Randomization and Dynamic Consistency*

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Abstract

Raiffa (1961) has suggested that ambiguity aversion will cause a strict preference for randomization. We show that dynamic consistency implies that individuals will be indifferent to *ex-ante* randomizations. On the other hand, it is possible for a dynamically-consistent ambiguity averse preference relation to exhibit a strict preference for some *ex-post* randomizations. We argue that our analysis throws some light on the recent debate on the status of the smooth model of ambiguity We show that this rests on whether the randomizations implicit in the set-up are viewed as being resolved before or after the (ambiguous) uncertainty.

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1 Introduction

1.1 Background

> But why did big shareholders not move to stop over-leveraging before it reached dangerous levels. Why did legislators not demand regulatory intervention? The answer I believe is that they had no sense of Knightian uncertainty. So they had no sense of the possibility of a huge break in housing prices and no sense of the fundamental inapplicability of the risk management models used in the banks.
>
> Risk came to mean volatility over the recent past.

Edmund Phelps, Financial Times 12th May 2007

An ambiguous uncertainty (also known as Knightian Uncertainty) is one for which objective probabilities are not available and individuals cannot or do not assign (conventional) subjective probabilities. As the above quote shows, many economists believe that ambiguity has an important role in financial crises. Ambiguity is also believed to play a central role in entrepreneurship, Knight (1921), and macroeconomic instability, Keynes (1936). Ellsberg (1961) argued that individuals would behave differently when faced with ambiguous uncertainties.[1]

At the same time, a number of economists have maintained that ambiguity will not give rise to observable differences in behaviour. A leading example is Raiffa (1961). We shall illustrate Raiffa's argument in the context of a variant of the 3-colour Ellsberg problem.[2] Consider an urn containing one hundred balls, thirty-three of which are red, while the remaining sixty-seven balls are black and yellow in unknown proportions. One ball is to be drawn at random from the urn. Let R, (respectively, B or Y) denote a 'bet' on red (respectively, black or yellow) which pays £100 if a red (respectively, black or yellow) ball is drawn from the urn and £0 otherwise. Furthermore, let $\bar{R}$, (respectively, $\bar{B}$ or $\bar{Y}$) denote a 'bet' against red (respectively, black or yellow) which pays £0 if a red (respectively, black or yellow) ball is drawn from the urn and £100 otherwise. Reasoning analogous to that of Ellsberg (1961) suggests that many individuals might prefer bets where probabilities are more precisely defined and hence display the preferences $R \succ B$ and $\bar{R} \succ \bar{B}$.

[1] Numerous experiments have confirmed this observation; Camerer and Weber (1992), Trautmann and van de Kuilen (2015).

[2] Raiffa (1961) considers the original composition of the Ellsberg urn with 30 red balls and 60 balls which were either black or yellow. Unlike the standard 3-colour Ellsberg urn, in the modified example, slightly less than one third of the balls are red.

In his reply to Ellsberg (1961), Raiffa (1961) tries to convince people of the "reasonableness" of the axioms of Savage (1954). For this purpose, he asks the reader to consider two options, A and B, which condition the choice of the Ellsberg acts on the outcome of a fair coin flip (Raiffa (1961), p. 694). Denote by $C^{\{h\}}D$ a randomization between a bet C and a bet D corresponding to the conditional bet: bet on colour C if the flipped coin comes up heads, otherwise bet on colour D. Option A corresponds to the act $R^{\{h\}}\bar{R}$ and Option B to the act $B^{\{h\}}\bar{B}$. Behaviour as observed in the Ellsberg paradox, $R \succ B$ and $\bar{R} \succ \bar{B}$, would suggest that Option A dominates Option B, while "accounting by analyzing the implications of the options conditional on the colour of the withdrawn ball" (Raiffa (1961), p. 694) shows that, for each colour drawn from the urn, the outcome is one of the lotteries $100^{\{h\}}0$ or $100^{\{t\}}0$. For a fair coin, one obtains the same lottery conditional on each colour. Hence, in Raiffa (1961)'s opinion, "this reasoning should lead everyone to assert that options A and B are *objectively identical!*" (p. 694).

Though Raiffa (1961) compares two randomizations, Option A and Option B, using his "accounting by analyzing the implications" it is easy to interpret the preferences of an ambiguity averse agent as a preference for randomization. If the individual makes her choice on whether to bet on either a black or a yellow ball, conditional on the flip of a 'fair' coin, $B^{\{h\}}Y$, then the (overall) probability of winning is 67/200, independent of the actual proportion of black and yellow balls in the urn. Hence, she is facing only conventional risk. This would seem to suggest that ambiguity aversion causes individuals to strictly prefer such a randomization to the pure bet R which yields a winning probability of 33/100 and, hence, to the two pure ambiguous acts B or Y as well.

1.2 Ex-Ante versus Ex-Post Randomizations

Raiffa's argument does not appear to make a clear distinction between ex-ante and ex-post randomizations, that is, whether the coin is flipped before or after the ball is drawn from the ambiguous urn. On the one hand, the argument that the individual should flip a coin to determine whether to bet on B or Y seems to imply an ex-ante randomization. On the other hand, the claim that the individual faces the same lottery regardless of the proportion of black and yellow balls, suggests an ex-post randomization is intended. It is quite possible, however, that an individual might react differently to ex-ante and ex-post randomizations, Saito (2013). An individual who followed subjective expected utility theory (henceforth SEU) would be indifferent between the two. However this does not necessarily extend to preferences which are non-linear in the probabilities.

We study these issues in more detail using a model which makes a clear distinction between ex-post and ex-ante randomizations. The model makes explicit the timing of randomizations and state uncertainty. We show that dynamic consistency with respect to ex-ante randomizations implies indifference to randomization. In many respects our argument is quite general. We use a Savage framework. Apart from this, our results are model free in the sense that they do not assume any particular functional form for preferences.

The relation between dynamic consistency and preference for randomization may be explained intuitively as follows. Assume the decision maker is initially indifferent between a pair of ambiguous acts a and b. Let d be an act formed by taking an ex ante $\alpha : (1 - \alpha)$ randomization of a and b and suppose that she prefers d to both a and b. Let c be a constant act which is strictly preferred to a and b but is strictly inferior to d. Then before the randomization the individual prefers d to c. However, after she learns the outcome of the randomizing device she will be holding either a or b, both of which she views as inferior to c. This is a clear violation of dynamic consistency.

We shall argue that indifference to randomization is especially intuitive when the result of the randomization is made known to the individual *before* the resolution of the ambiguous bet. Assume the decision maker dislikes the acts a and b due to ambiguity. We argue that it is implausible that flipping a coin before the uncertainty is resolved will make these acts more attractive. After the flip, the decision maker remains exposed to either the ambiguous bet a or the ambiguous bet b as determined by the coin.

