INVARiANCE PROPERTIES OF PERSISTENT EQUILIBRIA
AND RELATED SOLUTION CONCEPTS

Dieter Balkenborg, Mathijs Jansen and Dries Vermeulen

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Abstract

Kohlberg and Mertens argued that a solution concept to a game should be invariant under the addition or deletion of an equivalent strategy and not require the use of weakly dominated strategies. In this paper we study which of these requirements are satisfied by Kalai and Samet’s concepts of persistent equilibria and persistent retracts. While none of these concepts has all the invariance properties, we show that a slight rephrasing of the notion of a persistent retract leads to a notion satisfying them all.

JEL classification: C72
Keywords: persistent equilibrium, admissibility, reduced normal form, ordinal invariance

1. Introduction

Kalai and Samet (1984) introduced the notion of a persistent retract to select among the Nash equilibria of a game. They call a Nash equilibrium contained in persistent retract a persistent equilibrium. However, persistent retracts are also of interest in themselves because they can be viewed as sets of rules that are self-enforcing in a strong sense. By a rule we mean hereby a restriction of the possible behavior of a
player to a subset of his strategies, not necessarily to a single strategy. One can think for instance of traffic rules like "drive on the right side of the road". Traffic rules restrict the behavior of car drivers, but they are not intended to determine their behavior in all circumstances, as a single strategy would. Similarly participation in a market typically requires adherence to certain rules which still leave room for individual decisions. We cannot expect rational players to follow a set of rules unless it is self enforcing in the sense that no player has an incentive to violate the rules as long as he expects everyone else to follow the rules. The idea to consider self-enforcing sets of rules rather than just self enforcing strategy combinations, i.e. Nash equilibria, is appealing because it will often be impossible or undesirable to agree on social conventions that perfectly determine behavior.

A set of rules is strongly self-enforcing if no player wants to violate them even if he has a slight uncertainty about everyone else following the rules. Whereas Nash equilibria are self-enforcing considering strongly self-enforcing sets of rules leads to a different, set valued concept, the persistent retracts. The latter are more akin to some set-valued concepts discussed in the literature like the primitive formations used in the theory of equilibrium selection by Harsanyi and Selten (1988), the curb sets of Basu and Weibull (1991) and the sets closed under better replies by Ritzberger and Weibull (1996). Balkenborg (1992) discusses a unified approach to compare these concepts. Whereas simple learning processes do often not converge to Nash-equilibrium, see e.g. Nachbar (1990), Hofbauer and Swinkels (1995), there are a number positive convergence results for these set valued concepts, see e.g. Hurkens (1995) Sanchirico (1996) and Ritzberger and Weibull (1996). Hurkens (1995) in particular discusses a simple stochastic learning process which always converges to a persistent retract.

What distinguishes persistent retracts from the other set valued concepts are their invariance properties. Kohlberg and Mertens (1986) argued that a solution concept should not depend on modelling details. In particular they made the following two requirements on solution concepts for normal form games: a) The solutions should be invariant to the addition or deletion of payoff-equivalent strategies. b) The solutions should not prescribe the use of weakly dominated strategies. Mertens (1987) went further and argued that we lack a theory of ordinal equivalence of games and that the solutions we consider should respect ordinal equivalence. In the theory he developed ordinal invariance can be derived from invariance requirements closely related to those above. However, b) is replaced by the slightly different requirement that the solutions should only depend on the "admissible best reply structure" of a game
which we will use here.

Regardless of whether one considers these requirements as indispensable or not, it is certainly useful to
know whether a solution concepts satisﬁes them or how it violates them. We discuss here the invariance
properties of persistent equilibria and related concepts. For other solution concepts we refer to Mertens
(1987) and Vermeulen and Jansen (1997a). Some results on the invariance properties of persistent retracts
and equilibria are already given in Mertens (1987). They are summarized here for completeness.

To obtain existence of persistent equilibria, Kalai and Samet require a persistent retract to be convex. If one is primarily interested in self-enforcing sets of rules, this restriction is not necessary. Without the
convexity requirement one obtains a slightly different concept which we call \persistent set\". However, convexity still plays an indirect role, via the formation of beliefs. We will show that persistent retracts
can be viewed as the set of beliefs associated with persistent sets.

To select among equilibria as sharply as possible, the deﬁnition of a persistent retract entails a
minimality condition. As in Basu and Weibull (1991) one can replace minimality by a weaker condition
called \tightness\" without altering our main conclusions.

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<tr>
<td>invariance</td>
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| abr-invariance      | yes                               | yes                                   | yes for n = 2
                                                                    | no for n > 2 |

Figure 1.1: The invariance properties of the solution concepts discussed in this paper.

Table 1.1 summarizes the results. We distinguish weak invariance, (full) invariance and abr-invariance.
Weak invariance refers to the behavior of a solution when a payo® equivalent strategy is deleted, invariance
to the behavior when such a strategy is added or deleted. abr-invariance refers to invariance under the
admissible best reply structure.

All concepts satisfy weak invariance. For persistent retracts and equilibria this was ﬁrst noted by
Mertens (1987). However, he showed also that they do not satisfy invariance. In contrast we show here that the notions of tight absorbing or persistent sets we introduce satisfy all the invariance properties and are hence ordinally invariant in the sense of Mertens (1987). Mertens observed that persistent retracts satisfy $abr$-invariance. We show in addition that persistent equilibria satisfy $abr$-invariance for two player games, but not for games with more than two players.

