities defined over her opponent's strategy set S_{-i} , corresponding to those beliefs that are admissible for this player to have ex ante in this strategic interaction. That is, C_i encodes both her perception of the strategic ambiguity as well as her attitude toward this ambiguity. In this case each player is characterized by a pair (u_i, C_i) , where u_i is her payoff function defined over terminal histories (and implicitly, over strategy profiles). An example of a set of admissible capacities would be the case where player i is a neo-expected payoff maximizer with neo-additive beliefs characterized by the pair of parameters $(\delta_i, \alpha_i) \in [0, 1] \times [0, 1]$ then her set of admissible beliefs $C_i = \{\nu = \nu(\cdot | \alpha, \delta, \pi_i) : \pi_i \in \Delta(S_{-i})\}$. Thus for a neo-expected payoff maximizer, we interpret $\delta_i \in [0, 1]$ as reflecting her (initial) perception of the degree of strategic ambiguity and $\alpha_i \in [0, 1]$ as her degree of (relative) pessimism. Given a "probabilistic theory" $\pi_i \in \Delta(S_{-i})$ she

evaluates the neo-expected payoff of playing strategy s_i as in equation (1). A standard expected payoff maximizer corresponds to the polar case $\delta_i = 0$.

3.2 Sub-histories, Continuation Strategies and Conditional Pay-offs.

A (*sub-*) *history* after a non-terminal history $h \in H$ is a sequence of actions h' such that $(h, h') \in \overline{H}$. Adopting the convention that (h, h^0) is identified with h , denote by \overline{H}^h the set of histories following h . Let Z^h denote the set of terminal histories following h . That is, $Z^h = \{z' \in \overline{H}^h : (h, z') \in Z\}$.

Consider a given individual, player i , (she). Denote by s_i^h a (continuation-) strategy of player i which assigns to each history $h' \in \overline{H}^h \setminus Z^h$ an action $a_i \in A_i^{(h, h')}$. We will denote by S_i^h the set of all those (continuation-) strategies available to player i following the history $h \in H$ and define $S^h = S_1^h \times S_2^h$ to be the set of (continuation-) strategy profiles. Each strategy profile $s^h = \langle s_1^h, s_2^h \rangle \in S^h$ defines a terminal history in Z^h . Furthermore, we can take $u_i^h : Z^h \rightarrow \mathbb{R}$, to be the pay-off function for player i given by $u_i^h(h') = u_i(h, h')$, and correspondingly set $u_i^h(s^h) := u_i^h(h')$ if the continuation strategy profile s^h leads to the play of the sub-history h' . Consider player i 's choice of continuation strategy s_i^h in S_i^h that starts in stage t . To be able to compute her conditional (Choquet) expected payoff, she must use Bayes' Rule to update her theory π_i (a probability measure defined on S_{-i}) to a probability measure defined on S_{-i}^h . In addition it is necessary to update her perception of ambiguity represented by the parameter δ_i . Now, since ζ^{t-1} is a surjection, there exists a well-defined pre-image $S(h) := (\zeta^{t-1})^{-1}(h) \subseteq S$ for any history $h \in H^{t-1}$. The event $S_{-i}(h)$ is the marginal of this event on S_{-i} given by

$$S_{-i}(h) := \left\{ s_{-i} \in S_{-i} \mid \exists s_i \in S_i, (\zeta^{t-1})^{-1}(s_i, s_{-i}) = h \right\}.$$

Similarly, the event $S_i(h)$ is the marginal of this event on S_i given by

$$S_i(h) := \left\{ s_i \in S_i \mid \exists s_{-i} \in S_{-i}, (\zeta^{t-1})^{-1}(s_i, s_{-i}) = h \right\}.$$

Suppose that player i 's initial belief about how the opponent is choosing a strategy is given by a capacity ν_i . Then, her evaluation of the Choquet expected payoff associated with her continuation strategy s_i^h is given by:

$$V_i^h(s_i^h | \nu_i) = \int u_i(s_i^h, s_{-i}^h) d\nu_i^h(s_{-i}^h),$$

where ν_i^h is the GBU of ν_i conditional on history h being reached. Hence, in particular, if she is a neo-

expected payoff maximizer with $\nu_i = \nu(\alpha_i, \delta_i, \pi_i)$ then her evaluation of the *conditional neo-expected payoff* of her continuation strategy s_i^h is given by:

$$V_i^h(s_i^h | \nu_i^h) = (1 - \delta_i^h) \mathbb{E}_{\pi_i^h} u_i(s_i^h, \cdot) + \delta_i^h \left[\alpha_i \min_{s_{-i}^h \in S_{-i}^h} u_i^h(s_i^h, s_{-i}^h) + (1 - \alpha_i) \max_{s_{-i}^h \in S_{-i}^h} u_i^h(s_i^h, s_{-i}^h) \right],$$

where $\delta_i^h = \frac{\delta_i}{\delta_i + (1 - \delta_i)\pi_i(S_{-i}(h))}$ (the GBU update of δ_i) and π_i^h is the Bayesian update of π_i whenever $\delta_i^h < 1$.

One-step deviations Consider a given a history $h \in H^{t-1}$ and a strategy profile $s \in S$. A *one-step deviation* in stage t by player i from her strategy s_i to the action $a_i \in A_i^h$ leads to the terminal history in Z^h determined by the continuation strategy profile $\langle a_i, s_i^h(-t), s_{-i}^h \rangle$, where $s^h \in S^h$, is the continuation of the strategy profile s starting in stage t from history h , and $s_i^h(-t)$ is player i 's component of that strategy profile except for her choice of action in stage t . This enables us to separate player i 's decision at stage t from the decisions of other players including her own past and future selves.

4 Equilibrium Concept: Consistent Planning

Our solution concept is an equilibrium in beliefs. Players choose pure (behaviour) strategies, but have possibly ambiguous beliefs about the strategy choice of their opponents. Each agent is required to choose at every decision node an action, which must be optimal with respect to his/her updated beliefs. When choosing an action a player treats his/her own future strategy as given. Consistency is achieved by requiring that the support of these beliefs is concentrated on the opponent's best replies. Thus it is an solution concept in the spirit of the agent normal form.

Definition 4.1 Fix a multi-stage game $\langle \{1, 2\}, \bar{H}, u_i, i = 1, 2 \rangle$. A Consistent Planning Equilibrium Under Ambiguity (CP-EUA) is a profile of capacities $\langle \nu_1, \nu_2 \rangle$ such that for each player $i = 1, 2$,

$$s_i \in \text{supp } \nu_{-i} \Rightarrow V_i^h(s_i^h | \nu_i^h) \geq V_i^h((a_i, s_i^h(-t)) | \nu_i^h),$$

for every $a_i \in A_i^h$, every $h \in H^{t-1}$, and every $t = 1, \dots, T$.

Remark 4.1 If $|\text{supp } \nu_i| = 1$ for $i = 1, 2$ we say that the equilibrium is singleton. Otherwise we say that it is mixed. Singleton equilibria are analogous to pure strategy Nash equilibria.

Remark 4.2 *A CP-EUA satisfies the one step deviation principle. No player may increase his/her conditional neo-expected payoff by changing his/her action in a single time period. We do not include a formal proof since the result follows directly from the definition.*

CP-EUA requires that the continuation strategy that player i is planning to play from history $h \in H^{t-1}$ is in the support of his/her opponent's beliefs ν_{-i} . Moreover, the only strategies in the support of his/her opponent's beliefs ν_{-i} are ones in which the action choice at history $h \in H^t$ is optimal for player i given her updated capacity ν_i^h . This rules out “incredible threats” in dynamic games. Thus our solution concept is an ambiguous analogue of sub-game perfection.¹⁴ Since we require beliefs to be in equilibrium in each subgame, an equilibrium at the initial node will imply optimal behaviour of each player at each decision node. In particular, players will have a consistent plan in the sense of Siniscalchi (2011).

Mixed equilibria should be interpreted as equilibria in beliefs. To illustrate this consider a given player (she). We assume that she chooses pure actions and any randomizing is in the mind of her opponent. We require beliefs to be consistent with actual behaviour in the sense that pure strategies in the support of the beliefs induce behaviour strategies, which are best responses at any node where the given player has the move. The combination of CEU preferences and GBU updating is not, in general, dynamically consistent. A consequence of this is that in a mixed equilibrium some of the *pure strategies*, which the given player's opponents believe she may play, are not necessarily optimal at all decision nodes. This arises because her preferences may change when they are updated. In particular equilibrium pure strategies will typically not be indifferent at the initial node. However at any node she will choose *actions* which are best responses. All behaviour strategies in the support of her opponents' beliefs will be indifferent. These issues do not arise with pure equilibria.

The following result establishes that when players have neo-additive beliefs, equilibrium exists for any exogenously given degrees of ambiguity and ambiguity attitudes.

Proposition 4.1 *Let Γ be a multi-stage game with 2 neo-expected payoff maximizing players. Then Γ has at least one CP-EUA for any given parameters $\alpha_1, \alpha_2, \delta_1, \delta_2$, where $0 \leq \alpha_i \leq 1$, $0 < \delta_i \leq 1$, for $i = 1, 2$.*

Since existence of equilibrium with beliefs restricted to be neo-additive implies existence without this constraint, our main result is a corollary.

¹⁴Recall that in a multi-stage game a new subgame starts after any given history h .

Theorem 4.1 *Let Γ be a multi-stage game with 2 players. Then Γ has at least one CP-EUA $\langle \nu_1, \nu_2 \rangle$. Moreover CP-EUA will still exist if we require preferences to be either strictly ambiguity-averse or strictly ambiguity-loving.*

4.1 Non-Existence of CP-EUA in Pure Strategies

Finite games of complete and perfect information always have a Nash equilibrium in pure strategies. This can be verified by backward induction. Here we show by example that this result may no longer hold when there is ambiguity. Thus a well-known property of Nash equilibria may no longer hold in CP-EUA. Consider Game A below.