We believe that some recent controversies may be clarified by paying attention to this distinction. As an illustration we consider the debate about the status of the smooth model of ambiguity by Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012) in section 4. A key issue in this discussion is the value of diversification. We show that diversification is valuable if it is achieved by ex-post randomization but using ex-ante randomization will not diversify ambiguous risks.

1.3 Previous Literature

This paper builds on our previous research. Eichberger and Kelsey (1996) showed that in the convex capacity model of Schmeidler (1989), there is a strict preference for randomization in the context of the Anscombe-Aumann (henceforth AA) framework, where outcomes are lotteries and so randomizations take place after the realization of the subjectively uncertain state. The preference for randomizations vanishes provided outcomes and acts are modelled in the Savage framework. To the contrary, this model implies that decision makers are *indifferent* to randomization.[3]

Eichberger and Kelsey (1996) also show that in the maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989), individuals may (or may not) have a strict preference for randomization. Klibanoff (2001) presents a preference-based notion of stochastically independent randomizing device. He shows that this notion is not compatible with representing beliefs by a convex capacity. In his view, MEU provides a better theory of ambiguity aversion since a preference for randomization remains a possibility.

These differences in preferences about ambiguous acts in the AA and Savage frameworks suggest that the sequence of the realizations of the outcomes of the randomizing device and the states matters. We argued above that Raiffa (1961) is assuming that the realization of the randomizing device comes first. In the AA framework, the randomizations are treated as a mixture of lotteries which take place state-wise and hence, after the realization of states. This point has also been made in a context without ambiguity by Kreps (1988) and, more recently, with ambiguity by Seo (2009) and Wakker (2010).[4] While Anscombe and Aumann (1963) introduce this interpretation of a randomization over acts as an explicit assumption, most of the subsequent literature takes this interpretation as an unquestioned characteristic of the AA approach itself.[5]

Organization of the paper In the next section we present our framework. Section 3 contains the main results on attitudes to ex-post and ex-ante randomization. We then apply them to the debate between Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012) in section 4. Section 5 relates our results to the recent literature and Section 6 concludes.

2 Framework and Definitions

There is a decision maker who faces uncertainty described by a finite state space S. A generic state is denoted by s. She also has access to a randomizing device with which she may conduct an ex-ante randomization before the state is revealed and/or an ex-post randomization after the state is revealed. The two randomizing devices are assumed to have objective probability distributions, which are known to the decision-maker. This is in the spirit of the Raiffa argument. These probability distributions are identically distributed and are (statistically) in-

[3]See also the discussion in Ghirardato (1997).

[4]See in particular Wakker (2010) sections 4.9 and 10.7.

[5]Anscombe and Aumann (1963) p. 201, Assumption 2 (Reversal of order of compound lotteries).

dependent of one another. We study whether she would wish to make use of either the ex-ante or ex-post randomization or both.

2.1 Uncertainty and Time

There are three time periods. In the first, the outcome of a randomizing device r contained in the sample space $R = [0, 1]$ is revealed. Without essential loss of generality, we assume the outcome of this randomizing device is distributed uniformly over the unit interval. A risky event is an element of $\mathcal{B}$, the Borel σ-algebra of subsets of $[0, 1]$, and we let μ denote Lebesgue measure on $\mathcal{B}$. For each $B \in \mathcal{B}$, the probability that the realization r of the randomizing device is in the event B, is given by $\mu(B)$. We say an event $C \in \mathcal{B}$ is null if $\mu(C) = 0$. Let $\mathcal{R} \subset \mathcal{B}$ denote the set of non-null or *relevant* events. We assume that all randomizations are conditional on non-null events. In the second time period, a possibly ambiguous state, $s \in S$, is revealed. Finally the outcome of a randomization device, identically and independently distributed to the first, $t \in T$ is realised in the third time period.

The grand state space is a Cartesian product $\Omega = R \times S \times T$. This is a complete description of how all uncertainty is resolved. Once r, s and t are known there is no further uncertainty. This framework allows us to distinguish between ex-ante and ex-post randomizations.

The space of outcomes, X, is a connected subset of $\mathbb{R}$. An act is function, $a : \Omega \to X$. (For simplicity, we assume that the pay-offs of acts are expressed in utility terms.) The set of all acts is denoted by $A(\Omega)$. The decision maker has preferences represented by a binary relation, $\succsim$, on $A(\Omega)$, which we take to be complete and transitive.

A *constant act* is one which assigns the same outcome $x \in X$, to every triple (r, s, t) in Ω. Let $A(R)$, $A(S)$ and $A(T)$ be the subsets of $A(\Omega)$, given by

$$
\begin{aligned}
A(R) &= \left\{a \in A(\Omega) : \forall r \in R, \forall \hat{s}, \tilde{s} \in S, \forall \hat{t}, \tilde{t} \in T, a\left(r, \hat{s}, \hat{t}\right) = a\left(r, \tilde{s}, \tilde{t}\right)\right\}; \\
A(S) &= \left\{a \in A(\Omega) : \forall \hat{r}, \tilde{r} \in R, \forall s \in S, \forall \hat{t}, \tilde{t} \in T, a\left(\hat{r}, s, \hat{t}\right) = a\left(\tilde{r}, s, \tilde{t}\right)\right\}; \\
\text{and } A(T) &= \left\{a \in A(\Omega) : \forall \hat{r}, \tilde{r} \in R, \forall \hat{s}, \tilde{s} \in S, \forall t \in T, a(\hat{r}, \hat{s}, t) = a(\tilde{r}, \tilde{s}, t)\right\}.
\end{aligned}
$$

The acts in $A(S)$ only depend on the resolution of the state uncertainty and are independent of the outcomes of the two randomizations. These are the acts in which the decision-maker is primarily interested. The randomizing devices are only present to help him/her if required. We shall denote generic elements of $A(S)$ by f, g, h (alluding to Savage acts that involve only the resolution of purely subjective uncertainty). When convenient we shall abbreviate $f(r, s, t)$

to $f(s)$, for $f \in A(S)$.

We shall denote generic elements of $A(R)$ by $\ell, \tilde{\ell}, \hat{\ell}$ and generic elements of $A(T)$ by $\lambda, \tilde{\lambda}, \hat{\lambda}$ (alluding to lotteries with 'objective' probabilities) and again when convenient we shall abbreviate $\ell(r,s,t)$ to $\ell(r)$ for any $\ell \in A(R)$ and $\lambda(r,s,t)$ to $\lambda(t)$ for $\lambda \in A(T)$.

2.2 Randomization

In this section we define ex-ante and ex-post randomizations. We then proceed to describe the related concept of a subjective mixture of two Savage acts.