In the remainder of this paper we will discuss cell by cell each entry in the table. Some basic terminology for normal form games is fixed in Section 2. In Section 3 we introduce the solution concepts. Loosely speaking our main result in this section states that persistent retracts correspond to the beliefs associated with a persistent set. The proof is given in the Appendix. Sections 4 and 5 start by defining invariance properties and then study them for the various solution concepts. Section 6 concludes.

2. Preliminaries

For a natural number $n$, a (finite $n$-person normal form) game is a pair $¡ = (A; u)$ such that $A := \prod_{i} A_{i}$ is a product of $n$ non-empty, finite sets and $u = (u_{i})_{i \in \mathbb{N}}$ is an $n$-tuple of functions $u_{i} : A \to \mathbb{R}$. $A_{i}$ is called the set of pure strategies of player $i$ and $u_{i}$ is his payoff function.

As usual, a game $¡ = (A; u)$ will be identified with its mixed extension. For this game, the mixed strategies of player $i$ are the elements of the set $\xi(A_{i})$ of probability distributions on $A_{i}$. By abuse of notation we will identify a pure strategy $a_{i} \in A_{i}$ with the mixed strategy in $\xi(A_{i})$ that puts all weight on $a_{i}$. So, $A_{i}$ will simply be viewed as a subset of $\xi(A_{i})$. Pure strategy profiles will be denoted by $a_{i} \in A_{i}$.

For a (mixed) strategy profile $\bar{x} = (x_{i})_{i \in \mathbb{N}} \in \mathbb{X} := \prod_{i \in \mathbb{N}} \xi(A_{i})$ we define, as usual, the (expected) payoff function of player $i$ by $u_{i}(\bar{x}) := \sum_{a_{i} \in A_{i}} \prod_{j \neq i} x_{j}(a_{j}) u_{i}(a)$ where $x_{j}(a_{j})$ is the probability with which $a_{j}$ is played in $x_{j}$. Furthermore, $(y_{j})_{j \neq i} \in \xi(A_{j})$ denotes the strategy profile where player $i$ uses $y_{i} \in \xi(A_{i})$ and his opponents use their strategies in $\bar{x} \in \mathbb{X}$. For a strategy profile $\bar{x} \in \mathbb{X}$

$$\bar{x} \in \mathbb{X} \quad \text{is the set of best replies (of player $i$) to $\bar{x}$. The strategy profiles in the set $\bar{x}$ are called best replies to $\bar{x}$. A strategy profile $\bar{x} \in \mathbb{X}$ is called a Nash equilibrium of $i$ if $\bar{x} \in \bar{x}$.}$$


3. Persistent equilibria and related solution concepts

A non-empty subset $R \subseteq \mathcal{A}$ of strategy profiles is called a product set if $R = \bigcap_{i} R_i$ with $R_i \subseteq \mathcal{A}_i$ for each player $i$. If the product set $R$ is convex we call it a retract. Notice that a product set is convex if and only if each of its components is convex.

Let $U$ be an open subset in $\mathcal{A}$. We say that a set of strategies $R_i$ of player $i$ absorbs $U$ if $R_i$ contains a best reply against each strategy combination in $U$. A product set $R$ is said to absorb $U$, if each of its components $R_i$ absorbs $U$.

**Definition 3.1.** A product set $R$ is an absorbing set if it absorbs some convex open neighborhood of itself.

**Remark 1.** The preceding definition requires a convex neighborhood for the following reason: A belief of a player $i \neq j$ over player $j$'s strategy choice is a probability distribution over the mixed strategy space $\mathcal{A}_j$. For the purpose of utility maximization this belief can be identified with the expected mixed strategy it induces. Player $i$ is certain that player $j$ will make a strategy choice in $R_j \subseteq \mathcal{A}_j$ if the support of his belief is in $R_j$. The expected mixed strategy will hence be in the closed convex hull of $R_j$. If we consider a small open neighborhood of this set, which we can assume to be convex, we allow for the possibility that player $i$ is slightly uncertain about whether player $j$ chooses a strategy in $R_j$ or not. Thus a product set $R$ is absorbing if no player $i$ has an incentive to violate the rule \"choose in $R_i$\", as long as he is almost certain that his opponents follow the rule \"choose in $R_j$\".

We can now proceed with the definitions of the solution concepts considered in this paper: A persistent set is an absorbing set that is minimal in the following sense:

**Definition 3.2.** A persistent set is an absorbing set which does not properly contain another absorbing set.

**Definition 3.3.** Kalai and Samet (1984) A persistent retract is an absorbing retract which does not properly contain another absorbing retract.

**Definition 3.4.** Kalai and Samet (1984) A persistent equilibrium is a Nash equilibrium contained in a persistent retract.
Kalai and Samet (1984) showed that every game has a persistent retract and every persistent retract contains a Nash equilibrium. In particular, a persistent equilibrium always exists.

If the primary interest is not in selecting among Nash equilibria, but in strongly self-enforcing sets of rules one can replace the minimality condition in the definition of a persistent set / retract as follows:

Definition 3.5. An absorbing set (retract) $R \subseteq A_i$ of a game $i$ is called tight if no open convex neighborhood of $R$ is absorbed by a product set (retract) properly contained in $R$.

Our main result in this section states that the persistent retracts correspond to the sets of beliefs associated with persistent sets. We need the following additional notation: We denote the convex hull of a set $X$ in some Euclidean space by $\text{conv}(X)$. Conversely, for a closed and convex set $Y$ we denote by $\text{ext}(Y)$ its set of extremal points, i.e., the set of points in $Y$ which cannot be written as proper convex combinations of other points in $Y$. For general convex sets $Y$ we define $\text{ext}(Y)$ as the set of extremal points of the closure of $Y$.