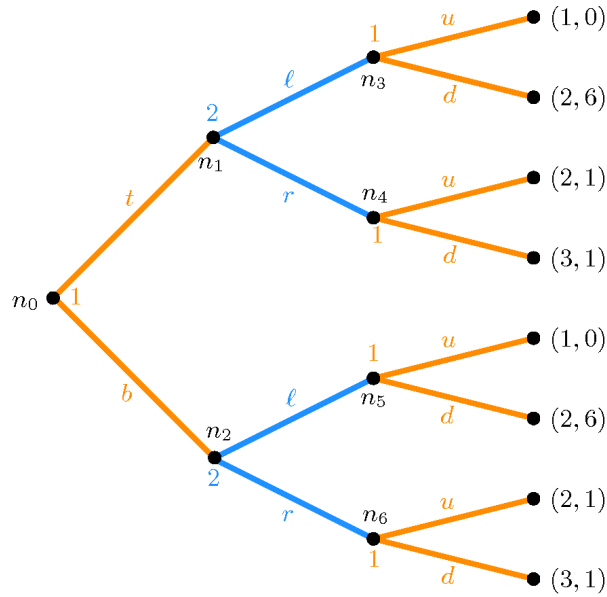


Figure 2: Game A

There are two Nash equilibria in pure strategies which are compatible with backward induction. Player 1 may either play t or b at node n_o , and plays d at nodes n_3 - n_6 . Player 2 chooses ℓ at nodes n_1 and n_2 .

Proposition 4.2 *Assume that Players 1 and 2 have neo-additive preferences with the same parameters, α and δ . For $1 \geq \alpha > \frac{5}{6}$ and $\frac{1}{4} \geq \delta \geq 0$ there is no singleton CP-EUA in Game A.*

Proof. Suppose, if possible, that a singleton equilibrium exists. Then either n_1 or n_2 (but not both) must be on the equilibrium path. Without loss of generality assume that n_1 is on the equilibrium path. First note that d is a dominant strategy for Player 1 at nodes n_3, n_4, n_5 and n_6 . Since CEU preferences respect dominance, Player 1 will choose action d at these nodes. Then player 2's neo-expected payoff

from his actions at node n_1 are:

$$V_2(\ell, \nu_2(\cdot|\alpha, \delta, \pi_1)|n_1) = 6\delta(1-\alpha) + \delta\alpha \cdot 0 + (1-\delta)6 = (1-\delta\alpha)6, \quad V_2(r, \nu_2(\cdot|\alpha, \delta, \pi_1)|n_1) = 1.$$

Thus 2 will choose ℓ at node n_1 provided $\frac{5}{6} \geq \alpha\delta$, which is implied by the assumption $\frac{1}{4} \geq \delta$.

Now consider Player 2's decision at node n_2 . In a singleton CP-EUA if node n_1 is on the equilibrium path node n_2 must be off the equilibrium path. Applying the formula for GBU updating we find that 2's preferences at node n_2 are represented by

$$\alpha \min \{u^2(\ell, u), u^2(r, d)\} + (1-\alpha) \max \{u^2(\ell, u), u^2(r, d)\}.$$

Thus $V^2(\ell, \nu_2|n_2) = \alpha \cdot 0 + (1-\alpha) \cdot 6 = 6 - 6\alpha$ and $V^2(r, \nu_2|n_2) = 1$. Hence 2 will choose action r provided $\alpha > \frac{5}{6}$.

Now consider 1's decision at node n_o . Her neo-expected payoff is:

$$V^1(t, \nu_2|n_o) = \delta(1-\alpha)3 + (1-\delta)2, \quad V^1(b, \nu_2|n_o) = 2\delta\alpha + (1-\delta)3.$$

Hence 1's unique best response is b provided $1 \geq \delta(4-5\alpha)$, which always holds provided $\delta \leq \frac{1}{4}$.

However this contradicts the original assumption that n_1 is on the equilibrium path. The result follows. ■

The parameter restrictions imply that both players perceive positive degrees of ambiguity and have high levels of ambiguity-aversion.¹⁵ The crucial feature of this example is that Player 2 has different ordinal preferences at node n_2 depending whether or not it is on the equilibrium path. Ambiguity-aversion causes Player 2 to choose the relatively safe action r off the equilibrium path. However, on the equilibrium path, he is prepared to accept an uncertain chance of a higher pay-off since he perceives this as less ambiguous. The difference between choices on and off the equilibrium path prevents us from applying the usual backward induction logic.¹⁶

¹⁵ Ambiguity-aversion is not crucial. One may construct a similar example where players have high degrees of ambiguity-loving.

¹⁶ This game is non-generic since it is symmetric. However the example is robust since if a small perturbation were applied to all the pay-offs the game would cease to be symmetric. In this case, similar reasoning would still imply the non-existence of a pure equilibrium.

5 The Centipede Game

In this section, we apply our analysis to the centipede game. This is a two-player game with perfect information. One aim is to see whether strategic ambiguity can contribute to explaining observed behaviour. In this section we shall assume that both players have neo-additive preferences.¹⁷ The centipede game was introduced by Rosenthal (1981) and studied in laboratory experiments by McKelvey and Palfrey (1992). A survey of subsequent experimental research can be found in Krokow, Colman, and Pulford (2016).

5.1 The Game

The centipede game may be described as follows. There are two people, Player 1 (she) and Player 2 (he). Between them is a table which contains $2M$ one-pound coins and a single two-pound coin. They move alternately. At each move there are two actions available. The player whose move it is may either pick up the two-pound coin in which case the game ends; or (s)he may pick up two one-pound coins keep one and give the other to his/her opponent; in which case the game continues. In the final round there is a single two-pound coin and two one-pound coins remaining. Player 2, who has the move, may either pick up the two-pound coin, in which case the game ends and nobody gets the one-pound coins; or may pick up the two one-pound coins keep one and give the other to his opponent, in which case the opponent also gets the two pound coin and the game ends. We label an action that involves picking up two one-pound coins by r (right) and an action of picking up the two-pound coin by d (down). The diagram below shows the final four decision nodes.

A standard backward induction argument establishes that there is a unique iterated dominance equilibrium. At any node the player, whose move it is, picks up the 2-pound coin and ends the game. There are other Nash equilibria. However these only differ from the iterated dominance equilibrium off the equilibrium path.

¹⁷One might criticise these preferences on the grounds that they only allow the best and worst outcomes to be over-weighted but do not allow over-weighting of other outcomes. In many cases the worst outcome is death. However it is likely that individuals would also be concerned about other bad outcomes such as serious injury and/or large monetary losses. Thus in many cases individuals may over-weight a number of bad outcomes rather than just the very worst outcome. Despite this potential problem, we believe this model is suitable for application to strategic situations in particular the centipede game. Our reason is that this game has focal best and worst outcomes, that is, the high payoff at the end and the low payoff from stopping the game.

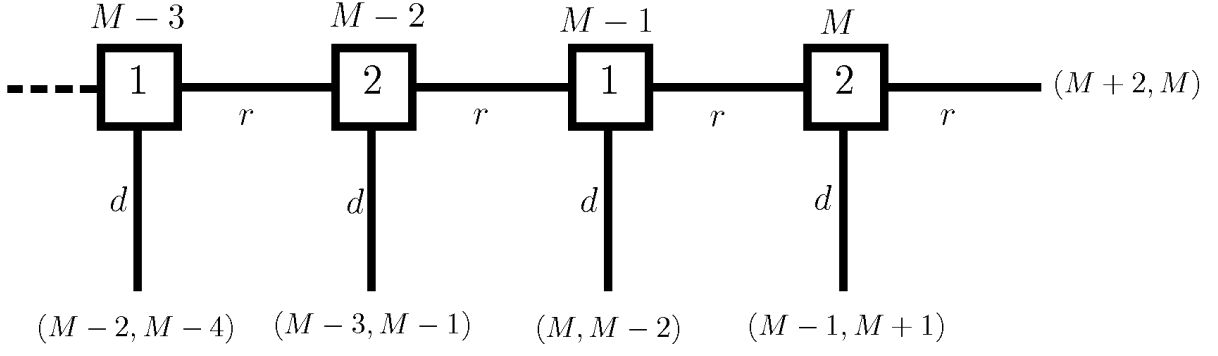


Figure 3: Last stages of the centipede game

5.2 Notation

Given the special structure of the centipede game, we can simplify our notation. The tree can be identified with the non-terminal nodes $H = \{1, \dots, M\}$. For simplicity we shall assume that M is an even number. The set of non-terminal nodes can be partitioned into the two player sets $H_1 = \{1, 3, \dots, M-1\}$ and $H_2 = \{2, 4, \dots, M\}$. It will be a maintained hypothesis that $M \geq 4$.

Strategies and Pay-offs A (pure) strategy for player i is a mapping $s_i : H_i \rightarrow \{r, d\}$. Given a strategy combination (s_1, s_2) set $m(s_1, s_2) := 0$ if d is never played, otherwise set $m(s_1, s_2) := m' \in H$ where m' is the first node where action d is played. The pay-off of strategy combination (s_1, s_2) is: $u_1(s_1, s_2) = M + 2$, if $m(s_1, s_2) = 0$; $u_1(s_1, s_2) = m(s_1, s_2) + 1$ if $m(s_1, s_2)$ is odd and $u_1(s_1, s_2) = m(s_1, s_2) - 1$ otherwise. Likewise $u_2(s_1, s_2) = M$, if $m(s_1, s_2) = 0$; $u_2(s_1, s_2) = m(s_1, s_2) + 1$ if $m(s_1, s_2)$ is even; $u_2(s_1, s_2) = m(s_1, s_2) - 1$ otherwise. For each player $i = 1, 2$ and each node ρ in H_i , let s_i^ρ denote the threshold strategy defined as $s_i^\rho(m) = r$, for $m < \rho$; and $s_i^\rho(m) = d$ for $m \geq \rho$. Let s_i^∞ denote the strategy to play r always. Since threshold strategies s_i^ρ , $1 \leq \rho \leq M-1$, are weakly dominant they deserve special consideration. As we shall show, all equilibrium strategies are threshold strategies.