First we define an ex-ante randomization between acts f and g to be a contingent plan in which the act f or g is chosen depending on the outcome of the randomizing device. For instance if the randomizing device is a fair die one could consider a plan to choose f if the die shows $1, 2, 3$ or 4 and g if it shows 5 or 6.

Definition 2.1 (Ex–ante randomization) *Suppose $f,g \in A(S)$ and $C \in \mathcal{R}$, define the act $f^C g$ by $f^C g(r,s,t) = f(s)$ if $r \in C$, and $f^C g(r,s,t) = g(s)$ if $r \notin C$.*

Analogously one may define an ex-post randomization.

Definition 2.2 (Ex–post randomization) *Suppose $f,g \in A(S)$ and $D \in \mathcal{R}$, define the act $f_D g$ by $f_D g(r,s,t) = f(s)$ if $t \in D$, and $f_D g(r,s,t) = g(s)$ if $t \notin D$.*[6]

Next we define attitudes to randomization. We say that an individual is indifferent to randomization if whenever (s)he is indifferent between a pair of acts f, g, she also views f as being indifferent to any randomization between f and g.

Definition 2.3 (Indifference to randomization) *An individual is indifferent to ex-ante (resp. ex-post) randomization if for all $C \in \mathcal{R}$, such that $\mu(C) \in (0,1)$, and all $f,g \in A(S)$, $f \sim g$ implies, $f^C g \sim f$, (resp. $f_C g \sim f$).*

We say that an individual has a strict preference for randomization if *there exist* two indifferent acts f, g such that some randomization between them is preferred to either of the original acts. The reason we do not require a strict preference for all randomizations between indifferent acts, is that the Raiffa argument only applies to complementary acts such as betting on the

[6] The ex-post randomizations in this framework may be identified with a subset of the horserace-lottery acts in the AA framework. In particular, the ex-post randomization $f_D g$ may be identified with the horserace-lottery act H where for each s in S, $H(s) = \mu(D)\,\delta_{f(s)} + (1-\mu(D))\,\delta_{g(s)}$ and δ_x is the degenerate lottery that yields the outcome x with probability 1. That is, an ex-post randomization can be naturally identified with a horserace-lottery act in which the second stage is a binary lottery.

red or black balls from the same Ellsberg urn. Raiffa's intuition does not obviously extend to non-complementary acts such as betting on the black balls from two different Ellsberg urns.

Definition 2.4 (Preference for randomization) *An individual exhibits a strict preference for ex-ante (resp. ex-post) randomization if there exists $C \in \mathcal{R}$, such that $\mu(C) \in (0,1)$, and $\exists f, g \in A(S)$ such that $f \sim g$ and $f^C g \succ f$, (resp. $f_C g \succ f$).*

We maintain throughout the analysis that the decision maker does not discriminate between lotteries over *certain* outcomes, which have the same cumulative distribution function, whether they are resolved before or after the subjective uncertainty.[7] An implication of this is that a gamble which pays £100 if a fair coin shows heads, £30 otherwise; is viewed as equivalent to a gamble which pays £100 if a fair die comes up 4, 5 or 6 and £30 otherwise.[8] This is captured in the next assumption.

For any $\ell \in A(R)$, let $F_\ell : X \to [0,1]$ denote the cumulative distribution function associated with this lottery. That is, for any $x \in X$, $F_\ell(x) = \mu(\{r \in R : \ell(r) \leqslant x\})$. Similarly, for any $\lambda \in A(T)$, let $F_\lambda : X \to [0,1]$ denote the cumulative distribution function associated with λ.

Assumption 2.1 (Distribution Invariance)

1. *For any $\ell \in A(R)$ and any $\lambda \in A(T)$, $F_\ell = F_\lambda \Rightarrow \ell \sim \lambda$;*

2. *if $a, b \in A(\Omega)$ are such that $\forall r \in R$, $\forall s \in S$, $F_{a(r,s,\cdot)} = F_{b(r,s,\cdot)}$ then $a \sim b$.*[9]

Notice in particular, part 2 of distribution invariance implies that for any pair of lotteries $\lambda, \tilde{\lambda} \in A(T)$, if $F_\lambda = F_{\tilde{\lambda}}$ then $\lambda \sim \tilde{\lambda}$. Given the transitivity of indifference, it also readily follows that for any pair of lotteries $\ell, \tilde{\ell} \in A(R)$, if $F_\ell = F_{\tilde{\ell}}$ then $\ell \sim \tilde{\ell}$. Hence preferences do not depend on the particular randomizing device used. Any randomizing device with the same probabilities will give rise to the same preference.

We define a subjective mixture of two acts as follows.

Definition 2.5 (Subjective mixture of acts) *For any pair f, $g \in A(S)$ and $E \subseteq S$, the* subjective mixture *of f and g, is an act $fEg \in A(S)$ defined by $fEg(s) = f(s)$ if $s \in E$ and $fEg(s) = g(s)$ if $s \notin E$.*

Thus the subjective mixture of f and g coincides with f if s is in the uncertain event E and coincides with g otherwise. We say an event $E \subseteq S$ is *null* if $gEf \sim f$ for all $f, g \in A(S)$. Let $\mathcal{N}$ denote the set of null subsets of S.

[7] This can be seen as a weak version of the AA reversal of order assumption, which only applies to acts which are independent of the S-states.

[8] This assumption is similar in spirit to the device independence assumption in Eichberger and Kelsey (1996).

[9] If $a \in A(\Omega)$ then for given r, s, $a(r,s,\cdot)$ is an act whose outcome only depends on the ex-post randomizing device and hence may be described by a distribution function $F_{a(r,s,\cdot)}$.

2.3 Assumptions

If one imposes dynamic consistency and consequentialism globally then one rules out ambiguity-aversion and indeed most preferences other than SEU. However there are ambiguity-averse preferences which satisfy dynamic consistency and consequentialism with respect to a specific filtration, Sarin and Wakker (1998) and Eichberger, Grant, and Kelsey (2005). We take this filtration to be the one in which the ex-ante randomisation is resolved first, then the state-uncertainty and finally the ex-post randomisation.[10]

2.3.1 Dynamic Consistency

We assume that in addition to the unconditional preference $\succsim$ on $A(\Omega)$, there is, for all (non-null) events $C \in \mathcal{R}$, a conditional preference $\succsim^C$ defined on $A(\Omega)$. We say preferences are dynamically consistent with respect to the ex-ante randomization if the following holds.

Assumption 2.2 (R-dynamic consistency) *For any relevant event $C \in \mathcal{R}$:*

(i) if $\mu(C) = 1$, then $\succsim^C = \succsim$;

(ii) if $\mu(C) < 1$, then for any pair of acts $a, b \in A(\Omega)$: $a \succsim^C b$ and $a \succsim^{R \setminus C} b$ implies $a \succsim b$.