Proposition 3.6. Let $i$ be a game and let $R \subseteq A_i$ be a product set. Then the following three statements are equivalent

(i) $R$ is a persistent retract of $i$.

(ii) $R$ is the convex hull of a persistent set of $i$.

(iii) $R$ is closed and convex and $\text{ext}(R)$ is a persistent set of $i$.

The same holds if \texttt{"persistence"} is replaced by \texttt{"tight absorbing"} in the previous statement.

The proof, which uses and extends several central findings in Kalai and Samet (1984), is given in the appendix. The central tool will be a careful analysis of the minimal sets of strategies of a player that absorb an open set.

It will further be shown in the appendix that every absorbing set contains a persistent set. From the above proposition it follows that every absorbing retract contains a persistent retract. Hence the persistent sets (retracts) are exactly the minimal tight absorbing sets (retracts).

Call two strategies of a player $x_i; x_i^0 \subseteq (A_i)$ player-equivalent if they yield the same payoff for the player regardless of the strategies played by the opponents, i.e., if we have for all $y \subseteq A$ and $i$

$$u_i(y, x_i) = u_i(y, x_i^0):$$
We may still have \( u_j(y_j x_i) \neq u_j(y_j x_i^0) \) for some other player \( j \neq i \).

There will be no player-equivalent pure strategies in our examples. The following further consequence from the results in the appendix is therefore useful when studying them:

**Proposition 3.7.** A tight absorbing set is finite. If no player has two distinct player-equivalent pure strategies, then a tight absorbing set contains only pure strategy combinations.

![Figure 3.1: The games Battle-of-the-Sexes (left), Battle-of-the-Sexes with a weakly dominated strategy added (middle) and Matching Pennies (right).](image)

**Example 3.8.** The Battle-of-the-Sexes game depicted in Figure 3.1 on the left has two persistent sets, namely \( (T;L) \) and \( (B;R) \). There is another tight absorbing set, namely the set of all pure strategy combinations \( fT;B g \subseteq fL;R g \), but it is not minimal and hence not persistent. These solutions do not change if we add a weakly dominated strategy \( T^0 \) for player 1 to obtain the game in the middle of Figure 3.1. Although \( (T;L) \) is no longer a strict equilibrium, i.e., it does not have a unique best reply, it is still a persistent equilibrium: As long as player 1 is almost certain that player 2 chooses \( L \), he has no reason to switch to \( T^0 \). However, the Nash equilibrium \( (T^0;L) \) is not even contained in any tight absorbing retract. Finally, the unique persistent retract of the Matching-Pennies game on the right of Figure 3.1 is the whole strategy space \( \emptyset fT;B g \subseteq \emptyset fL;R g \). Consequently the unique mixed strategy equilibrium of this game is persistent.
4. Invariance

For the purposes of this paper we can call any map which assigns to each game $i$ a collection of subsets of $\mathcal{A}$ a solution concept. Kohlberg and Mertens (1986) called a solution concept invariant if, roughly speaking, the addition or deletion of pure strategies that are convex combinations of other pure strategies does not affect the solutions of a game. To make this precise, we have to compare (solution) sets contained in strategy spaces of different dimensions. The method suggested by Mertens (1987) uses reduction maps to identify strategy profiles of the original game $i$ and the game $i^0$, where "duplicate" strategies are deleted. This method can be formalized as follows.

Let $i = hA; u_i i$ and $i^0 = hB; v_i i$ be two games. A map $f = (f_i)_i \in \mathbb{N}$ from $\mathcal{A}$ to $\mathcal{B}$ is called a reduction map from $i$ to $i^0$ (denoted by $i \stackrel{R}{\rightarrow} f i^0$) if for every player $i$ $\in \mathbb{N}$

1. $f_i : \mathcal{A}(A_i) \rightarrow \mathcal{B}(B_i)$ is affine and onto
2. $u_i = v_i \circ f_i$

The function $v_i \circ f_i$ denotes the composition of $v_i$ and $f_i$. For a detailed elaboration of the motivation for this particular formulation we refer to Mertens (1987) and Vermeulen and Jansen (1997b).

Definition 4.1. A solution concept $\mathcal{Q}$ is weakly invariant if for any triplet $(i; i^0; f)$ with $i \stackrel{R}{\rightarrow} f i^0$,

$$\mathcal{Q}(i^0) = \{ S : f(S) = T \}$$

This property states that every image under $f$ of a solution of the game $i$ must be a solution of the game $i^0$ and, conversely, that every solution of $i^0$ is the image under $f$ of a solution of $i$.

Definition 4.2. A solution concept $\mathcal{Q}$ is called invariant if it is weakly invariant and, moreover, for all $T \in \mathcal{Q}(i^0)$,

$$f(i^0(T)) = f(S) \text{ for all } S \in \mathcal{Q}(i)$$

This additional property states that every strategy profile of the game $i$ whose image under $f$ is an element of a solution $T$ of $i^0$ must be an element of a solution $S$ of $i$ whose image under $f$ equals $T$.

The two invariance properties will now be checked for the solution concepts introduced in the previous section. We will repeatedly use the following (straightforward) Lemma:
Lemma 4.3. If $f$ is a reduction map from a game $\pi$ to a game $\pi^0$, then $y$ is a best reply to $x$ in the game $\pi$ if and only if $f(y)$ is a best reply to $f(x)$ in the game $\pi^0$.