Subgames and Continuation Strategies For any node $m' \in H$ set $m(s_1, s_2 | m') := 0$ if d is not played at m' or thereafter. Otherwise set $m(s_1, s_2 | m') := m'' \in \{m', \dots, M\}$ where m'' is the first node where action d is played in the subgame starting at m' . We will write $u^i(s_1, s_2 | m')$ to denote the pay-off of the continuation strategy from node m' , that is, $u^1(s_1, s_2 | m') = M + 2$ if $m(s_1, s_2 | m') = 0$; $u_1(s_1, s_2 | m') = m(s_1, s_2 | m') + 1$ if $m(s_1, s_2 | m')$ is odd; $u_1(s_1, s_2 | m') = m(s_1, s_2 | m') - 1$ otherwise. Similarly define $u_2(s_1, s_2 | m') = M$ if $m(s_1, s_2 | m') = 0$; $u_2(s_1, s_2 | m') = m(s_1, s_2 | m') + 1$ if

$m(s_1, s_2 \mid m')$ is even; $u_2(s_1, s_2 \mid m') = m(s_1, s_2 \mid m') - 1$ otherwise.

5.3 Consistent Planning Equilibria Under Ambiguity

In this section we characterise the CP-EUA of the centipede game with symmetric neo-expected payoff maximizing players. That is, throughout this section we take Γ to be an M stage centipede game, where M is an even number no less than 4, and in which both players are neo-expected payoff maximizers with $\delta_1 = \delta_2 = \delta \in [0, 1]$ and $\alpha_1 = \alpha_2 = \alpha \in [0, 1]$.

There are three possibilities, cooperation continues until the final node, there is no cooperation at any node or there is a mixed equilibrium. As we shall show that a mixed equilibrium also involves a substantial amount of cooperation. The first proposition shows that if there is sufficient ambiguity and players are sufficiently optimistic the equilibrium involves playing “right” until the final node. At the final node Player 2 chooses “down” since it is a dominant strategy.

Proposition 5.1 *For $\delta(1 - \alpha) \geq \frac{1}{3}$, there exists a CP-EUA $\langle \nu_1(\cdot \mid \alpha, \delta, \pi_1), \nu_2(\cdot \mid \delta, \alpha, \pi_2) \rangle$, with $\pi_1(s_2^*) = \pi_2(s_1^*) = 1$ for the strategy profile $\langle s_1^*, s_2^* \rangle$ in which $m(s_1^*, s_2^*) = M$. This equilibrium will be unique provided the inequality is strict.*

This confirms our intuition. Ambiguity-loving preferences can lead to cooperation in the centipede game. To understand this result, observe that $\delta(1 - \alpha)$ is the decision-weight on the best outcome in the Choquet integral. Cooperation does not require highly ambiguity loving preferences. A necessary condition for cooperation is that ambiguity-aversion is not too high i.e. $\alpha \leq \frac{2}{3}$. Such ambiguity-attitudes are not implausible, since Kilka and Weber (2001) experimentally estimate that $\alpha = \frac{1}{2}$.

Recall that players do not cooperate in Nash equilibrium. We would expect that ambiguity-aversion makes cooperation less likely, since it increases the attractiveness of playing down which offers a low but ambiguity-free payoff. The next result finds that, provided players are sufficiently ambiguity-averse, non-cooperation at every node is an equilibrium.

Proposition 5.2 *Let For $\alpha \geq \frac{2}{3}$, there exists a CP-EUA $\langle \nu_1(\cdot \mid \alpha, \delta, \pi_1), \nu_2(\cdot \mid \delta, \alpha, \pi_2) \rangle$, with $\pi_1(s_2^*) = \pi_2(s_1^*) = 1$ for the strategy profile $\langle s_1^*, s_2^* \rangle$, in which $m(s_1^*, s_2^* \mid m') = m'$ at every node $m' \in H$. This equilibrium will be unique provided $\alpha > \frac{2}{3}$.*

It is perhaps worth emphasizing that pessimism must be large in order to induce players to exit at every node. If $\frac{1}{2} < \alpha < \frac{2}{3}$ the players overweight bad outcomes more than they overweight good

outcomes. However non-cooperation at every node is not an equilibrium in this case even though players are fairly pessimistic about their opponents' behaviour.

We proceed to study the equilibria when $\alpha < \frac{2}{3}$ and $\delta(1 - \alpha) < \frac{1}{3}$. This case is interesting, since Kilka and Weber (2001) estimate parameter values for α and δ in a neighbourhood of $\frac{1}{2}$. The next result shows that there is no singleton equilibrium for these parameter values and characterises the mixed equilibria which arise. Interestingly, the equilibrium strategies imply continuation for most nodes. This supports our hypothesis that ambiguity-loving can help to sustain cooperation.

Proposition 5.3 *Assume that $\delta(1 - \alpha) < \frac{1}{3}$ and $\alpha < \frac{2}{3}$. Then:*

1. Γ does not have a singleton CP-EUA;
2. there exists a CP-EUA $\langle \nu_1(\cdot|\alpha, \delta, \pi_1), \nu_2(\cdot|\delta, \alpha, \pi_2) \rangle$ in which,
 - (a) player 1 believes with degree of ambiguity δ that player 2 will choose his strategies with (ambiguous) probability $\pi_1(s_2) = p$ for $s_2 = s_2^M$; $1 - p$, for $s_2 = s_2^{M-2}$; $\pi_1(s_2) = 0$, otherwise, where $p = \frac{\delta(2-3\alpha)}{1-\delta}$;
 - (b) player 2 believes with degree of ambiguity δ that player 1 will choose her strategies with (ambiguous) probability $\pi_2(s_1) = q$ for $s_1 = s_1^{M-1}$; $1 - q$, for $s_1 = s_1^{M-3}$; $\pi_2(s_1) = 0$, otherwise, where $q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}$;
 - (c) The game will end at $M - 2$ with player 2 exiting, at $M - 1$ with player 1 exiting, or at M with player 2 exiting.

Notice that for the profile of admissible capacities $\langle \nu_1(\cdot|\alpha, \delta, \pi_1), \nu_2(\cdot|\delta, \alpha, \pi_2) \rangle$ specified in Proposition 5.3 to constitute a CP-EUA, we require Player 2's "theory" about the "randomization" of Player 1's choice of action at node $M - 1$ should make Player 2 at node $M - 2$ indifferent between selecting either d or r . That is, $M - 1 = (1 - \delta)((1 - q)(M - 2) + q(M + 1)) + \delta(\alpha(M - 2) + (1 - \alpha)(M + 1))$, which solving for q yields,

$$q = \frac{1 - 3\delta(1 - \alpha)}{3(1 - \delta)}. \quad (3)$$

This is essentially the usual reasoning employed to determine the equilibrium 'mix' with standard expected payoff maximizing players.

The situation for Player 1 is different, however, since her perception of the "randomization" undertaken by Player 2 over his choice of action at node $M - 2$ increases the ambiguity Player 1

experiences at node $M - 1$. Given full Bayesian updating, this should generate enough ambiguity for Player 1 so that she is indifferent between her two actions at node $M - 1$ given her “theory” that Player 2 will choose d at node M . More precisely, given the GBU of Player 1’s belief conditional on reaching node $M - 1$, Player 1 should be indifferent between selecting either d or r ; that is, $M = (1 - \delta^{M-1}(1 - \alpha))(M - 1) + \delta^{M-1}(M + 2)$, where $\delta^{M-1} = \frac{\delta}{\delta + (1-\delta)p}$. Solving for p yields,

$$p = \frac{\delta(2 - 3\alpha)}{1 - \delta}. \quad (4)$$

Thus substituting p into the expression above for δ^{M-1} we obtain $\delta^{M-1} = \frac{1}{3(1-\alpha)}$ and $\delta^{M-1}(1 - \alpha) = \frac{1}{3}$, as required.

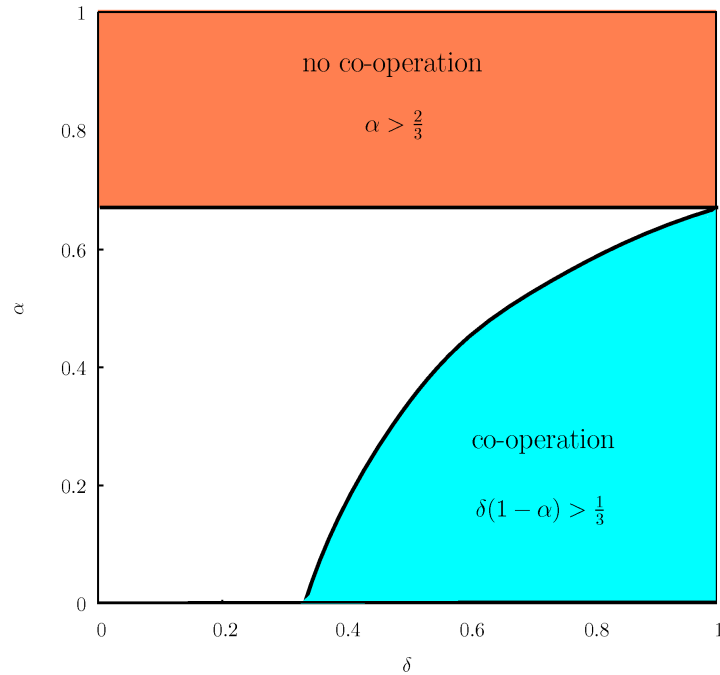
Remark 5.1 *It may at first seem puzzling that as $\delta \rightarrow 0$, we have $q \rightarrow \frac{1}{3}$, $p \rightarrow 0$ and $\delta^{M-1} = \frac{1}{3(1-\alpha)}$, for all $\delta \in \left(0, \frac{1}{3(1-\alpha)}\right)$, and yet for $\delta = 0$ (that is, with standard expected payoff maximizing players) by definition $\delta^{M-1} = 0$ and the unique equilibrium entails both players choosing d at every node, so in particular, $q = p = 0$. This discontinuity, is simply a consequence of the fact that (for fixed δ) $\frac{\delta}{\delta + (1-\delta)p} \rightarrow 1$ as $p \rightarrow 0$ in contrast to an intuition that the updated degree of ambiguity δ^{M-1} should converge to zero as $\delta \rightarrow 0$. Notice that for any (constant) $p > 0$, $\delta \rightarrow 0$ would indeed imply $\frac{\delta}{\delta + (1-\delta)p} \rightarrow 0$. However, to maintain an equilibrium of the type characterized in Proposition 5.3, p has to increase sufficiently fast to maintain $\delta^{M-1} = \frac{1}{3(1-\alpha)}$.*

The discontinuity at $\delta = 0$ is puzzling if the intuition is guided by what one knows about mixed strategies and exogenous randomizations of pay-offs in perturbed games. Moreover this argues against interpreting any limit of a sequence of CP-EUA as $\delta \rightarrow 0$ as constituting a possible refinement of subgame perfect (Nash) equilibrium. Without optimism, there is no discontinuity, but then we are no longer able to explain the observed continuation in centipede games.¹⁸

The mixed equilibria occur when $\alpha < \frac{2}{3}$ and $\delta(1 - \alpha) < \frac{1}{3}$. These parameter values could be described as situations of low ambiguity and low pessimism. On the equilibrium path players are not optimistic enough, given the low degrees of ambiguity, in order to play “right” at all nodes. However low pessimism makes them optimistic enough for playing “right” once they are off the equilibrium path whenever it is not a dominated strategy. This difference in behaviour on and off the equilibrium path is the reason for non-existence of a singleton equilibrium.