Part (i) of R-dynamic consistency says that if C is sure to occur then the conditional preference should be the same as the unconditional preference. As is standard, part (ii) of R-dynamic consistency says that an individual who conditionally expresses a preference between two acts whether or not the randomization based on R yields a realization in the event C, should also unconditionally express that preference. This is a weak dynamic consistency axiom since it only requires consistency *after observing a non-null event* of R.[11]

2.3.2 Consequentialism

The next assumption states that preferences over ex-ante randomizations satisfy a weak version of consequentialism.

Assumption 2.3 (R-consequentialism) *For any $C \in \mathcal{R}$ and any $f, g \in A(S)$, $f \sim^C f^C g$.*

[10] Since we use a specific filtration our axioms are not symmetric between the ex-post and ex-ante randomizations.

[11] In this sense it may also be viewed as a weaker form of the "coherence" condition of Skiadas (1997), expression (6) p.353.

This assumption requires that the conditional preference $\succsim^C$ only depends on the outcomes in $C \times S \times T$. Hence R-consequentialism may be viewed as a weak version of consequentialism in the sense that it only applies to events defined by the ex ante randomizing device. In particular it does not exclude possible violations of consequentialism due to ambiguity.

2.3.3 Independence of the Randomizing Devices

In the Raiffa argument the randomizing device does not affect anything the decision-maker cares about. It is there to help him/her. Hence it should be independent of the process which determines the ambiguous states. The next assumption captures these ideas formally. In our opinion, the key features of an independent randomizing device are that its output conveys no useful information about the likelihood of the states and that the decision maker does not care directly about the outcome of the device. We believe that for acts which only depend on the S states, preferences conditional on events defined by the randomizing device should coincide with the unconditional preference.

It is logically possible that preferences might depend directly on the outcome of the randomizing device even though it is independent of the state-uncertainty. For instance an individual may be more willing to accept an ambiguous risk if her lucky number comes up. The Machina's Mom example, concerning the allocation of an indivisible treat, is another case where the conditional preference may depend on the outcome of an independent randomizing device, see Machina (1989). The following assumption rules out such preferences, thus focusing our model on the use of randomization to hedge ambiguity.

Assumption 2.4 (R-independent randomizations) *For any relevant event $C \in \mathcal{R}$, and any $f, g \in A(S)$, $f \succsim g \Leftrightarrow f \succsim^C g$.*

2.3.4 Ambiguity-Aversion

The following assumption says that the decision-maker is ambiguity-averse in the sense that there is at least one occasion in which (s)he expresses the usual preferences in the two-ball Ellsberg paradox. It says that there exists an event $E \subseteq S$, such that the decision-maker is indifferent between betting on E and its complement. However (s)he views both of these bets as being strictly inferior to a bet with probability $\frac{1}{2}$ with the same prizes.

Assumption 2.5 (Ellsberg Ambiguity-Aversion (EAA)) *A decision maker with preferences $\succsim$ on $A(\Omega)$ is* Ellsberg ambiguity-averse *if, for some (non-null) event $E \notin \mathcal{N}$ and some event $C \in \mathcal{B}$, such that $\mu(C) = \frac{1}{2}$ and some pair of outcomes $x \succ y$, $x_C y \sim y_C x \succ xEy \sim yEx$.*

Notice that the indifferences $xEy \sim yEx$ and $x_C y \sim y_C x$ for some $x \succ y$, suggests the decision maker views both the events $R \times E \times T$ and $R \times S \times C$ as equally likely as their respective complements. If the decision-maker were probabilistically sophisticated in the sense of Machina and Schmeidler (1992) this would require an indifference between the act xEy and the lottery $x_C y$. Instead the lack of indifference indicates a strict preference of the decision maker to prefer the two-outcome symmetric lottery rather than two-outcome symmetric Savage acts.[12] The opposite preference could be taken as a characterization of ambiguity-loving behaviour.

3 Attitudes to Randomization

3.1 Ex-ante Randomization

Our main motivation is to study ex-ante randomizations with known probabilities, which is the case suggested in Raiffa (1961). Suppose the randomizing device is a fair coin, hence $R = \{r_1, r_2\}$, where r_1 and r_2 are equally likely.[13] The spirit of the Raiffa argument seems to suggest that if $f, g \in A(S)$ are ambiguous bets on complementary events such that $f \sim g$ then $f^{r_1} g \succ f$ and $f^{r_2} g \succ f$. In other words both 'f if heads g if tails' and 'f if tails g if heads' are preferred to f. Our first result says that it cannot be the case that the decision-maker always prefers randomizations. If f and g are indifferent and some randomizations are preferred to the pure acts there must be other ex-ante randomizations which are not superior to the pure acts.[14]

Proposition 3.1 *Let $\{\succsim, \succsim^C : C \in \mathcal{R}\}$ be a family of conditional preference orders that satisfies R-dynamic consistency (2.2) and R-consequentialism (2.4). Consider a given event $C \in \mathcal{R}$ and two acts $f, g \in A(S)$ such that $f \sim g$, if $f^C g \succ f$ then $f \succsim g^C f$.*

Proof. Suppose, if possible, $f \succsim^{R \backslash C} g$. Then by R-dynamic consistency $f \succsim^C f$ and $f \succsim^{R \backslash C} g$ would imply $f \succsim f^C g$, which contradicts the assumption that $f^C g \succ f$. Hence we must have $g \succ^{R \backslash C} f$.

[12] This may be viewed as a special case of 'issue preference', as in Ergin and Gul (2009), or 'source dependence', as in Chew and Sagi (2008).

[13] This can be made compatible with our assumption that the R-space is a continuum by identifying r_1 with $[0, \frac{1}{2}] \subseteq R$, and r_2 with $(\frac{1}{2}, 1] \subseteq R$.

[14] One may find this result less compelling in a normative sense than a strict preference which violates dynamic consistency. If preferences satisfy an appropriate continuity property, however, then we can construct a violation of dynamic consistency which only involves strict preferences. The argument in the introduction provides an outline of the proof.

Notice that R-consequentialism implies $f \sim^{R\backslash C} g^C f$, which together with $g \succ^{R\backslash C} f$ implies $g \succ^{R\backslash C} g^C f$. A second application of R-consequentialism implies $g \sim^C g^C f$. Applying R-dynamic consistency to $g \succ^{R\backslash C} g^C f$ and $g \sim^C g^C f$, we may deduce $g \succsim g^C f$. However, by assumption $f \sim g$ so we may conclude that $f \succsim g^C f$. ■

An implication of this is that the decision-maker must be indifferent to all 50:50 randomizations. More generally it is not possible to have a strict preference for ex-ante randomization and distribution invariance at the same time, that is, Assumptions 2.1*,* 2.2 and 2.3 imply indifference to randomization.