Let $(\pi; \pi^0; f)$ be a triplet with $f : \pi \rightarrow \pi^0$. Let $V^0$ be a convex open neighborhood of a product set $R^0$ in the strategy space $\mathcal{A}$ of the game $\pi^0$ and let $V$ be a convex neighborhood of a product set $R$ in the strategy space $\mathcal{A}$ of the game $\pi$ such that $f(V) = V^0$ and $f(R) = R^0$.

Lemma 4.4. $R$ absorbs $V$ in the game $\pi$ if and only if $R^0$ absorbs $V^0$ in $\pi^0$. Consequently, $R$ is absorbing if and only if $R^0$ is absorbing.

Proof. (a) Suppose that $R^0$ absorbs $V^0$ in $\pi^0$. We will show that $R$ absorbs $V$. To this end, take a strategy profile $x \in V$. Then obviously $f(x) \in V^0$. So, since $R^0$ absorbs $V^0$, there exists a best reply $y^0$ to $f(x)$. Furthermore, there exists a strategy profile $y \in R$ with $f(y) = y^0$ since $f(R) = R^0$. Then, by Lemma 4.3, $y$ is a best reply to $x$.

(b) Conversely, suppose that $R$ absorbs $V$ in $\pi$. We will show that $R^0$ absorbs $V^0$. To this end, let $x^0$ be a strategy profile in $V^0$. Then there exists a strategy profile $x \in V$ with $f(x) = x^0$ since $f(V) = V^0$. Furthermore, there exists a best reply $y \in R$ to $x$, since $R$ absorbs $V$. Then $f(y) \in R^0$ is a best reply to $x^0$ by Lemma 4.3.

(c) To prove the second statement of the Lemma, suppose that $R^0$ absorbs a convex open neighborhood $V^0$ of itself. Let $R$ be a product set with $f(R) = R^0$. Then $V := f^{-1}(V^0)$ is a convex open neighborhood of $R$, since $\mathcal{A}$ is convex and $f$ is affine and onto. Furthermore, $f(V) = V^0$. Hence, $R$ absorbs $V$ by (a). Conversely, suppose that the product set $R$ absorbs a convex open neighborhood $V$ of itself. Then $V^0 := f(V)$ is a convex open neighborhood of the product set $R^0 = f(V)$. Hence, $R^0$ absorbs $V^0$ by (b).

Proposition 4.5. Persistent sets and tight absorbing sets are invariant.

Proof. (a) Let $(\pi; \pi^0; f)$ be a triplet with $f : \pi \rightarrow \pi^0$. Assume first that $R$ is a tight absorbing or persistent set of the game $\pi$. Then the product set $R^0 := f(R)$ is absorbing by Lemma 4.4. In order to show that $R^0$ is also tight absorbing or persistent, suppose that $S^0 \subseteq R^0$ is a product set absorbing a convex neighborhood $V^0$ of $R^0$. First note that $S := f^{-1}(S^0) \setminus R$ is a product set since both $f^{-1}(S^0)$ and $R$ are product sets. Furthermore, since $S^0 \subseteq R^0$ and $R^0 = f(R)$, we obtain $f(S) = S^0$. Therefore,
by Lemma 4.4, \( S \) is a product set absorbing the convex open neighborhood \( V := f^{-1}(V^0) \) of \( R \). If \( R \) is a persistent set, it cannot contain a proper product set absorbing \( V \) and hence \( S = R \) since \( S \subseteq R \) by definition. If \( R \) is a tight absorbing set, we obtain \( S = R \) as well, since \( S \) absorbs any sufficiently small neighborhood of \( R \). We conclude in both cases that \( S^0 = f(S) = f(R) = R^0 \). Hence, \( R^0 \) is also either tight absorbing or persistent.

(b) Suppose now that \( R^0 \) is a tight absorbing or persistent set of the game \( i^0 \). Fix \( x_0 \in f^{-1}(R^0) \) and define \( y_0 := f(x_0) \). Choose any product set \( R \) in \( \Gamma_A \) which contains \( x_0 \) and for which \( f : R \to R^0 \) is a bijection. By Lemma 4.4 \( R \) is an absorbing set of \( i \). In order to show that \( R \) is tight absorbing or persistent, let \( S \) absorb a convex neighborhood of \( R \). Then \( f(S) \) absorbs the open neighborhood \( f(V) \) of \( R^0 \) by Lemma 4.4. As in (a) we conclude \( f(S) = R^0 \) since \( R^0 \) is tight absorbing or persistent. Since \( S \subseteq R \) and since \( f : R \to R^0 \) is a bijection we conclude \( R = S \). Therefore \( R \) is tight absorbing or persistent.

Mertens (1987) already noticed that persistent retracts and persistent equilibria are weakly invariant.

**Proposition 4.6.** Tight absorbing retracts, persistent retracts and persistent equilibria are weakly invariant.

**Proof.** Notice that Nash equilibria are invariant by Lemma 4.3. We will only give a proof for tight absorbing retracts, since the proof for persistent retracts is completely analogous. Let \( (i^0 ; f) \) be a triplet with \( i^0 \) an absorbing set of \( i \).