In the mixed equilibrium the support of the original beliefs would contain two pure strategies,

¹⁸We thank stimulating comments and suggestions from David Levine and Larry Samuelson for motivating this remark.



which Player 1 has a strict preference between. However at any node where they differ the behaviour strategies which they induce are indifferent. (In these circumstances Player 2 might well experience ambiguity concerning which strategy Player 1 is following.)

The conditions $\alpha \gtrless \frac{2}{3}$ and $\delta(1 - \alpha) \gtrless \frac{1}{3}$ characterise the parameter regions for the three types of *CP-EUA* equilibria. These are shown in figure ???. For strong pessimism ($\alpha > \frac{2}{3}$) players will always exit (red region), while for sufficient optimism and ambiguity ($\delta(1 - \alpha) > \frac{1}{3}$) players will always continue (blue region).

Kilka and Weber (2001) experimentally estimate the parameters of the neo-additive model as $\delta = \alpha = \frac{1}{2}$. For parameters in a neighbourhood of these values only the mixed equilibrium exists. This would be compatible with a substantial degree of cooperation.

6 Bargaining

The alternating offer bargaining game, was developed by Stahl (1972) and Rubinstein (1982), has become one of the most intensely studied models in economics, both theoretically and experimentally. In its shortest version, the ultimatum game, it provides a prime example for a subgame perfect Nash equilibrium prediction at odds with experimental behaviour. The theoretical prediction is of an initial offer of the smallest possible share of a surplus (often zero) followed by acceptance. However experimental results show that the initial offers range around a third of the surplus which is often,

but by far not always, accepted.

In bargaining games lasting for several rounds, the same subgame perfect equilibrium predicts a minimal offer depending on the discount rate and the length of the game, which will be accepted in the first round. Experimental studies show, however, that players not only make larger offers than suggested by the equilibrium but also do not accept an offer in the first round (Roth (1995), p. 293). In a game of perfect information rational agents should not waste resources by delaying agreement.

In order to accommodate the observed delays, game-theoretic analysis has suggested incomplete information about the opponent's pay-offs. Though it can be shown that incomplete information can lead players to reject an offer, the general objection to this explanation advanced in Forsythe, Kennan, and Sopher (1991) remains valid:

“In a series of recent papers, the Roth group has shown that even if an experiment is designed so that each bargainer knows his opponent's utility pay-offs, the information structure is still incomplete. In fact, because we can never control the thoughts and beliefs of human subjects, it is impossible to run a complete information experiment. More generally, it is impossible to run an incomplete information experiment in which the experimenter knows the true information structure. Thus we must be willing to make conjectures about the beliefs which subjects might plausibly hold, and about how they may reasonably act in light of these beliefs. (p.243)”

In this paper we suggest another explanation. Following Luce and Raiffa (1957), p.275, we will assume that players view their opponent's behaviour as ambiguous. Though this uncertainty will be reduced by their knowledge about the pay-offs of the other player and their assumption that opponents will maximise their pay-off, players cannot be completely certain about their prediction. As we will show such ambiguity can lead to delayed acceptance of offers.

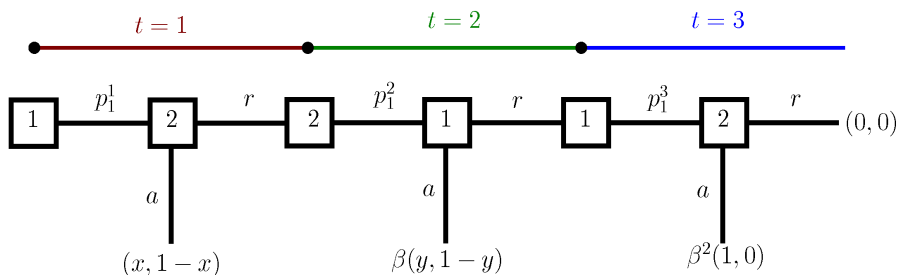


Figure 4: The bargaining game

Consider the bargaining game in figure 4. Without ambiguity, backward induction predicts a split of $\langle \beta(1 - \beta), 1 - \beta(1 - \beta) \rangle$ which will be accepted in period $t = 1$. Delay is not sensible because the

best a player can expect from rejecting this offer is the same payoff (modulo the discount factor) a period later. Depending on the discount factor β the lion's share will go to the player who makes the offer in the last stage when the game turns into an ultimatum game.

Suppose now that a player feels some ambiguity about such equilibrium behaviour. Such ambiguity appears particularly reasonable because the incentives of the two players are delicately balanced. If a player has even a small degree of optimism, (s)he may consider it possible that, by deviating from the expectations of the equilibrium path, the opponent may accept an offer which is more favourable for her/him. Hence, there may be an incentive to “test the water” by deviating from the equilibrium path. Note that this may be a low-cost deviation since, by returning to the previous path, just the discount is lost. Hence, if the discount is low, i.e., the discount factor β is high, a small degree of optimism may suffice.

Decision makers with preferences represented by CEU with a neo-additive capacity who face ambiguity $\delta > 0$ and update their beliefs according to the GBU rule give some extra weight $1 - \alpha$ to the best expected payoff and to the worst expected payoff α and update their beliefs to complete ambiguity, $\delta = 1$, if an event occurs which has probability zero according to their focal (additive) belief π . Hence, off the equilibrium path updates are well-defined but result in complete uncertainty. A decision maker with neo-additive beliefs will evaluate her/his strategies following an out-of-equilibrium and, therefore probability zero event of π by their best and worst outcomes. Hence, from an optimistic perspective, asking for a high share may have a chance of being accepted resulting in some expected gain which can be balanced against the loss of discount associated with a rejection. Whether a strategy resulting in a delay is optimal will depend on the degree of ambiguity δ , the degree of optimism $1 - \alpha$ and the discount factor β .

The following result supports this intuition. With ambiguity and some optimism delayed agreement along the equilibrium path may occur in a CP-EUA equilibrium.

Proposition 6.1 *If $\frac{\alpha - (1 - \alpha)\beta}{1 - (1 - \alpha)[\max\{1 - (1 - \alpha)\beta, \beta\} + \beta]} \geq \delta$, then there exists a CP-EUA*

$\langle (\delta, \alpha, \pi_1), (\delta, \alpha, \pi_2) \rangle$ such that $\pi_1(s_2^) = \pi_2(s_1^*)$ for the following strategy profile (s_1^*, s_2^*) :*

- *at $t = 1$, Player 1 proposes division $\langle x^*, 1 - x^* \rangle = (1, 0)$, Player 2 accepts a proposed division $\langle x, 1 - x \rangle$ if and only if $x \leq 1 - [(1 - \delta)(1 - (1 - \alpha)\beta) + \delta(1 - \alpha)\beta \max\{1 - (1 - \alpha)\beta, \beta\}]$;*
- *at $t = 2$,*

- if Player 1's proposed division in $t = 1$ was $\langle x, 1 - x \rangle = \langle 1, 0 \rangle$, then Player 2 proposes division $\langle y^*, 1 - y^* \rangle = \langle (1 - \alpha)\beta, 1 - (1 - \alpha)\beta \rangle$ and Player 1 accepts a proposed division $\langle y, 1 - y \rangle$ if and only if $y \geq (1 - \alpha)\beta$;
- otherwise, Player 2 proposes division $\langle \hat{y}, 1 - \hat{y} \rangle = \langle 0, 1 \rangle$, Player 1 accepts the proposed division $\langle y, 1 - y \rangle$ if and only if $y \geq (1 - \alpha)\beta$;
- at $t = 3$, Player 1 proposes division $\langle z^*, 1 - z^* \rangle = \langle 1, 0 \rangle$ and Player 2 accepts any proposed division $\langle z, 1 - z \rangle$.¹⁹

7 Relation to the Literature

This section relates the present paper to the existing literature. First we consider our own previous research followed by the relation to other theoretical research in the area. Finally we discuss the experimental evidence.

7.1 Ambiguity in Games

Most of our previous research has considered normal form games e.g. Eichberger and Kelsey (2014). The present paper extends this by expanding the class of games. Two earlier papers study a limited class of extensive form games, Eichberger and Kelsey (2004) and Eichberger and Kelsey (1999). These focus on signalling games in which each player only moves once. Consequently dynamic consistency is not a major problem. Signalling games may be seen as multi-stage games with only two stages and incomplete information. The present paper relaxes this restriction on the number of stages but has assumed complete information. The price of increasing the number of stages is that we are forced to consider dynamic consistency.

Hanany, Klibanoff, and Mukerji (2016) (henceforth HKM) also present a theory of ambiguity in multi-stage games. However they have made a number of different modelling choices. Firstly they consider games of incomplete information. In their model, there is ambiguity concerning the type of the opponent while his/her strategy is unambiguous. In contrast in our theory there is no type space and we focus on strategic ambiguity. However we believe that there is not a vast difference between strategic ambiguity and ambiguity over types. It would be straightforward to add a type space to our model, while HKM argue that strategic uncertainty can arise as a reduced form of a model with type

¹⁹Note this is irrespective of Player 1's proposed division $\langle x, 1 - x \rangle$ at $t = 1$, and irrespective of Player 2's proposed division $\langle y, 1 - y \rangle$ at $t = 2$:

uncertainty. Other differences are that HKM represent ambiguity by the smooth model, they use a different rule for updating beliefs and strengthen consistent planning to dynamic consistency. A cost of this is that they need to adopt a non-consequentialist decision rule. Thus current decisions may be affected by options which are no longer available.

