Dynamic Consistency and Preference for Randomization

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This paper studies the relation between ambiguity and preference for randomization.

Consider the 2-ball Ellsberg urn:

The urn contains 100 balls which are either red or black in unknown proportions.

The subject is given the choice of option $a$ (resp. $b$) $100 if the ball drawn is red (resp. black).

Ellsberg argues that betting on either ball will be viewed as inferior to an objective 50:50 gamble between $100 and $0.
Raiffa (1961) argues that ambiguity can be avoided by flipping a coin and betting on red (resp. black) if it shows heads (resp. tails). Thus ambiguity-aversion gives rise to a strict preference for randomization.

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However Raiffa does not explicitly model the randomizing device.
A Savage Framework

- In a Savage framework, the randomizing device must be modelled in the state space.

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It is no longer obvious that the randomizations $c$ and $d$ should be preferred.

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- Eichberger and Kelsey (1996) show that in the convex capacity model, individuals must be indifferent to randomization.
- ?? find necessary and sufficient conditions for dynamic consistency in the convex capacity model.
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We find necessary and sufficient conditions for CEU preferences to be dynamically consistent.
Set-Up

- A decision-maker faces uncertainty described by a state space $S$.
- There is also a randomizing device. Its output, $r$, is contained in a set $R$.
- The grand state space, $\Omega$, is a Cartesian product $\Omega = R \times S$.
- An act is function, $a : \Omega \rightarrow X$, where $X$ denotes the space of outcomes.
- Preferences are represented by a binary relation, $\succ$, on the set of all acts $A(\Omega)$.
- For any measurable event $E \subseteq R$, the decision maker has a conditional preference $\succ_E$ defined on $A(\Omega)$.
- $A(S) = \{ a \in A(\Omega) : \forall \hat{r}, \tilde{r} \in R, \forall s \in S, a(\hat{r}, s) = a(\tilde{r}, s) \} \subseteq A(\Omega)$ is the set of acts which do not depend upon the randomizing device.
We define a randomization between two acts as follows.

**Definition**

Suppose \( a, b \in A(S) \) and \( F \) is a measurable subset of \( \mathbb{R} \), define an act \( a^F b \) by

\[
a^F b(r, t) = a(r, s) \text{ if } s \in F, = b(r, s) \text{ if } s \not\in F.
\]

**Definition**

We say that an individual has a strict preference for (resp. indifference to) randomization if for all non-null \( F \subseteq \mathbb{R} \) such that \( F^c \) is also non-null, and all \( a, b \in A(S) \), such that \( a \sim b, a^F b \succ a \), (resp. \( a^F b \sim a \)).
Assumption (Consequentialism, Conseq)

For any non-null measurable event \( E \subseteq R \), and any \( a, b, \in A(\Omega) \),
\[ a \sim_E a^E b. \]

Assumption (Dynamic Consistency)

For any non-null measurable event \( E \subseteq R \), such that its complement \( E^c \) is also non-null and any pair of acts \( a, b \in A(\Omega) \): \( a \succ_E b \) and \( a \succ_{E^c} b \)
implies \( a \succ b \).

As usual dynamic consistency says that a decision-maker will keep to an ex-ante optimal plan at subsequent events. This is a weak dynamic consistency axiom since it only applies after observing on an event defined by the output of the randomizing device. We can get similar results if dynamic consistency is only required for strict preferences.
Assumption (Independent Randomizations, IR)

For any non-null measurable event $E \in \mathcal{A}$, and any $a, b \in A(S)$, $a \gneq b \iff a \gneq_E b$.

- The acts $a, b \in A(S)$ do not depend directly on the realization of the randomizing device.
- Moreover the randomizing device does not convey useful information which might improve the decision.
- Hence it seems reasonable that preferences between $a$ and $b$ will not be changed once the outcome of the randomizing device is known.
- Independent randomizations does rule out some kinds of behaviour. For instance if an individual is more willing to take a gamble if his/her lucky number comes up.
Proposition

Let \( \succsim_{E}: E \in \mathcal{A} \) be a family of conditional preference orders which satisfies Consequentialism and Dynamic Consistency. Then the unconditional preference order \( \succsim \) cannot display a uniform strict preference for (or aversion to) randomization.

This result does not make use of Independent Randomizations.
Proposition

Let \( \{\succeq_E : E \in \mathcal{A}\} \) be a family of conditional preference orders which satisfies Consequentialism, Dynamic Consistency, and Independent Randomizations. Then the unconditional preference order \( \succeq \) must be indifferent to randomization.

Proof.

Suppose \( a \sim b \). Then \( a \sim_E b \) and \( a \sim_{E^c} b \) by Independent Randomizations.

By Consequentialism \( a^E b \sim_E a \) and \( a^E b \sim_{E^c} b \).

Combining \( a^E b \sim_E a \) and \( a^E b \sim_{E^c} a \) which implies \( a^E b \sim a \) by Dynamic Consistency.
Now assume that $\succsim$ is a CEU preference with capacity $\nu$ and utility function $u$.

We assume that beliefs are updated by a procedure which satisfies the following assumption.