Proposition 3.1 does not make use of R-independent randomizations, (2.4). If we add this to our previous assumptions we can deduce that preferences must exhibit indifference to randomization.

Proposition 3.2 *Let* $\left\{\succsim, \succsim^C : C \in \mathcal{R}\right\}$ *be a family of conditional preference orders that satisfies R-dynamic consistency (2.2), R-consequentialism (2.3) and R-independent randomizations (2.4). Then the unconditional preference order* $\succsim$ *must be indifferent to R-randomization.*

Proof. Fix a pair of acts $f, g \in A(S)$ such that $f \sim g$ and a $C \in \mathcal{R}$. Then $f \sim g \Rightarrow f^C g \sim^C f$, by R-consequentialism. Also $f \sim g \Rightarrow f \sim^{R\backslash C} g$, by R-independent randomizations. Now $f \sim^{R\backslash C} g \Rightarrow f \sim^{R\backslash C} f^C g$, by R-consequentialism. But if $f^C g \sim^C f$ and $f^C g \sim^{R\backslash C} f$ then by R-dynamic consistency $f^C g \sim f$. Hence we have shown $f \sim g$ implies $f^C g \sim f$ for all non-null $C \subsetneq R$, and thus $\succsim$ is indifferent to randomization. ■

The proof does not use the full strength of the dynamic consistency assumption but only applies it to indifferences. Proposition 3.2 does not assume beliefs over the S-space are ambiguous. Hence it also applies to some non-expected utility preferences which are probabilistically sophisticated in the sense of Machina and Schmeidler (1992). Its proof also does not exploit the fact that beliefs about the randomizing device are additive (and that the preferences satisfy Distribution Invariance (A. 2.1)). Hence even if an ambiguous randomizing device is available, the decision maker would not wish to use it.[15] None of the results in this section assume ambiguity aversion. Thus ambiguity-loving individuals who satisfy our assumptions will also be indifferent to randomization.

[15] Bade (2011) makes a related point. She shows that, in a strategic setting, an ambiguous randomizing device would not help a player in a game. In her model the equilibria with ambiguous randomizations coincide with conventional Nash equilibria.

3.2 Ex-post Randomization

In this section we show Ellsberg Ambiguity-Aversion (A.2.5) and Distribution Invariance (A.2.1) imply a strict preference for ex-post randomization. Suppose that one takes a randomization over AA acts as a state-wise randomization over the outcome lotteries, as almost all of the literature does. Then ambiguity-aversion implies that there exist ex-post randomizations that are preferred to the ambiguous act itself. In particular, a decision maker who prefers to bet on the unambiguous urn as described in Ellsberg's two-colour urn (Ellsberg (1961)) will prefer some ex-post randomization.

As the next proposition shows, in conjunction with *Distribution Invariance*, such an individual will exhibit a strict preference for some ex-post randomization over some pairs of acts in $A(S)$.

Proposition 3.3 *Let $\succsim$ be a preference on $A(\Omega)$ that satisfies Assumptions 2.1 (distribution invariance) and 2.5 (Ellsberg ambiguity-aversion). Then there exists a relevant event $D \in \mathcal{R}$, and a pair of acts f and g in $A(S)$, such that $f_D g \succ f \sim g$.*

Proof. Since the agent is Ellsberg ambiguity-averse, there exists an event $C \in \mathcal{R}$ with $\mu(C) = \frac{1}{2}$, a non-null event $E \notin \mathcal{N}$ and some pair of outcomes $x \succ y$, such that $x_C y \sim y_C x \succ xEy \sim yEx$. Set $f := xEy$ and $g := yEx$. Consider the ex post randomization $f_C g$:

$$f_C g \equiv (xEy)_C (yEx) \equiv (x_C y) E (y_C x).$$

By applying Assumption 2.1, it follows that $f_C g = (x_C y) E (y_C x) \sim (x_C y) E (x_C y) = x_C y \succ xEy = f$.[16] Hence $f_C g \succ f$, as required. ■

Theoretical arguments based on hedging between complementary acts imply that individuals are likely to have a strict preference for ex-post randomization, Schmeidler (1989). It is an empirical question whether people will actually prefer ex-post randomizations.

3.3 Multiple Priors

This section contains an example of preferences which satisfy R-dynamic consistency, R-consequentialism and Ellsberg Ambiguity Aversion. They exhibit indifference to ex-ante randomization but a strict preference for ex-post randomization. This demonstrates that our

[16] The indifference $(x_C y) E (y_C x) \sim (x_C y) E (x_C y)$ arises because $\mu(C) = 1/2$ implies that $(x_C y)$ and $(y_C x)$ give the same outcomes with the same probabilities. Intuitively they should be indifferent conditional on E^c. Indifference is implied by Assumption 2.1 which says objective randomizations with the same probabilities are indifferent and if they have the same conditional probabilities they are indifferent in the corresponding conditional preferences.

axioms are mutually consistent. The preferences in the example have the recursive multiple priors form with rectangular beliefs. These are Maxmin Expected Utility (MEU) preferences with a specific set of beliefs. MEU is defined below.

Definition 3.1 *We say that a preference relation $\succsim$ on $A(\Omega)$ is* Maxmin Expected Utility *(MEU) if there exists a closed convex set of probability distributions, $\mathcal{P}$, on Ω such that preferences may be represented by the function*

$$V(a) = \min_{p\in\mathcal{P}} \int_{\Omega} a(\omega)\, dp(\omega).$$

Furthermore, if for each $p \in \mathcal{P}$, we write p_R for its marginal distribution on R and $p(\cdot|\cdot)$ for the distribution on $S \times T$ conditional on the realization r, and we set

$$\begin{aligned} \mathcal{P}_R &:= \{q \in \Delta(R) : \ q = p_R \text{ for some } p \in \mathcal{P}\}, \\ \mathcal{P}_r &:= \{q' \in \Delta(S\times T) : \ q' = p(\cdot|r) \ \text{ for some } p \in \mathcal{P}\} \end{aligned}$$

then we say $\mathcal{P}$ satisfies the rectangularity property of Epstein and Schneider (2003) with respect to the ex ante randomization, if

$$\mathcal{P} = \{q(r)p(s,t|r) \ \mid p(\cdot|r) \in \mathcal{P}_r, \quad q \in \mathcal{P}_R\}.$$

For a MEU decision maker with a rectangular set of priors, it follows that

$$\begin{aligned} V(a) \quad &: = \min_{p\in\mathcal{P}} \int_{\Omega} a(\omega)\, dp(\omega) \\ &= \min_{q\in\mathcal{P}_R} \int_R \left[\min_{q'\in\mathcal{P}_r} \int_{S\times T} a(r,s,t)\, dq'(s,t)\right] dq(r). \end{aligned}$$