We conjecture that similar results could have been obtained using the smooth model. However, the GBU rule has the advantage that it defines beliefs both on and off the equilibrium path. In contrast, with the smooth rule, beliefs off the equilibrium path are to some extent arbitrary. In addition, we note that since there is little evidence that individuals are dynamically consistent, this assumption is more suitable for a normative model rather than a descriptive theory. As HKM show, dynamic consistency imposes strong restrictions on preferences and how they are updated.

Jehiel (2005) proposes a solution concept which he refers to as analogy-based equilibrium. In this a player identifies similar situations and forms a single belief about his/her opponents behaviour in all of them. These beliefs are required to be correct in equilibrium. For instance in the centipede game a player might consider his/her opponent's behaviour at all the non-terminal nodes to be analogous. Thus (s)he may correctly believe that the opponent will play right with high probability at the average node, which increases his/her own incentive to play right. (The opponent perceives the situation similarly.) Jehiel predicts that either there is no cooperation or cooperation continues until the last decision node. This is not unlike our own predictions based on ambiguity.

What is common between his theory and ours is that there is an "averaging" over different decision nodes. In his theory this occurs through the perceived analogy classes, while in ours averaging occurs via the decision-weights in the Choquet integral. We believe that an advantage of our approach is that the preferences we consider have been derived axiomatically and hence are linked to a wider literature on decision theory.

7.2 Experimental Papers

Our paper predicts that ambiguity about the opponent's behaviour may significantly increase cooperation above the Nash equilibrium level in the centipede game. This prediction is broadly confirmed by the available experimental evidence, (for a survey see Krokow, Colman, and Pulford (2016)).

McKelvey and Palfrey (1992) study 4 and 6-stage centipede games with exponential pay-offs. They find that most players play right until the last 3-4 stages, after which cooperation appears to break down randomly. This is compatible with our results on the centipede game which predict that

cooperation continues until near the end of the game.

Our paper makes the prediction that either there will be no cooperation in the centipede game or that cooperation will continue until the last three stages. In the latter case it will either break down randomly in a mixed equilibrium or break down at the final stage in a singleton equilibrium. In particular the paper predicts that cooperation will not break down in the middle of a long centipede game. This can in principle be experimentally tested. However we would note that it is not really possible to refute our predictions in a 4-stage centipede as used by McKelvey and Palfrey (1992) . Thus there is scope for further experimental research on longer games.²⁰

8 Conclusion

This paper has studied extensive form games with ambiguity. This is done by constructing a thought experiment, where we introduce ambiguity but otherwise make as few changes to standard models as possible. We have proposed a solution concept for multi-stage games with ambiguity. An implication of this is that singleton equilibria may not exist in games of complete and perfect information. This is also demonstrated by the fact that the centipede game only has mixed equilibria for some parameter values. We have shown that ambiguity-loving behaviour may explain apparently counter-intuitive properties of Nash equilibrium in the Centipede game and non-cooperative bargaining. It also produces predictions closer to the available evidence than Nash equilibrium.

8.1 Irrational Types

As mentioned in the introduction, economists have been puzzled about the deviations from Nash predictions in a number of games such as the centipede game, the repeated prisoners' dilemma and the chain store paradox. In the present paper we have attempted to explain this behaviour as a response to ambiguity. Previously it has been common to explain these deviations by the introduction of an "irrational type" of a player. This converts the original game into a game of incomplete information where players take into consideration a small probability of meeting an irrational opponent. An "irrational" player is a type whose pay-offs differ from the corresponding player's pay-offs in the original game. In such modified games of incomplete information, it can be shown that the optimal

²⁰There are a number of other experimental papers on the centipede game. However many of them do not study the version of the game presented in this paper. For instance they may consider a constant sum centipede or study the normal form. It is not clear that our predictions will apply to these games. Because of this, we do not consider them in this review. For a survey see Krokow, Colman, and Pulford (2016).

strategy of a “rational” player may involve imitating the “irrational” player in order to induce more favourable behaviour by his/her opponents. This method is used to rationalize observed behaviour in the repeated prisoner’s dilemma, (Kreps, Milgrom, Roberts, and Wilson (1982)), and in the centipede game (McKelvey and Palfrey (1992)).

There are at least two reasons why resolving the conflict between backward induction and observed behaviour by introducing “irrational” players may not be the complete answer. Firstly, games of incomplete information with “irrational” players predict with small probabilities that two irrational types will confront each other. Hence, this should appear in the experimental data. Secondly, in order to introduce the appropriate “irrational” types, one needs to know the observed deviations from equilibrium behaviour. Almost any type of behaviour can be justified as a response to some kind of irrational opponent. It is plausible that individuals may play tit for tat in the repeated prisoners’ dilemma. Thus an intuitive account of cooperation in the repeated prisoners dilemma may be based on a small probability of facing an opponent who plays tit for tat. However, for most games, there is no such focal strategy which one can postulate for an irrational type to adopt. Theory does not help to determine which irrational types should be considered and hence does not make usually clear predictions in most games. In contrast our approach is based on axiomatic decision theory and can be applied to any multi-stage game.

8.2 Directions for Future Research

In the present paper we have focused on multi-stage games. There appears to be scope for extending our analysis to a larger class of games. For instance, we believe that it would be straightforward to add incomplete information by including a type space for each player. Extensions to multi-player games are possible. If there are three or more players it is usual to assume that each one believes that his/her opponents act independently. At present it is not clear as to how one should best model independence of ambiguous beliefs.²¹

It should also be possible to extend the results to a larger class of preferences. Our approach is suitable for any ambiguity model which maintains a separation between beliefs and tastes and allows a suitable support notion to be defined. In particular both the multiple priors and smooth models of ambiguity fit these criteria. These models represent beliefs by a set of probabilities. A suitable support notion can be defined in terms of the intersection of the supports of the probabilities in this

²¹There are still some differences of opinion among the authors of this paper on this point.

set of beliefs. This is the inner support of Ryan (2002).

A natural application is to finitely repeated games. Such games have some features in common with the centipede game. If there is a unique Nash equilibrium then backward induction implies that there is no scope for cooperation in the repeated game. However in examples, such as the repeated prisoners' dilemma, intuition suggests that some cooperation should be possible.

We believe the model is suitable for applications in financial markets. In particular phenomena such as asset price bubbles and herding have some features in common with the centipede game. Thus we believe that our analysis could be used to study them. In an asset price bubble the value of a security rises above the level, which can be justified by fundamentals. Individuals continue buying even though they know the price is too high since they believe it will rise still further. Thus at every step a buyer is influenced by the perception that the asset price will continue to rise even though they are aware it cannot rise for ever. This is somewhat similar to observed behaviour in the centipede game, where players choose "right" many times even though they know that cooperation cannot last for ever. Reasoning analogous to that of the present paper may be useful to explain an asset price bubble in terms of ambiguity-loving behaviour.

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A Appendix: Proofs

A.1 Existence of Equilibrium

In this appendix we present the proofs of the existence results Proposition 4.1 and Theorem 4.1.

The strategy of our proof is to associate with Γ a modified game Γ' , which is based on the agent-normal form of Γ . We show that Γ' has a Nash equilibrium and then use this to construct a CP-EUA for Γ . The game Γ' has 2θ players.²² A typical player is denoted by $i_{h(t)}$, $h(t) \in H \setminus Z$, $i = 1, 2$. Thus there are 2 players for each decision node in Γ .

The strategy set of Player $i_{h(t)}$, $\Sigma_{i_{h(t)}} = \Delta(A_i^{h(t)})$ is the set of all probability distributions over $A_i^{h(t)}$, with generic element $s_{i_{h(t)}} \in \Sigma_{i_{h(t)}}$, for $i = 1, 2$. Hence in game Γ' , Player $i_{h(t)}$ may choose any mixed strategy over the set of actions $A_i^{h(t)}$. Let $\pi(h(t), \rho)$ denote the probability of history $h(t)$ when the strategy profile is ρ . This is calculated according to the usual rules for reducing compound lotteries to simple lotteries. We shall suppress the arguments and write $\pi = \pi(h(t), \rho)$ when convenient. Let π_i denote the marginal of π on S_{-i} .

The pay-off of Player $i_{h(t)}$ is $\phi_{i_{h(t)}} : \Sigma_{i_{h(t)}} \rightarrow \mathbb{R}$, defined by $\phi_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) = \int u_i(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) d\nu_{i_{h(t)}}$, where ν_i is the neo-additive capacity on S_{-i} defined by $\nu_i(\emptyset) = 0$; $\nu_i(A) = \delta_i(1 - \alpha_i)\pi_i(A)$, $\emptyset \subsetneq A \subsetneq S_{-i}$; $\nu_i(S_{-i}) = 1$ and $\nu_{i_{h(t)}}$ is the GBU update of ν_i conditional on $h(t)$. Since $\nu_{i_{h(t)}}$ is neo-additive:

$$\begin{aligned} \phi_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) &= \delta_{i_{h(t)}}(1 - \alpha_i) M_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) + \delta_{i_{h(t)}}\alpha_i m_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) \\ &\quad + (1 - \delta_{i_{h(t)}}) \mathbf{E}_{\pi_{h(t)}^i} u_i(h(t), a_i^{h(t)}, s_i^{t+1}, s_{-i}^t), \end{aligned} \quad (5)$$

where $M_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) = \max_{s_{-i}^t \in S_{-i}^t} u_i(h^t, a_i^{h(t)}, s_i^{t+1}, s_{-i}^t)$, and $m_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t) = \min_{s_{-i}^t \in S_{-i}^t} u_i(h^t, a_i^{h(t)}, s_i^{t+1}, s_{-i}^t)$. Here $\pi_{h(t)}^i$ denotes the Bayesian update of π^i , given that node $h(t)$ has been reached and $h(t)$ has positive probability. (If $h(t)$ has probability 0, then $\delta_{i_{h(t)}} = 1$ and $\pi_{h(t)}^i$ can be any probability distribution over S_{-i}^t .)

²²Recall $\theta = |H \setminus Z|$ denotes the number of non-terminal histories.