(a) Suppose that \( R \) is a persistent retract of \( i^0 \). Then \( \text{ext}(R) \) is a persistent set of \( i \) by Proposition 3.6. So, by Proposition 4.5 we know that \( f(\text{ext}(R)) \) is a persistent set of \( i^0 \). Hence,

\[
    f(R) = f(\text{conv}(\text{ext}(R))) = \text{conv}(f(\text{ext}(R)))
\]

is a persistent retract of \( i^0 \) by Proposition 3.6.

(b) Suppose that \( R^0 \) is a persistent retract of \( i^0 \). Then \( \text{ext}(R^0) \) is a persistent set of \( i^0 \) with convex hull \( R^0 \) by Proposition 3.6. So, there exists a persistent set \( S \) of \( i \) with \( f(S) = \text{ext}(R^0) \) by Proposition 4.5. Hence, \( \text{conv}(S) \) is a persistent retract of \( i \) by Proposition 3.6, while \( f(\text{conv}(S)) = \text{conv}(f(S)) = \text{conv}(\text{ext}(R^0)) = R^0 \).

The next example shows that persistent or tight absorbing retracts are not invariant.
Example 4.7. (Mertens (1987)) Persistent retracts and equilibria are not invariant. Consider the games in Figure 4.1

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<tr>
<td>B^0</td>
<td>0, 0</td>
<td>0, 0</td>
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Figure 4.1: Matching pennies (left) and Matching Pennies with a mixed strategy duplicated (right).

The first game is again Matching pennies, where the unique persistent (tight absorbing) retract of the first game is \( \xi (A_1^0) \subseteq \xi (A_2^0) \). In the second game the mixed strategy \( \frac{1}{2}; \frac{1}{2} \) of player 1 is duplicated. Let \( f = (f_1; f_2) : \xi_{A_1} \ni \xi_{A_2^0} \) be the affine map where \( f_2 \) equals the identity on \( \xi (A_2) = \xi (A_2^0) \) and where for \( (q_1; q_2; q_3) \in \xi (A_1) \)

\[
    f_1(q_1; q_2; q_3) := (q_1 + \frac{1}{2}q_3; q_2; q_3 + \frac{1}{2}q_3);
\]

(\( q_3 \) denotes the probability for the added strategy.) We have \( i^{PQV}_{f \xi_i^0} \). Propositions 3.6 and 4.5 imply that \( f q 2 \subseteq (A_1) \vdash q_3 = 0 g E \subseteq (A_2) \) is the only persistent or tight absorbing retract of the second game. \( ((\frac{1}{2}; \frac{1}{2}); (\frac{1}{2}; \frac{1}{2})) \) is the only persistent equilibrium of the game \( i^0 \). \( ((\frac{1}{2}; \frac{1}{2}); (0; 0; 1)) \) is a Nash-equilibrium which is mapped by \( f \) onto the persistent equilibrium in \( i \), although it does itself not belong to a persistent retract or even a tight absorbing retract. Hence neither persistent or tight absorbing retracts nor persistent equilibria are invariant.

5. ABr-Invariance

In order to define the third notion of invariance considered in this paper, we first introduce an equivalence relation for games with the same strategy spaces. Two games \( i = hA; u_i \) and \( i'' = hA; u''_i \) are called
admissible-best-reply-equivalent (abr-equivalent) if, for every completely mixed strategy profile $x$, the set of best replies to $x$ in the game $\gamma$ coincides with the set of best replies to $x$ in the game $\gamma^u$.

Definition 5.1. A solution concept $\mathcal{Q}$ is called abr-invariant if

$$\mathcal{Q}(\gamma) = \mathcal{Q}(\gamma^u)$$

for any pair of abr-equivalent games $\gamma$ and $\gamma^u$.

Remark 2. Ab-invariance requires first of all that a solution depends only on the best-reply structure of a game and not on the actual utilities of the players. For instance, the solution concept which assigns to a game the set of all Pareto-efficient Nash-equilibria is not abr-invariant. Secondly, a solution may actually depend only on the admissible best reply correspondence of a game. Hereby a strategy $x_i$ is called an admissible best reply against a strategy combination $y$ if there exist a sequence of strategy combinations $(y_k)_k$ converging to $y$ such that $x_i$ is a best reply against each strategy combination in the sequence. (Notice the similarity to the definition of a normal-form perfect equilibrium in Selten (1975).) Because a weakly dominated strategy is never a best reply against completely mixed strategy combination, abr-invariance implies that a solution does not use weakly dominated pure strategies. (One could always decrease the payoffs to such a strategy such that it becomes strictly dominated. The resulting game would be abr-equivalent.) An abr-invariant solution concept may however contain weakly dominated mixed strategies, as we will show in Example 5.5 below. Thus abr-invariance differs slightly from the admissibility requirement discussed in Kohlberg and Mertens (1986). See Mertens (1987) and Vermeulen and Jansen (1997a) for further discussions on the notion of abr-invariance.

Lemma 5.2. Suppose a product set $P$ absorbs a non-empty open set $V$ of strategy combinations in a game $\gamma$. Then $P$ also absorbs $V$ in any abr-equivalent game $\gamma^u$.

Proof. Let $V^0$ be the set of all completely mixed strategy combinations in $V$. Then $V^0$ is open and dense in $V$. Since $P$ absorbs $V^0$ in $\gamma$ and $\gamma^u$ and since the best reply correspondence is upper hemi-continuous, it follows that $P$ absorbs $V$ in $\gamma^u$. 

Using the lemma it follows straightforwardly from the definitions:

Proposition 5.3. Tight absorbing sets or retracts and persistent sets or retracts are abr-invariant.
Proposition 5.4. Persistent equilibria are $abr$-invariant for two-player games.