Example 3.1 *Suppose $R = \{r_1, r_2\}$ with $\rho(r_1) = \rho(r_2) = \frac{1}{2}, S = \{s_1, s_2\}, T = \{t_1, t_2\}$ and $\tau(t_1) = \tau(t_2) = \frac{1}{2}$, where ρ (resp. τ) denotes the probability of the ex-ante (resp. ex-post) randomizing device. This models a situation where the ex-post and ex-ante randomizing devices are independent tosses of a fair coin. Define $Q \subseteq \Delta(S\times T)$ by*

$$Q = \{q \in \Delta(S\times T) : q(s_1,t) = \frac{1}{4} - \frac{\epsilon}{2}, q(s_2,t) = \frac{1}{4} + \frac{\epsilon}{2} : -\alpha \le \epsilon \le \alpha\}.$$

Let the decision-maker's set of priors be given by:

$$\mathcal{P} = \{p \in \Delta(\Omega) : p(r,s,t) = \rho(r)\, q_r(s,t) : q_r \in Q\}.$$

By construction, the set is rectangular with respect to the partition defined by the ex-ante randomization, i.e. $r_1 \times S \times T, r_2 \times S \times T$. As the following result shows these preferences satisfy our axioms, thus, inter alia, demonstrating that they are consistent.

Proposition 3.4 *If the DM has MEU preferences with beliefs represented by $\mathcal{P}$ then he will satisfy R-dynamic consistency (A.2.2) R-consequentialism (A.2.3) and Ellsberg Ambiguity-Aversion (A.2.5). He will be indifferent to ex-ante randomization but will have a strict preference for ex-post randomization.*

Proof. Define acts $a, b \in A(S)$ by:

$$a(r,s,t) = \begin{cases} 1 \text{ if } s = s_1, \\ 0 \text{ if } s = s_2; \end{cases} \qquad b(r,s,t) = \begin{cases} 0 \text{ if } s = s_1, \\ 1 \text{ if } s = s_2. \end{cases}$$

Let $\tilde{p}$ denote a typical element of $\mathcal{P}$. By construction $\tilde{p}$ has the form

$$\tilde{p}(r,s,t) = \begin{cases} \frac{1}{2}\left(\frac{1}{4} - \frac{\epsilon}{2}\right) \text{ if } \langle s,t\rangle = \langle s_1,t_1\rangle \text{ or } \langle s_1,t_2\rangle, \\ \frac{1}{2}\left(\frac{1}{4} + \frac{\epsilon}{2}\right) \text{ if } \langle s,t\rangle = \langle s_2,t_1\rangle \text{ or } \langle s_2,t_2\rangle, \end{cases} \text{ for } r \in \{r_1, r_2\}.$$

The expected value of a with respect to $\tilde{p}$ is
$\mathbf{E}_{\tilde{p}}a = \tilde{p}(r_1,s_1,t_1) + \tilde{p}(r_2,s_1,t_1) + \tilde{p}(r_1,s_1,t_2) + \tilde{p}(r_2,s_1,t_2) = \frac{1}{2} - \epsilon$. Thus $\min_{p\in\mathcal{P}} \mathbf{E}_p a = \frac{1}{2} - \alpha$. Similarly $\min_{p\in\mathcal{P}} \mathbf{E}_p b = \frac{1}{2} - \alpha$, hence $a \sim b$.

Preferences for Ex-post Randomization and Ellsberg Ambiguity Aversion Let C denote the event $t = t_1$, and let E denote the event $s = s_1$. Consider the ex-post randomization $a_C b$, then $a_C b(r,s_1,t_1) = a_C b(r,s_2,t_2) = 1$ and $a_C b = 0$ otherwise.

Thus $\mathbf{E}_{\tilde{p}} a_C b = \tilde{p}(r_1,s_1,t_1) + \tilde{p}(r_2,s_1,t_1) + \tilde{p}(r_1,s_2,t_2) + \tilde{p}(r_2,s_2,t_2) = \left(\frac{1}{4} - \frac{\epsilon}{2}\right) + \left(\frac{1}{4} + \frac{\epsilon}{2}\right) = \frac{1}{2}$, (recall $\tilde{p}$ denotes a generic element of $\mathcal{P}$). Hence $V(a_C b) = \min_{p\in\mathcal{P}} \mathbf{E}_p a_C b = \frac{1}{2} > V(a) = V(b) = \frac{1}{2} - \alpha$, which implies that the individual displays a strict preference for ex-post randomization. Since for $x \succ y$, $x_C y \sim y_C x \succ xEy \sim yEx$, Ellsberg ambiguity aversion is also satisfied.

Indifference to Ex-ante Randomization Consider the ex-ante randomization $a^{r_1}b$. Then $a^{r_1}b(r_1, s_1, t) = a^{r_1}b(r_2, s_2, t) = 1$ and $a^{r_1}b = 0$ otherwise. As before let $\tilde{p}$ denote a generic element of P. Then $\tilde{p}(r_1, s_1, t_1) = \tilde{p}(r_1, s_1, t_2) = \frac{1}{2}\left(\frac{1}{4} - \epsilon_{r_1}\right)$ and $\tilde{p}(r_2, s_2, t_1) = \tilde{p}(r_2, s_2, t_2) = \frac{1}{2}\left(\frac{1}{4} + \epsilon_{r_2}\right)$.

Thus $V(a^{r_1}b) = \min_{p \in \mathcal{P}} \mathbf{E}_p a^{r_1}b = \frac{1}{2}.\min_{q_{r_1} \in \mathcal{Q}} \mathbf{E}_p a^{r_1}b(r_1, s, t) + \frac{1}{2}.\min_{q_{r_2} \in \mathcal{Q}} \mathbf{E}_p a^{r_1}b(r_2, s, t)$
$= \frac{1}{2}\min_{-\alpha \leqslant \epsilon_{r_1} \leqslant \alpha}\left(\frac{1}{4} - \epsilon_{r_1}\right) + \frac{1}{2}\min_{-\alpha \leqslant \epsilon_{r_2} \leqslant \alpha}\left(\frac{1}{4} + \epsilon_{r_2}\right) = \frac{1}{2}\left(\frac{1}{4} - \alpha\right) + \frac{1}{2}\left(\frac{1}{4} - \alpha\right) = \frac{1}{2} - \alpha.$
Since $V(a) = V(b) = V(a^{r_1}b)$, the individual is indifferent to ex-ante randomization.

R-Dynamic Consistency The beliefs are rectangular with respect to the partition created by the ex-ante randomizing device. Thus by Epstein and Schneider (2003) they satisfy R-dynamic consistency.