If Player $i_{h(t)}$ plays a mixed strategy then his/her action may be described by a probability distribution ρ over $A_i^{h(t)}$, which is treated as an *ex-ante* randomization. Eichberger, Grant, and Kelsey (2016) show that individuals will be indifferent to ex-ante randomizations. Hence it is evaluated as

$$\sum_{a \in A_i^{h(t)}} \rho(a) \phi_{i_{h(t)}}(a, s_i^{t+1}, s_{-i}^t). \quad (6)$$

It follows that $i_{h(t)}$'s preferences are linear and hence quasi-concave in his/her own strategy.²³

Likewise if one of $i_{h(t)}$'s "future selves" randomizes this is evaluated as

$\sum_{s_i^{t+1} \in S_i^{t+1}} \xi(a) \phi_{i_{h(t)}}(a, s_i^{t+1}, s_{-i}^t)$, where ξ is the probability distribution over S_i^{t+1} induced by future randomizations. This is treated as an ex-ante randomization because it is resolved before the strategic ambiguity arising from the choice of i 's opponent in the relevant subgame. We do not need to specify $i_{h(t)}$'s reaction to randomizations by his/her past selves since these are, by definition, already resolved at the point where the decision is made.

Lemma A.1 *The function $\phi_{i_{h(t)}}(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t)$ is continuous in s , provided $1 \geq \delta_i > 0$, for $i = 1, 2$.*

Proof. Consider equation (5). First note that $\phi_{i_{h(t)}}$ depends directly on s via the $(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t)$ term. It also depends indirectly on s since the degree of ambiguity $\delta_{i_{h(t)}}$ and $\pi_{i_{h(t)}}$ are functions of s .

It follows from our assumptions that the direct relation between s and ϕ is continuous. By equation (6), ϕ is continuous in $\pi_{i_{h(t)}}$. Thus we only need to consider whether $\delta_{i_{h(t)}}$ and $\pi_{i_{h(t)}}$ are continuous in s . Recall that $\pi_{i_{h(t)}}$ is the probability distribution over terminal nodes induced by the continuation strategies s_i^t, s_{-i}^t . Since this is obtained by applying the law of compound lotteries it depends continuously on s . By definition $\delta_i^h = \frac{\delta_i}{\delta_i + (1 - \delta_i)\pi_i(S_{-i}(h))}$. This is continuous in $\pi(h(t))$ provided the denominator is not zero, which is ensured by the condition $\delta_i > 0$. Since $\pi(h(t))$ is a continuous function of s , the result follows. ■

The next result establishes that the associated game Γ' has a standard Nash equilibrium.

Lemma A.2 *The associated game Γ' has a Nash equilibrium provided $1 \geq \delta_i > 0$, for $i = 1, 2$.*

Proof. In the associated game Γ' , the strategy set of a typical player, $i_{h(t)}$, is the set of all probability distributions over the finite set $A_i^{h(t)}$ and is thus compact and convex. By equation (6) the pay-off, $\phi_{i_{h(t)}}$, of Player $i_{h(t)}$ is continuous in the strategy profile σ . Moreover $\phi_{i_{h(t)}}$ is quasi concave in own

²³To clarify these remarks about randomization apply to the modified game Γ' . In the original game Γ there is an equilibrium in beliefs and no randomization is used.

strategy by equation (6). It follows that Γ' satisfies the conditions of Nash's theorem and therefore has a Nash equilibrium in mixed strategies. ■

Proposition 4.1 *Let Γ be a 2-player multi-stage game. Then Γ has at least one CP-EUA for any given parameters $\alpha_1, \alpha_2, \delta_1, \delta_2$, where $1 \leq \alpha_i \leq 0, 0 < \delta_i \leq 1$, for $i = 1, 2$.*

Proof. Let $\rho = \langle \rho^{i_h(t)} : i = 1, 2, h \in H \setminus Z \rangle$ denote a Nash equilibrium of Γ' . We shall construct a CP-EUA $\hat{\sigma}$ of Γ based on ρ . Note that ρ may be viewed as a profile of behaviour strategies in Γ . Let \hat{s} denote the profile of mixed strategies in Γ , which corresponds to ρ , and let π denote the probability distribution which ρ induces over S . (If ρ is an equilibrium in pure strategies then π will be degenerate.) The beliefs of player i in profile $\hat{\sigma}$ are represented by a neo-additive capacity ν_i on S_{-i} , defined by $\hat{\nu}_i(B) = \delta_i(1 - \alpha_i) + (1 - \delta_i)\pi_i(B)$, where $B \subseteq S_{-i}$ and π_i denotes the marginal of π on S_{-i} .

Let $\hat{\nu}$ denote the profile of beliefs $\hat{\sigma} = \langle \hat{\nu}_1, \hat{\nu}_2 \rangle$. We assert that $\hat{\nu}$ is a CP-EUA of Γ . By Remark 4.2 it is sufficient to show that no player can increase his/her current utility by a one-step deviation. Consider a typical player j . Let $\hat{t}, 0 \leq \hat{t} \leq T$, be an arbitrary time period and consider a given history $\hat{h}(\hat{t})$ at time \hat{t} . Let $\hat{a}_j^{\hat{t}} \in A_j^{h(\hat{t})}$ denote an arbitrary action for j at history $\hat{h}(\hat{t})$. Since ρ is an equilibrium of Γ' ,

$$\phi_{j_{\hat{h}(\hat{t})}}(\hat{a}_j^{\hat{t}}, s_j^{\hat{t}+1}, s_{-j}^{\hat{t}}) \geq \phi_{j_{\hat{h}(\hat{t})}}(\hat{a}_j^{\hat{t}}, s_j^{\hat{t}+1}, s_{-j}^{\hat{t}})$$

for any $\hat{a}_j^{\hat{t}} \in \text{supp } \rho(j) = \text{supp } \hat{\nu}_j^{\hat{h}(\hat{t})}$, where $\rho(j)$ denotes the marginal of ρ on $A_j^{h(\hat{t})}$. Without loss of generality we may assume that $\hat{a}_j^{\hat{t}} = \hat{s}_j^{\hat{h}(\hat{t})} \in \text{supp } \hat{\nu}_j$. By definition $\phi_{j_{\hat{h}(\hat{t})}}(\hat{a}_j^{\hat{t}}, s_j^{\hat{t}+1}, s_{-j}^{\hat{t}}) = \int u_j(\hat{s}_j^{\hat{h}(\hat{t})}, \hat{s}_{\langle j, 1 \rangle}^{t+1}, s_{-j}^t) d\nu_j^{\hat{h}(\hat{t})}$, where $\nu_j^{\hat{h}(\hat{t})}$ is the GBU update of ν_j conditional on $\hat{h}(\hat{t})$. Since the behaviour strategy $\hat{s}_j^{\hat{h}(\hat{t})}$ is by construction a best response at $\hat{h}(\hat{t})$, $\int u_j(\hat{s}_j^{\hat{h}(\hat{t})}, \hat{s}_j^{t+1}, s_{-j}^t) d\nu_j^{\hat{h}(\hat{t})} \geq \int u_j(\hat{a}_j^{\hat{t}}, \hat{s}_j^{t+1}, s_{-j}^t) d\nu_j^{\hat{h}(\hat{t})}$, which establishes that $\hat{s}_j^{\hat{h}(\hat{t})}$ yields a higher pay-off than the one step deviation to $\hat{a}_j^{\hat{t}}$. Since both j and $\hat{h}(\hat{t})$ were chosen arbitrarily this establishes that it is not possible to improve upon $\hat{\sigma}$ by a one-step deviation. Hence by Remark 4.2, $\hat{\sigma}$ is a CP-EUA of Γ . ■

A.2 The Centipede Game Proofs

Proof of Proposition 5.1 We shall proceed by (backward) induction. The final node is M . At this node 2 plays d_M , which is a dominant strategy. This yields pay-offs $\langle M - 1, M + 1 \rangle$.

Node $M - 1$ Now consider Player 1's decision at node $M - 1$. Assume that this node is on the equilibrium path.²⁴ The (Choquet) expected value of her pay-offs are:

²⁴This will be proved once we have completed the induction.

$$V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)) = M,$$

$$V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \delta(1-\alpha)(M+2) + \delta\alpha(M-1) + (1-\delta)(M-1) = M-1+3\delta(1-\alpha).$$

Thus r_{M-1} is a best response provided $\delta(1-\alpha) \geq \frac{1}{3}$. To complete the proof we need to show that $s_i^*(\rho) = r_\rho$ is the preferred action for all $\rho \in H_i$ with $i = 1, 2$.

Inductive step Consider node ρ . Assume ρ is on the equilibrium path. We make the inductive hypothesis that r_κ is a best response at all nodes κ , $\rho < \kappa < M-1$. There are two cases to consider:

Case 1 $\rho = 2\tau + 1$ Player 1 has the move. The expected value of her pay-offs are:

$$V_1^\rho(d_\rho|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \rho + 1, \quad V_1^\rho(r_\rho|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \delta(1-\alpha)(M+2) + \delta\alpha\rho + (1-\delta)(M-1).$$

Thus r_ρ is a best response provided $(1-\delta\alpha)M + 2\delta(1-\alpha) - (1-\delta) \geq (1-\delta\alpha)\rho + 1$,

$\Leftrightarrow (1-\delta\alpha)(M-\rho) \geq 2-3\delta+2\delta\alpha$. Now $(1-\delta\alpha)(M-\rho) \geq 3(1-\delta\alpha) = 3-3\delta\alpha$. Thus a sufficient condition is, $3-3\delta\alpha \geq 2-3\delta+2\delta\alpha \Leftrightarrow 1 \geq 2\delta\alpha-3\delta(1-\alpha)$, which always holds since $\delta(1-\alpha) \geq \frac{1}{3}$.

Case 2 $\rho = 2\tau, 1 \leq \tau \leq M-2$. Player 2 has the move. The expected value of her pay-offs are:

$$V_2^\rho(d_\rho|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \rho + 1, \quad V_2^\rho(r_\rho|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \delta(1-\alpha)(M+1) + \delta\alpha\rho + (1-\delta)(M+1).$$

Thus r_ρ is a best response provided: $(1-\delta\alpha)(M+1) + \delta\alpha\rho \geq \rho + 1 \Leftrightarrow (1-\delta\alpha)(M-\rho) \geq \delta\alpha$.