Proof. Let $x = (x_1; x_2)$ be a persistent equilibrium of the game $j$. We have to show that $x$ is also a persistent equilibrium of any $abr$-equivalent game $j''$. By Proposition 3.6 each strategy $x_i$ is in the convex hull of a set $R_i$, where $R_1 \subseteq R_2$ is a persistent set of $j$. By Proposition 5.3 $R_1 \subseteq R_2$ is also a persistent set in $j''$. Thus it remains to show that $x$ is a Nash equilibrium of $j''$. For this purpose we write each $x_i$ as a convex combination $\sum_{z \in R_i^0} \alpha(z) z$ where $R_i^0$ is a subset of $R_i$ and each coefficient $\alpha(z)$ is strictly positive. The claim follows, if we can show that every $z \in R_i^0$ is a best reply to $x$ in $j$.

According to Proposition A.3 in the appendix, $R_i$ is a sparse set, i.e., no two strategies in $R_i$ are player-equivalent and each of them is robust. Lemma A.1 i) in the appendix implies that the support of a strategy in $R_i$ consists only of player-equivalent pure strategies. Every pure strategy in the support of a strategy $z \in R_i^0$ is therefore in the support of $x_i$ and hence a best reply to $x$. Consequently each $z \in R_i^0$ is a best reply to $x$ in $j$. Moreover, since every strategy $z \in R_i^0$ is robust, i.e., a best reply against a non-empty open subset of strategies in $j$, it is a best reply against some completely mixed strategy combination $y$ in $j$. For a two-player game the function $u_i(y^0; z)$ is linear in $y^0$. The strategy $z_i$ is therefore a best reply in $j$ against each of the completely mixed strategy combinations $y(,) := (1 \cdot , \cdot ) cx + \cdot cy$ with $0 < c < 1$. Consequently, it is a best reply against each $y(,) (0 < c < 1)$ in the $abr$-equivalent game $j''$. By continuity, $z_i$ is also a best reply against $x = y(0)$, which concludes the proof.

Example 5.5. Although persistent retracts and (for two-player games) persistent equilibria are $abr$-invariant, they may use weakly dominated mixed strategies: In the example of Figure 5.1 the unique persistent retract consists of all strategies of player 1 and only of the strategy $R$ for player 2. It contains the persistent equilibrium $i_1 \frac{1}{2}, 0; 1 \frac{1}{2}; R^q$. However, the strategy $i_1 \frac{1}{2}, 0; 1 \frac{1}{2}$ is weakly dominated by $C$. The appendix shows that in contrast tight absorbing or persistent retracts consist only of robust strategies, which are never weakly dominated.

Finally, the solution concept which assigns to every game the set of all persistent equilibria is not $abr$-invariant for games with more than two players:

\footnotetext[1]{In particular the equilibrium is not perfect. Kalai and Samet (1984) showed that a persistent equilibrium does not have to be perfect using a three-player game somewhat simpler than the one discussed below.}
Figure 5.1: A game with a persistent equilibrium where a weakly dominated strategy is used.

Counterexample. Consider the following three-person game $\mathcal{I}$:

\begin{align*}
\begin{array}{c|ccc}
 & L & M & R \\
\hline
T & 3, 0 & 0, 0 & 0, 1 \\
C & 2, 0 & 2, 0 & 0, 1 \\
B & 0, 0 & 3, 0 & 0, 1 \\
\end{array}
\end{align*}

Player 1 picks the matrix, player 2 the row and player 3 the column. The second game $\mathcal{I}$ differs from $\mathcal{I}$ in the sense that player 1's payoffs are slightly changed if he takes his third strategy:

\begin{align*}
& e_3^1 : \\
& \begin{array}{c|c}
 e_3^1 & e_3^2 \\
\hline
 1; 0 & 0; 0 \\
 1; 1 & 0; 1 \\
\end{array} \\
& e_3^2 : \\
& \begin{array}{c|c}
 e_3^1 & e_3^2 \\
\hline
 0; 1; 1 & 0; 0; 0 \\
 0; 0; 0 & 0; 0; 0 \\
\end{array} \\
& e_3^3 : \\
& \begin{array}{c|c}
 e_3^1 & e_3^2 \\
\hline
 1; 1; 1 & 0; 0; 0 \\
 0; 0; 0 & 1; 0; 0 \\
\end{array}
\end{align*}

Obviously, $(e_3^1; e_3^2; e_3^3)$ is a Nash-equilibrium of the first, but not of the second game. We will show next that $\mathcal{I}$ and $\mathcal{I}$ are $ab\epsilon$-equivalent and then that the whole set of strategy profiles is a persistent...
retract in both games. Hence, the above strategy profile is a persistent equilibrium of , but not of the abr-equivalent game .

Figure 5.2: Graphics for the games and discussed in the counterexample.

 and are abr-equivalent: Since the payoffs for players 2 and 3 are not changed in the two games it suffices to show that the set of pure best replies of player 1 against any completely mixed strategy combination is the same in the two games. We can denote a strategy combination for the opponents by \((p; q) \in [0; 1]^2\), where \(p = \text{prob } e_2^1\) and \(q = \text{prob } e_3^1\). Figure 5.2 illustrates for both games how player 1’s payoff with one of his pure strategies \(e_1^1\) \((1 2 f 1; 2; 3g)\) varies with the strategy combinations \((p; q)\) in the unit square \([0; 1]^2\).