R-Consequentialism R-Consequentialism is clearly satisfied because the definition of the conditional preferences makes no reference to outcomes in other states. Consider the event $\{r_1\}$, (for convenience we abbreviate $\{r_1\} \times S \times T$ to $\{r_1\}$). The set of updates of $\mathcal{P}$ conditional on $\{r_1\}$ is Q. Conditional preferences are defined by:

$$a \succsim_{\{r_1\}} b \Leftrightarrow \min_{q \in Q} E_q a(r_1, s, t) \geqslant \min_{q \in Q} E_q b(r_1, s, t).$$

By inspection, preferences are independent of outcomes on the complement of $\{r_1\}$. By similar reasoning we can show that preferences conditional on $\{r_2\}$ are consequentialist.

■

It can be seen that this construction generalizes. Take a set of multiple prior preferences over $S \times T$, which give a strict preference for ex-post randomization. Then extend them by imposing rectangularity with respect to the partition created by the ex-ante randomizing device. The result will be a set of preferences which satisfies our axioms. Thus a large class of multiple prior preferences is compatible with our framework.

4 Example: A Paradox for the Smooth Ambiguity Model

In this section we study the ex-post ex-ante distinction in the context of the Epstein-KKM debate about the smooth ambiguity model. Epstein (2010) provides examples of acts for which the intuitive ranking is incompatible with the representation by the smooth model proposed by Klibanoff, Marinacci, and Mukerji (2005), (henceforth KKM). In response KKM have offered their own intuition for a different ranking of the same acts. We believe that making a distinction between ex-ante and ex-post randomizations can clarify some of the issues in this debate.

Epstein's argument involves a toss of a fair coin. He does not specify whether this is to be interpreted as an ex-ante or an ex-post randomization. To accommodate this we use the symbol $*$ to denote a 50:50 randomization without specifying whether it is ex-post or ex-ante. Thus $\phi * \psi$ denotes a randomization which yields ϕ (resp. ψ) if the coin shows heads (resp. tails). In terms of our notation, Epstein's thought experiment may be expressed as follows.

Example 4.1 (Epstein (2010)) *You are given two urns, numbered* 1 *and* 2*, each containing* 50 *balls that are either black or white. In addition you are told that the two are generated independently, for example, they are set up by administrators from opposite sides of the planet who have never been in contact with one another. One ball will be drawn from each urn. In addition there is a fair coin which can come up either heads or tails.*

Consider the following acts $\hat{f}$ (resp. $\tilde{f}$) is a bet on the event that a black ball will be drawn from urn 1 (resp. 2), $\hat{f} * \tilde{f}$ is an act which selects $\hat{f}$ if the coin comes up heads and $\tilde{f}$ if tails; g is a bet on the event that a white ball will be drawn from urn 1*. Finally $\hat{a}$ (resp. $\tilde{a}$) is a randomization which selects $\hat{f}$ (resp. $\tilde{f}$) if the coin shows heads and g otherwise.*

To interpret this in our framework assume that the ex-ante and ex-post randomizations are independent tosses of a fair coin. Let $R = \{r_1, r_2\}$ and $T = \{t_1, t_2\}$, where r_1 denotes the event that the ex-ante coin shows heads etc. The grand state space is $\Omega = R \times S \times T$, where $S = \{b_1, w_1\} \times \{b_2, w_2\}$. The colour of the ball drawn from the first (respectively, second) urn is denoted by $s_1 \in \{b_1, w_1\}$ (respectively, $s_2 \in \{b_2, w_2\}$).

The various acts from Epstein's thought experiment may be represented as follows:

Bets for Experiment 2

	$(s_1, s_2) \in S$			
	(b_1, b_2)	(b_1, w_2)	(w_1, b_2)	(w_1, w_2)
$\hat{f}$	x	x	y	y
$\tilde{f}$	x	y	x	y
$\hat{f} * \tilde{f}$	x	$x * y$	$y * x$	y
g	y	y	x	x
$\hat{a}$	$x * y$	$x * y$	$y * x$	$y * x$
$\tilde{a}$	$x * y$	y	x	$y * x$

where $x > y$.

Epstein (2010) argues for the preference ranking $\hat{f} \sim \tilde{f} \sim \hat{f} * \tilde{f}$ and $\hat{a} \succ \tilde{a}$ and shows that this ranking of acts cannot be represented by the smooth model, since $\hat{a} = \hat{f} * g$ and $\tilde{a} = \tilde{f} * g$. In order to support this ranking Epstein (2010) writes:[17]

> "Symmetry suggests indifference between $\hat{f}$ and $\tilde{f}$. If it is believed that the compositions of the two urns are unrelated, then $\hat{f}$ and $\tilde{f}$ do not hedge one another. If, as in the multiple-priors model, hedging ambiguity is the only motivation for randomizing, then we are led to the rankings $\hat{f} \sim \tilde{f} \sim \hat{f} * \tilde{f}$. Ambiguity aversion suggests $\hat{a} \succ \tilde{a}$."

In their reply, Klibanoff, Marinacci, and Mukerji (2012) do not question the result of Epstein, but challenge his intuition:[18]

> "Epstein argued that $\hat{f} * \tilde{f} \sim \hat{f} \sim \tilde{f}$ and $\hat{a} \succ \tilde{a}$ are natural for a strictly ambiguity averse individual. .... We agree with the intuition for $[\hat{a} \succ \tilde{a}]$, but disagree that $\hat{f} * \tilde{f} \sim \hat{f} \sim \tilde{f}$ is natural for an ambiguity averse individual and think there is good reason to expect $\hat{f} * \tilde{f} \succ \hat{f} \sim \tilde{f}$. ... The evaluation of $\hat{f} * \tilde{f}$ depends on the colour compositions of both urns, but has half the exposure to the uncertainty about the ratio in each urn compared to $\hat{f}$ and $\tilde{f}$. Recall that the determination of the two urn compositions is viewed as independent. The act $\hat{f} * \tilde{f}$ thus diversifies

[17] In the quotation, the notation of events and acts has been adapted to the one used in this paper.
[18] As in the previous quotation, we have adapted the notation to the one used in this paper.

the individual's exposure across the urns: it provides a hedging of the two independent ambiguities in the same sense as diversifying across bets on independent risks provides a hedging of the risks. To an individual who is averse to ambiguity (i.e., to subjective uncertainty about relative likelihoods), such diversification is naturally valuable."