**Assumption**

Let $\nu$ be a convex capacity on $\Omega$ and let $E$ be an event. Then if $\nu_E$ denotes the update of $\nu$ conditional on $E$, we assume that,

$$\nu(E) + \nu(E^c) = 1 \text{ implies, } \nu_E(A) = \frac{\nu(A \cap E)}{\nu(E)} \text{ for } A \subseteq \Omega.$$  

Show that commonly used updating rules such as the Generalized Bayesian Update, the Dempster-Shafer Updating rule and the Optimistic Update all satisfy this assumption.
Technical Assumptions

Assumption (Continuity)

The utility function \( u : X \rightarrow \mathbb{R} \) is continuous.

Assumption (Strong Monotonicity)

For two acts \( a, b \in A(S) \), if \( \exists \hat{s} \in S \), such that \( u(a(\hat{s})) > u(b(\hat{s})) \) and \( \forall s \in S, u(a(s)) \geq u(b(s)) \) then \( a \succ b \).
Dynamic Consistency for CEU

Theorem

Let $E_1, \ldots, E_K$ be a non-trivial partition of $\Omega$. If a decision-maker has CEU preferences, which satisfy Consequentialism, Continuity and Strong Monotonicity with beliefs represented by a capacity $\nu$ and (s)he updates his/her beliefs with an updating rule which satisfies Assumption 2.1 then the following conditions are equivalent:

1. (s)he is dynamically consistent,
2. $A \subseteq \Omega, \nu (A) = \sum_{k=1}^{K} \nu (A \cap E_k)$.
Proposition

Let $\mathcal{R} = \{ R_1, ..., R_K \}$ be a given partition of $R$ and let $E_k = R_k \times S$ for $1 \leq k \leq K$. If $\succsim$ is a CEU preference, which is dynamically consistent with respect to $\mathcal{R}$, then for any event $F$, which is measurable with respect to $\mathcal{R}$ and any acts $a, b \in A(T)$, $a^F b \succ a \Rightarrow b \succ a^{F^c} b$.

* Thus a uniform strict preference for randomization is not possible.
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- Thus a uniform strict preference for randomization is not possible.
- This result does not use our independence axiom.
Strict Preference for Randomization

**Proposition**

Let $\mathcal{R} = \{R_1, \ldots, R_K\}$ be a given partition of $R$ and let $E_k = R_k \times S$ for $1 \leq k \leq K$. If $\succsim$ is a CEU preference, which is dynamically consistent with respect to $\mathcal{R}$, then for any event $F$, which is measurable with respect to $\mathcal{R}$ and any acts $a, b \in A(T)$, $a^F b \succ a \Rightarrow b \succ a^{F^c} b$.

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Proposition

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- Thus a uniform strict preference for randomization is not possible.
- This result does not use our independence axiom.
- Convexity is not assumed.
- No assumptions are imposed on the capacity other than those implied by dynamic consistency.
Corollary

Let $\succ$ be a CEU preference over $A(\Omega)$, which satisfies Assumptions Consequentialism, Continuity, Strong Monotonicity, Dynamic Consistency and Independent Randomizations then $\succ$ must be indifferent to randomization.
In a single time period multiple priors preferences may or may not exhibit a strict preference for randomization.

- A core of a convex capacity is an example of a set of priors which does not exhibit preference for randomization.

There is a dynamic extension of multiple priors known as recursive multiple priors.

These preferences are dynamically consistent if the set of priors is rectangular.

- A rectangular set of priors implies indifference to randomization.
An individual with a strict preference for randomization will have a negative value of information.

In the two-ball Ellsberg urn, suppose an individual asks a third party to flip a coin and bet on red (resp. black) if it comes up heads (resp. tails).

Before the draw from the urn, if the third party does not tell the individual the outcome of the coin flip, (s)he will perceive herself as holding the randomization.

However if the third party does reveal the outcome of the coin flip, the given individual will either be holding a bet on the red or black ball, which (s)he views as inferior.

Thus the given individual has a negative value for information.

In contrast indifference to randomization implies zero value for information. SEU preferences also have zero value for information, Kelsey (1992).
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Nature will always choose the least favourable probability which is available.

If the decision-maker believes nature has moved before his/her decision then a strict preference for randomization makes sense.

In contrast if nature moves after the individual has chosen, then randomizing does not help and the decision-maker will be indifferent to randomization.
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Our results show that there is no normative theory which predicts a strict preference for randomization.

Experimental studies find little evidence of dynamic consistency. This suggests there may be a role for a strict preference for randomization in a descriptive theory. However, the available evidence suggests that subjects do not display a strict preference for randomization.
CONCLUSION

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- Our results show that is no normative theory which predicts a strict preference for randomization.
- Experimental studies find little evidence of dynamic consistency.
- This suggests there may be a role for a strict preference for randomization in a descriptive theory.
- However the available evidence suggests that subjects do not display a strict preference for randomization, Dominiak and Schnedler (2011).
• We should be careful when using the Anscombe-Aumann framework to study non-expected utility models.

• EUT is linear in probabilities. Thus the order in which different kinds of uncertainty are resolved is unimportant.

• Non-EU theories (including ambiguity) are non-linear in the probabilities. In this case the order in which different kinds of uncertainty are resolved matters.

• In a Savage framework these issues do not arise. All uncertainty is resolved in one stage.