Since $M-\rho \geq 2$, a sufficient condition for r_ρ to be a best response is $2-2\delta\alpha \geq \delta\alpha \Leftrightarrow \delta\alpha \leq \frac{2}{3}$. Now $\delta(1-\alpha) \geq \frac{1}{3} \Rightarrow (1-\delta) + \delta\alpha \leq \frac{2}{3}$. Thus r_ρ is a best response under the given assumptions.

Having considered all possible cases we have established the inductive step. Thus there exists an equilibrium in which cooperation continues until the final node when $\delta(1-\alpha) \geq \frac{1}{3}$. Moreover if $\delta(1-\alpha) > \frac{1}{3}$ then r_ρ is the unique best response at the relevant nodes, which establishes uniqueness of the equilibrium. \blacksquare

Proof of Proposition 5.2 We shall proceed by (backward) induction. At node the final node M , d_M is a dominant strategy. Now consider the decision at node $M-1$. Assume this node is off the equilibrium path. (This assumption will be confirmed when the proof is complete.)

Recall that, at nodes off the equilibrium path, the GBU updated preferences may be represented by the function, $W(a) = (1-\alpha)\max u(a) + \alpha\min u(a)$. Player 1 moves at this node. Her (Choquet) expected pay-offs from continuing are:

$$\begin{aligned}
V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)) &= (1-\alpha)(M+2) + \alpha(M-1) = M+2-3\alpha < M \\
&= V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)), \quad \text{since } \alpha \geq \frac{2}{3} \text{ we may conclude that } d_{M-1} \text{ is a best response.}
\end{aligned}$$

Inductive step The inductive hypothesis is that d_κ is a best response at all nodes $2M-1 > \kappa > \rho > 1$. Now consider the decision at node ρ . Assume this node is off the equilibrium path and that player i has the move at node ρ . His/her expected pay-offs are given by:

$$V_i^\rho(d_\rho|\nu_1(\cdot|\alpha, \delta, \pi_i)) = \rho + 1, \quad V_i^\rho(r_\rho|\nu_1(\cdot|\alpha, \delta, \pi_i)) = (1-\alpha)(\rho+3) + \alpha\rho = \rho + 3(1-\alpha).$$

(Player i perceives no ambiguity about his/her own move at node $\rho+2$.) Thus d_ρ is a best response provided: $1 \geq 3(1-\alpha) \Leftrightarrow \alpha \geq \frac{2}{3}$. This establishes by induction that d_ρ is a best response at all nodes ρ such that $1 < \rho \leq 2M$, provided they are off the equilibrium path.

Node 1 Finally we need to consider the initial node, which is different since it is on the equilibrium path. Player 1 has to move at this node. Her expected pay-offs are:

$$V_1^1(d_1|\nu_1(\cdot|\alpha, \delta, \pi_1)) = 2, \quad V_1^1(r_1|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \delta(1-\alpha)4 + \delta\alpha + (1-\delta) = 3\delta(1-\alpha) + 1.$$

Since $\alpha \geq \frac{2}{3}$ implies $\delta(1-\alpha) \leq \frac{1}{3}$, which implies that d_1 is a best response at the initial node. This confirms our hypothesis that subsequent nodes are off the equilibrium path. The result follows. ■

Proposition A.1 For $\alpha < \frac{2}{3}$ and $\delta(1-\alpha) < \frac{1}{3}$, Γ does not have a singleton CP-EUA.

The proof of Proposition A.1 follows from Lemmas A.3, A.4 and A.5.

Lemma A.3 Assume $\alpha < \frac{2}{3}$ then at any node $\tau, M > \tau \geq 2$, which is off the equilibrium path, the player to move at node τ will choose to play right, i.e. r_τ .

Proof. We shall proceed by (backward) induction. To start the induction consider the decision at node $M-1$.

Node $M-1$ Assume this node is off the equilibrium path. Player 1 has the move. Her expected pay-off from continuing is, $V^1(r_{2M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)) = (1-\alpha)(M+2) + \alpha(M-1) = M+2-3\alpha \geq M = V^1(d_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1))$, since $\alpha < \frac{2}{3}$.

Inductive step Since τ is off the equilibrium path so are all nodes which succeed it. The inductive hypothesis is that r_κ is a best response at all nodes $M-1 > \kappa > \rho \geq \tau$. Now consider the decision at node ρ . First assume that Player 2 has the move at node ρ , which implies that ρ is an even number. His expected pay-offs are given by:

$$V^2(d_\rho|\nu_2(\cdot|\alpha, \delta, \pi_2)) = \rho + 1, \quad V_2^\rho(r_\rho|\nu_2(\cdot|\alpha, \delta, \pi_2)) = (1 - \alpha)(M + 1) + \alpha\rho.$$

Thus r_ρ is a best response provided: $(1 - \alpha)(M + 1 - \rho) \geq 1$. Note that $\rho \leq M - 2$ and since $\alpha < \frac{2}{3}$, $3(1 - \alpha) > 1$ hence $(1 - \alpha)(M + 1 - \rho) \geq 1$ which establishes that right is a best response in this case.

Now assume that Player 1 has the move at node ρ . Her expected pay-offs are given by:

$$V_1^\rho(d_\rho|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \rho + 1, \quad V_1^\rho(r_\rho|\nu_1(\cdot|\alpha, \delta, \pi_1)) = (1 - \alpha)(M + 2) + \alpha\rho.$$

The analysis for Player 2 shows that r_ρ is also a best response in this case.

This establishes the inductive step. The result follows. ■

Lemma A.4 *Assume $\alpha < \frac{2}{3}$ and $\delta(1 - \alpha) < \frac{1}{3}$. Let Γ be a M -stage centipede game, where $M \geq 4$. Then there does not exist a singleton equilibrium in which Player 1 plays d_1 at node 1.*

Proof. Suppose if possible that there exists a singleton equilibrium in which Player 1 plays d_1 at node 1. Then all subsequent nodes are off the equilibrium path. By Lemma A.3 players will choose right, r_ρ , at such nodes. Given this Player 1's expected pay-offs at node 1 are:

$$V_1^1(d_1|\nu_1(\cdot|\alpha, \delta, \pi_1)) = 2,$$

$$V_1^1(r_1|\nu_1(\cdot|\alpha, \delta, \pi_1)) = \delta(1 - \alpha)(M + 2) + \delta(1 - \alpha) \cdot 1 + (1 - \delta)(M - 1).$$

Thus d_1 is a best response if $2 \geq (1 - \delta\alpha)M + 3\delta(1 - \alpha) + \delta - 1$

$$\Leftrightarrow 3 - 3\delta(1 - \alpha) - \delta \geq (1 - \delta\alpha)M \geq 4 - 4\delta\alpha, \quad \text{since } M \geq 4.$$

$$\Leftrightarrow 7\delta\alpha - 4\delta \geq 1. \text{ Since } \alpha < \frac{2}{3}, 1 > \frac{2}{3}\delta = \frac{14}{3}\delta - 4\delta > 7\delta\alpha - 4\delta, d_1 \text{ cannot be a best response at node 1.}$$

■

Lemma A.5 *Assume $\alpha < \frac{2}{3}$ and $\delta(1 - \alpha) < \frac{1}{3}$. Let Γ be a M -stage centipede game, where $M \geq 4$. Then there does not exist a singleton equilibrium in which Player 1 plays r_1 at node 1.*

Proof. Suppose if possible such an equilibrium exists. Let τ denote the first node at which a player fails to cooperate. Since cooperation will definitely break down at or before node M , we know that $1 < \tau \leq M$. There are three possible cases to consider.

Case 1 $\tau = M$. Consider the decision of Player 1 at node $M - 1$. Her expected pay-offs are:

$$V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)) = M,$$

$$\begin{aligned} V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1)) &= \delta(1 - \alpha)(M + 2) + \delta\alpha(M - 1) + (1 - \delta)(M - 1) \\ &= M + 3\delta(1 - \alpha) - 1. \end{aligned}$$

However $\delta(1 - \alpha) < \frac{1}{3}$, implies that r_{M-1} is not a best response. Thus it is not possible that $\tau = M - 1$.

Case 2 $\tau \leq M - 1$ and $\tau = 2k + 1$ This implies that Player 1 moves at node τ and that all subsequent nodes are off the equilibrium path. By Lemma A.3, right is a best response at these nodes.

Hence her expected pay-offs are:

$$V_1^\tau(d_\tau | \nu_1(\cdot | \alpha, \delta, \pi_1)) = \tau + 1, \quad V_1^\tau(r_\tau | \nu_1(\cdot | \alpha, \delta, \pi_1)) = \delta(1 - \alpha)(M + 2) + \delta\alpha\tau + (1 - \delta)(M - 1).$$

Thus d_τ is a best response provided,

$$\tau + 1 \geq (1 - \delta\alpha)M + 2\delta(1 - \alpha) + \delta\alpha\tau - (1 - \delta) \Leftrightarrow 2 + 2\delta\alpha - 3\delta \geq (1 - \delta\alpha)(M - \tau).$$

Since $M - \tau \geq 3$, a necessary condition for d_τ to be a best response is:

$5\delta\alpha - 3\delta \geq 1 \Leftrightarrow 2\delta\alpha \geq 1 + 3\delta(1 - \alpha)$. However the latter inequality cannot be satisfied since $\alpha < \frac{2}{3}$, $3\delta(1 - \alpha) > \delta \geq \delta\alpha$, and $\delta\alpha < 1$. Hence it is not possible that τ is odd.

Case 3 $\tau \leq M - 2$ and $\tau = 2k$ This implies that Player 2 moves at node τ and that all subsequent nodes are off the equilibrium path. His pay-offs are

$$V_2^\tau(d_\tau | \nu_1(\cdot | \alpha, \delta, \pi_2)) = \tau + 1, \quad V_2^\tau(r_\tau | \nu_1(\cdot | \alpha, \delta, \pi_2)) = \delta(1 - \alpha)(M + 1) + \delta\alpha\tau + (1 - \delta)(M + 1).$$

For d_τ to be a best response we need

$$\tau + 1 \geq (1 - \delta\alpha)M + 1 - \delta\alpha + \delta\alpha\tau \Leftrightarrow \delta\alpha \geq (1 - \delta\alpha)(M - \tau).$$

Since $\alpha < \frac{2}{3}$, $\delta\alpha < \frac{2}{3}$ and $(1 - \delta\alpha) > \frac{1}{3}$. In addition $M - \tau \geq 2$, which implies that this inequality can never be satisfied. Thus τ cannot be even.