The graphic on the left illustrates this for the game . The payoff with \(e_1^1\) against \((p; q) \in [0; 1]^2\) is calculated as \(6pq + 3p + 3q + 1\). The graph of this function is the quadratic surface in the picture. With \(e_2^1\) player 1 obtains the constant payoff 0, which yields the horizontal plane. With \(e_3^1\) he obtains \(p + q + 1\), which yields the non-horizontal plane. When changing to , the payoffs for \(e_2^1\) and \(e_3^1\) are not altered, while those for \(e_1^1\) are changed to \(2(p + q + 1)\). The corresponding plane is flipped upward in the graphic on the right and no longer intersects the quadratic surface above the point \((1; 1)\).
Since \( e_1^3 \) yields strictly negative payoffs in the triangle \( f p + q < 1g \) and payo\-s 0 along the diagonal \( f p + q = 1g \) in both games, the best replies are the same in the region \( f p + q < 1g \). It therefore remains to study the best replies on the triangle \( f 1 < p + qg \). Here \( e_1^3 \) yields strictly positive payoffs and is hence better than \( e_2^3 \). To compare with the payoffs for \( e_1^1 \) we calculate

\[
6pq \quad 3p \quad 3q + 1 \quad < \quad p + q \quad 1
\]
\[
, \quad pq \quad \frac{2}{3}p \quad 2q + \frac{1}{3} \quad < \quad 0
\]
\[
, \quad pq \quad \frac{2}{3}p \quad \frac{2}{3}q + \frac{4}{9} \quad < \quad \frac{1}{9}
\]
\[
, \quad p \quad q \quad \frac{2}{3} \quad \frac{2}{3} \quad < \quad \frac{1}{9}
\]

The inequality is satisfied if \( p > \frac{2}{3}, q > \frac{2}{3} \); \( p \), \( q \) \( 1 \) and \( (p, q) \not\in (1;1) \). If \( p < \frac{2}{3} \), the first factor in the last inequality is negative and the second factor positive since \( q > 1 \). Similarly the inequality holds for \( q < \frac{2}{3} \). Thus \( e_1^3 \) is the unique best reply on the region \( f 1 < p + qg \) except for \( (1;1) \), where \( e_1^1 \) is also a best reply. When changing to \( i \) the payoffs for \( e_1^3 \) are further increased on the region and \( e_1^3 \) becomes the unique best reply even against \( (1;1) \).

In summary, the best replies on \( f 0 < p < 1g \) \( f 0 < q < 1g \) are the same in both games, i.e., the two games are abr-equivalent.

**Determination of the persistent retract:** It remains to be shown that the whole strategy space is a persistent retract in \( i \) \( i \) \( i \) \( i \). Since both games are abr-equivalent, it suffices to show that any persistent set of \( i \) must contain all pure strategy combinations. Since the games have no player-equivalent strategies, a persistent set can only contain pure strategy combinations.

Let \( R = R_1 \times R_2 \times R_3 \) be a persistent set of \( i \). Let \( V = V_1 \times V_2 \times V_3 \) be a convex open neighborhood of \( R \) which is absorbed by \( R \).

(i) Suppose \( R_1 \) contains \( e_1^1 \). Consider a strategy profile \( (e_1^1; x_3) \) \( R_1 \times V_3 \) with \( x_3 \) completely mixed. The unique best reply of player 2 against such a profile is \( e_2^3 \). Hence \( e_2^3 \) \( R_2 \). Similarly, \( e_3^3 \) \( R_3 \). Furthermore, \( e_1^3 \) is the unique best reply of player 1 to \( e_2^3 \). Hence, \( e_1^3 \) \( R_1 \). But then it follows as before that \( e_1^3 \) \( R_2 \) and \( e_1^3 \) \( R_3 \). Therefore \( R_2 = A_2 \) and \( R_3 = A_3 \). One calculates that \( e_1^3 \) is the unique best reply against the strategy pair \( ((\frac{3}{2}; \frac{2}{3}); (\frac{3}{2}; \frac{3}{2})) \) \( V_2 \times V_3 \). It is therefore contained in \( R_1 \). Thus \( R = A_1 \times A_2 \times A_3 \).
(ii) Suppose now that \( e_2^j \notin R_1 \) or that \( e_3^j \notin R_1 \). Then \( (e_2^j; e_3^j) \notin R_2 \), and therefore \( e_1^j \notin R_1 \). Step (i) implies \( R = A_1 \leq A_2 \leq A_3 \), which concludes the discussion of the example.

6. Conclusions

We have argued that persistent sets (or tight absorbing sets) can be viewed as strongly self-enforcing sets of rules. We have seen that these solution concepts satisfy all the invariance properties discussed here.

Mertens (1987) has noted that persistent retracts do not satisfy the invariance requirement related to the addition of a payoff-equivalent strategy. However, we have shown that persistent retracts correspond to the sets of beliefs associated with a persistent set. We therefore think that the lack of invariance of persistent retracts is caused by the identification of beliefs and mixed strategies.

The lack of invariance properties of persistent equilibria is more severe. It occurs because the concept does not refine among the equilibria within a persistent retract, although it restricts the set of equilibria to be considered. To satisfy all invariance properties it is hence necessary to amend further refinement requirements to the concept. For instance, Mertens (1992) has shown that every persistent retract contains a strategically stable set as reformulated in Mertens (1989). Such \( \setminus \) persistent stable sets satisfy all the invariance properties.