Epstein's argument that the indifference between f and $\tilde{f}$ cannot be improved upon by a randomization because "the compositions of the two urns are unrelated" suggests that he views the randomization as occurring ex-ante, i.e. $\hat{f} * \tilde{f} = \hat{f}^{\{h\}} \tilde{f}$ in our notation. In contrast, the hedging argument of KMM refers explicitly to the (ex-post) equivalence of the 'lottery' outcomes $x*y$ and $y*x$ in states (b, w) and (w, b), respectively, achieved by the 'randomization'. This suggests that they view the randomization $\hat{f} * \tilde{f}$ as occurring ex-post, in our notation $\hat{f} * \tilde{f}$ is equal to $\hat{f}_{\{t_1\}} \tilde{f}$. In combination with our analysis, this example shows that the smooth model can accommodate ex-post randomizations, however it appears to have difficulties with ex-ante randomizations.

5 Related Research

Much of the recent literature in this area is not directly comparable since they tend to use the AA model while we use a Savage framework.[19] We made this modelling choice since we believe that when studying attitudes to randomization it is important to model the randomizing device in the same way as other uncertainties.

Bade Bade (2015) has a related argument concerning experiments on ambiguity-aversion. She argues that subjects may use randomizing devices, which are part of the experimental design, to reduce their exposure to ambiguity. As a result the experiment may understate actual ambiguity-aversion. An example would be the commonly used Becker-de Groot-Marshak mechanism.

Consider an experiment which involves asking subjects a number of questions, one of which is selected randomly for payment. The question Bade asks is whether subjects will make the same answer as in a situation where they faced a single question for certain. It is well known that the answer is yes for expected utility preferences but no for some common types of non-expected utility preferences, Karni and Safra (1987). Bade investigates how these results can be extended to a class of ambiguity-averse preferences.

[19] There is also an earlier literature, see for instance, Brewer (1963) or Fellner (1963).

This is related to the question studied in the present paper. Consider the variant of the one-urn Ellsberg problem from the introduction. Recall R (respectively, B or Y) denotes the bet that pays £100 if a red (respectively, black or yellow) ball is drawn from the urn. Then an ambiguity-averse individual who has a strict preference for randomization may well have the preference

$$B_{\{h\}}Y \succ R \succ B \sim Y,$$

where as before h denotes the event that a fair coin shows heads when tossed and $B_{\{h\}}Y$ denotes the act which pays off on B (resp. Y) if the coin shows heads (resp. tails). Thus the subject prefers R to B and R to Y. However if (s)he is asked to express a preference in both of these situations and told that one of these choices will be played out for real with probability $\frac{1}{2}$ then choosing B in the first choice and Y in the second is equivalent to choosing $B_{\{h\}}Y$ which is strictly preferred to R. Hence the subject will express the opposite choices in the situation where one decision is randomly played for real, compared to the situation in which (s)he was asked either question in isolation. Thus in experiments on ambiguity it is necessary to be careful when introducing extra uncertainty.

Bade does not explicitly model the timing of randomizations. Our results suggest that if an explicit temporal structure is imposed on her model the distortion of preferences would arise with ex-post randomizations. It may be the case that subjects are not told the order of resolution of uncertainty. In this case the result would depend on the subjects' belief about whether the randomizations were ex-post or ex-ante.

Kuzmics Kuzmics (2015) argues that if a rational individual is ambiguity-averse, it will not be possible to demonstrate his/her ambiguity aversion experimentally. The paper starts by assuming a preference for ex-post randomization induced by hedging complementary acts. The AA axiom on reversal of order of randomization is then invoked to deduce a strict preference for ex-ante randomization. An extension to the basic argument shows that even asking subjects a finite sequence of questions is not capable of distinguishing ambiguity-averse and SEU preferences.

There are a number of differences between his theoretical framework and that in the present paper. Firstly he assumes that individuals are committed to their randomizations. This is reminiscent of our distinction between ex-ante and ex-post randomizations. An ex-post randomization embodies a form of commitment, since it is not possible to change actions after the randomization. Secondly the paper uses an AA framework. This involves making a particular modelling choice about which uncertainties are modelled using the state space and which are modelled by the AA scaling probabilities. It is not clear that the same result could be obtained with a different modelling choice.

We contend that the crucial assumption is the reversal of order of randomizations which rules out most of the issues studied in the present paper. One might note that none of the standard theories of ambiguity-aversion satisfy this assumption. As already noted, the spaces of ex-ante and ex-post randomizations are different mathematical objects. There is therefore no compelling reason to impose indifference between them.

Saito Saito (2013) characterizes preference for randomization in terms of preferences over *sets* of acts. In his model there is a single randomization. The subject is not told whether the randomization occurs ex-ante or ex-post. Hence they must then form subjective beliefs about the timing of the randomization. Saito axiomatizes preferences which depend on a parameter which reflects subjective beliefs as to whether the randomization is ex-post or ex-ante.

6 Conclusion

This paper studies the Raiffa argument for randomization as a response to ambiguity. It shows that preferences which are not indifferent to randomization must be dynamically inconsistent. If one views dynamic consistency as a fundamental normative principle, then these results suggest there is no normative theory of ambiguity aversion in which there is a strict preference for randomization.

Much of the previous literature on ambiguity has explicitly or implicitly assumed ambiguity-aversion. However our results in Section 3.1 do not assume ambiguity-aversion. Thus we conclude that an ambiguity-loving individual would also be indifferent to ex-ante randomization if they are R-dynamically consistent. Assumption 2.5 does imply a form of ambiguity-aversion. However the opposite strict preference could be taken to be a characterization of ambiguity-loving behaviour. If such an assumption were made we could conclude that an ambiguity-lover would be indifferent to ex-ante randomization and averse to ex-post randomization.

From the descriptive point of view there is little evidence that experimental subjects are dynamically consistent. Thus there is no logical reason why one cannot assume strict preference for randomization in a descriptive theory. However the experimental evidence does not suggest that individuals express a strict preference for ex-ante randomization, Dominiak and Schnedler (2011). There is also evidence against a preference for ex-post randomization, see Eichberger, Oechssler, and Schnedler (2015). Reconciling these experimental results with the theory is a topic for future research.

The preference for randomization argument is most plausible when applied to problems of balls and urns. In an experiment like the 2-ball Ellsberg urn it is reasonably clear that betting on red is a complementary act to betting on black. In real ambiguous decisions there are often no easily available acts which pay-off in the complementary events. Consider, for instance, decisions such as whether it is worth paying a large amount of money to protect against an uncertain environmental threat or whether to invade a rogue state which may have weapons of mass destruction. In such cases randomization is not obviously attractive. Ball and urn experiments capture some aspects of reality. However they also leave important things out.

In the convex capacity model it is not possible to have a strict preference for randomization. In more general models of ambiguity, such as multiple priors, one may or may not have a strict preference for randomization. Both preferences are possible, and which one uses is a modelling choice. We anticipate that some researchers will continue to assume a strict preference for randomization. The price of doing so is dynamic inconsistency.

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