Having considered all possibilities we may conclude that there is no pure equilibrium where Player 1 plays r_1 at node 1. ■

Proposition 5.3 *If $\alpha < \frac{2}{3}$ and $\delta(1 - \alpha) < \frac{1}{3}$, there is a CP-EUA in which the support of Player 1's (resp. 2's) beliefs is $\{s_2^M, s_2^{M-2}\}$ (resp. $\{s_1^{M-1}, s_1^{M+1}\}$). The game will end at $M - 2$ or M with Player 2 exiting or at $M - 1$ with Player 1 exiting.*

Proof. Assume that Player 1's beliefs are a neo additive capacity based on the additive probability π_1 defined by $\pi_1(s_2^M) = p$, for $\pi_1(s_2^{M-2}) = 1 - p$, $\pi_1(s_2) = 0$ otherwise, where $p = \frac{\delta(2-3\alpha)}{(1-\delta)}$. Likewise assume that Player 2's beliefs are a neo additive capacity based on the additive probability π_2 defined by $\pi_2(s_1^{M+1}) = q$, for $\pi_2(s_1^{M-1}) = 1 - q$, $\pi_2(s_1) = 0$ otherwise, where $q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}$.

First consider the updated beliefs. At any node ρ , $0 \leq \rho \leq M - 2$, the updated beliefs are a neo additive capacity with the same δ and α . The new probability π' is the restriction of the prior probability π to the set of continuation strategies. At node $M - 1$, the GBU rule implies the updated beliefs are a neo additive capacity with the same α , the updated δ given by $\delta'_1 := \frac{\delta}{\delta + (1-\delta)p}$. The

updated π assigns probability 1 to s_2^M .²⁵ We do not need to specify the beliefs at node M . Player 2 has a dominant strategy at this node which he always will choose.

Player 2 Consider Player 2's decision at node $M - 2$. His actions yield pay-offs,

$V_2^{M-2}(d_{M-2}|\nu_1(\cdot|\alpha, \delta, \pi_2))$ and $V_2^{M-2}(a_{M-2}|\nu_1(\cdot|\alpha, \delta, \pi_2))$. For both to be best responses they must yield the same expected pay-off, which occurs when $q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}$, by equation (3). Since $\delta(1-\alpha) \leq \frac{1}{3}$ by hypothesis, $q \geq 0$. In addition $q \leq 1 \Leftrightarrow 3(1-\delta) \geq 1-3\delta(1-\alpha)$ which holds since $\alpha < \frac{2}{3}$ implies $(1-\alpha) > \frac{1}{3}$ thus $1-3\delta(1-\alpha) < 3-3\delta$.

Player 1 Now consider Player 1's decision at node $M - 1$. His updated δ is given by: $\delta'_1 = \delta(p | m') := \frac{\delta}{\delta+(1-\delta)p}$. His strategies yield pay-offs $V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha, \delta, \pi_1))$ and $V_1^{M-1}(d_{M+1}|\nu_1(\cdot|\alpha, \delta, \pi_1))$. For both of these strategies to be best responses they must yield the same expected utility which occurs when $p = \frac{2\delta-3\delta\alpha}{1-\delta}$, by equation (4). Since $\frac{2}{3} > \alpha$ by hypothesis, $p \geq 0$. Moreover $1 \geq p \Leftrightarrow 1-\delta \geq 2\delta-3\delta\alpha \Leftrightarrow \frac{1}{3} \geq \delta(1-\alpha)$, which also holds by hypothesis. ■

A.3 Bargaining

Proof of Proposition 6.1 To establish that the strategy profile specified in the statement of Proposition 6.1 constitutes a CP-EUA, we work backwards from the end of the game.

At $t = 3$. Since this is the last period, it is a best response for player 2 to accept *any* proposed division $\langle z, 1-z \rangle$ by player 1. So in any consistent planning equilibrium, player 1 will propose a $\langle 1, 0 \rangle$ division after any history.

At $t = 2$. If player 1 rejects player 2's proposed division $\langle y, 1-y \rangle$, then the game continues to period 3 where we have established player 1 gets the entire cake. However, the neo-expected payoff to player 1 in period 2 of *rejecting* 2's offer is $(1-\alpha)\beta$. This is because by rejecting player 2's proposed division, player 1 is now in an "off-equilibrium" history h , with $\delta(h) = 1$. Thus her neo-expected payoff is simply an $(\alpha, 1-\alpha)$ -weighting of the best and worst outcome that can occur (given his own continuation strategy). Hence for player 2's proposed division $\langle y, 1-y \rangle$ to be accepted by player 1 requires $y \geq (1-\alpha)\beta$.

So the neo-expected payoff for player 2 of proposing the division $\langle (1-\alpha)\beta, 1-(1-\alpha)\beta \rangle$ in round 2 is $(1-\delta_2)(1-(1-\alpha)\beta) + \delta_2(1-\alpha)\max\{1-(1-\alpha)\beta, \beta\}$, where $\delta_2 = \delta/(\delta+(1-\delta)\pi)$ if $\langle x, 1-x \rangle = \langle 1, 0 \rangle$, =1 otherwise. Notice that the 'best' outcome for player 2 is either $(1-(1-\alpha)\beta)$

²⁵The form of the updated beliefs follows from the formulae in Eichberger, Grant, and Kelsey (2010).

or possibly β which is what she would secure if player 1 rejected his proposed division and then followed that by proposing in period 3 the (extraordinarily generous) division $\langle 0, 1 \rangle$.

If player 2 proposes $\langle y, 1 - y \rangle$ with $y < (1 - \alpha)\beta$, then according to our putative equilibrium, player 1 rejects and player 2 receives a pay-off of zero. However, his neo-expected payoff of that is $(1 - \alpha) \max\{1 - y, \beta\}$, where $1 - y$ is what he would get if player 1 actually accepted his proposed division. Thus his best deviation is the proposal $\langle y, 1 - y \rangle = \langle 0, 1 \rangle$.

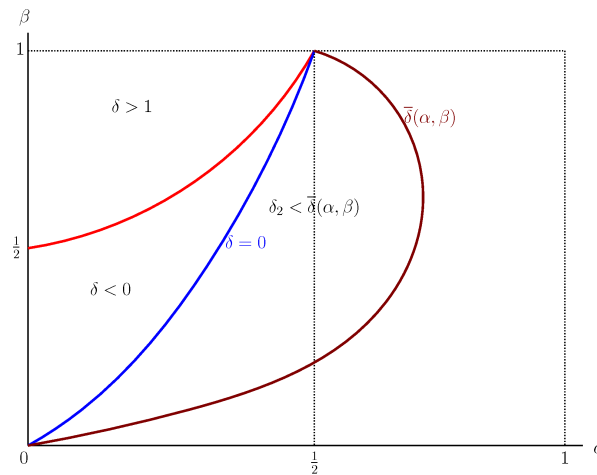
So we require:

$$(1 - \delta_2)(1 - (1 - \alpha)\beta) + \delta_2(1 - \alpha) \max\{1 - (1 - \alpha)\beta, \beta\} \geq (1 - \alpha). \quad (7)$$

This is equivalent to

$$\bar{\delta}(\alpha, \beta) := \frac{\alpha - (1 - \alpha)\beta}{1 - (1 - \alpha)[\max\{1 - (1 - \alpha)\beta, \beta\} + \beta]} \geq \delta_2.$$

The following figure shows the region of parameters (α, β) satisfying this constraint.



parameter region: $\delta_2 \leq \bar{\delta}(\alpha, \beta)$

Clearly, when $\delta_2 = 1$ (that is, when we are already off the equilibrium path), (7) cannot hold for any α strictly less than 1. So in that case, player 2's best action is indeed to propose $\langle y, 1 - y \rangle = \langle 0, 1 \rangle$, which is what our putative equilibrium strategy specifies in these circumstances.

At $t = 1$, if player 2 rejects player 1's proposed division $\langle x, 1 - x \rangle$, then the game continues to $t = 2$ in which case he offers player 1 the division $\langle y^*, 1 - y^* \rangle = \langle (1 - \alpha)\beta, 1 - (1 - \alpha)\beta \rangle$, which according to her strategy she accepts.

On the equilibrium path, $\delta_2 = \delta < 1$, the neo-expected payoff for player 2 in period 1 of rejecting player 1's proposed division $\langle x, 1 - x \rangle$ is $(1 - \delta)(1 - (1 - \alpha)\beta)\beta + \delta\alpha$. In this putative equilibrium Player 1 proposes the division $\langle x^*, 1 - x^* \rangle = \langle 1, 0 \rangle$ which player 2 rejects, since she anticipates an offer of $\langle (1 - \alpha)\beta, 1 - (1 - \alpha)\beta \rangle$ in period 2. So her period 1 neo-expected payoff is $(1 - \delta)(1 - \alpha)\beta^2 + \delta(1 - \alpha)$, since the 'best' thing that could happen is player 2 actually accepts her proposed division $\langle 1, 0 \rangle$ today!

Alternatively, player 1 might propose a division $\langle \tilde{x}, 1 - \tilde{x} \rangle$, that would just entice player 2 to accept today. That is, she could propose $\langle \tilde{x}, 1 - \tilde{x} \rangle$ where

$$\tilde{x} = 1 - [(1 - \delta)(1 - (1 - \alpha)\beta) + \delta(1 - \alpha)\beta \max\{1 - (1 - \alpha)\beta, \beta\}].$$

The neo-expected payoff for player 1 of this deviation is: $\delta(1 - \alpha)\max\{\tilde{x}, \beta\}$. In order for it not to be "profitable", we require $(1 - \delta)(1 - \alpha)\beta^2 + \delta(1 - \alpha) \geq \delta(1 - \alpha)\max\{\tilde{x}, \beta\}$, which holds since $\max\{\tilde{x}, \beta\} < 1$. ■