A. Appendix

In the following we fix an arbitrary game \( \hat{\gamma} \). We need some additional terminology before stating our first Lemma. A strategy \( x_i \in \zeta(\mathcal{A}_i) \) is called a robust best reply of player \( i \) against the strategy combination \( y \in \zeta(\mathcal{A}) \), if \( x_i \) is a best reply against all strategy combinations in a neighborhood of \( x \) in \( \zeta(\mathcal{A}) \). Any strategy which is a robust best reply against some strategy combination is called robust. We call \( x_i \) a semi-robust best reply against \( y \), if there exists a sequence \( (y_k)_{k=0}^\infty \) converging to \( y \) such that \( x_i \) is a robust best reply against each \( y_k \). Finally, a face of the strategy simplex \( \zeta(\mathcal{A}_i) \) is the convex hull of a non-empty subset of pure strategies.

Lemma A.1. i) The sets of robust best replies against a strategy combination \( y \in \zeta(\mathcal{A}) \) is either empty or a full player-equivalence class that forms a face of the strategy simplex \( \zeta(\mathcal{A}_i) \).

ii) There exists a semi-robust best reply against every strategy combination \( y \in \zeta(\mathcal{A}) \).
iii) If $x_i$ is a semi-robust best reply against $y$ then there is a strategy combination $z$ arbitrarily close to $y$ such that all best replies against $z$ are player-equivalent to $x_i$.

Proof. i) Suppose $x_i$ is a robust best reply against $y$. If $x_0^i$ is player-equivalent to $x_i$, then $x_0^i$ is a best reply whenever $x_i$ is and therefore $x_0^i$ is a robust best reply against $y$. If $x_0^i$ is a robust best reply against $y$ then $x_0^i$ is player-equivalent to $x_i$ by Lemma 4 in Kalai and Samet (1984). Hence the set of all robust best replies against $y$ is a player-equivalence class. Moreover, if $x_0^i$ is a robust best reply against $y$ and $a_i$ a pure strategy in the support of $x_0^i$, then $a_i$ is a best reply against a strategy-combination whenever $x_0^i$ is. Consequently $a_i$ is also a robust best reply against $y$ and therefore player-equivalent to $x_0^i$ and hence to $x_i$. It follows that the player equivalence class of $x_i$ is a face of $\xi(A_i)$.

ii) Consider any open neighborhood of $y$. By Lemma 5 in Kalai and Samet (1984) there exists a strategy combination $y_0$ in this neighborhood such that all best replies against $y_0$ are player-equivalent. Their proof reveals moreover that all best replies against $y_0$ are robust best replies against $y_0$. We can consequently find a sequence $i \cdot y_k, 1$ converging to $y$ such that the set of best replies against each $y_k$ is equal to the set of robust best replies against $y_k$. Since the set of best replies against $y_k$ is a face of the strategy simplex and since $\xi(A_i)$ has only finitely many faces, we can choose the sequence $i \cdot y_k, 1$ such that the set of (robust) best replies against $y_k$ consists of the same face $F$ of strategies for all $k \geq 1$. Therefore each $x_i \in F$ is a semi-robust best reply against $y$.

iii) Let $x_i$ be a semi-robust best reply against $y$ and choose an open neighborhood $U$ of $y$. By definition, there exists $z \in U$ such that $x_i$ is a robust best reply against $z$ and therefore an open neighborhood $V \mu U$ of $z$ such that $x_i$ is a best reply against each $z^0, 2 V$. By Lemma 5 in Kalai and Samet (1984) we can choose $z^0, 2 V$ such that all best replies against $z^0$ are player-equivalent to $x_i$. /
sparse set of strategies is finite. Since these faces are all disjoint, no strategy in a sparse set can be a convex combination of other strategies in the set.

Let \( U \) be a non-empty open subset of \( \xi_A \). We say that a subset of strategies for a player tightly absorbs \( U \) if it absorbs \( U \) while no proper subset absorbs \( U \). A product set tightly absorbs \( U \), if each of its components does.

**Lemma A.2.** Let \( U \) be a non-empty open subset of \( \xi_A \). Then:

a) Every strategy set of a player that absorbs \( U \) contains a subset that tightly absorbs \( U \).

b) A strategy set that tightly absorbs \( U \) is sparse.

**Proof.** Suppose \( R_i \) absorbs \( U \). Let \( R_i^0 \) denote the set of strategies in \( R_i \) that are semi-robust best replies against some strategy combination in \( U \). Lemma A.1 ii) and iii) implies that \( R_i^0 \) absorbs \( U \). Let \( R_i^{(0)} \) be a set obtained by selecting for each face \( F \) \( F_i \) a single strategy in \( F \setminus R_i^0 \), if this intersection is not empty. Then \( R_i^{(0)} \) is sparse and absorbs \( U \) by construction. Moreover, the following argument shows that it tightly absorbs \( U \): Let \( x_i \in R_i^{(0)} \). Then \( x_i \) is a semi-robust best reply against some \( y \in U \). By Lemma A.1 iii) we can find \( y^0 \in U \) such that all best replies against \( U \) are player-equivalent. Therefore \( x_i \) is the only best reply against \( y^0 \) contained in \( R_i^{(0)} \).

Suppose that moreover \( R_i \) tightly absorbs \( U \). Then it follows \( R_i = R_i^{(0)} \). Hence \( R_i \) is sparse. /

The definitions imply:

**Proposition A.3.** A non-empty product set \( R = \prod_{i=1}^{n} R_i \) in \( \xi_A \) is a tight absorbing set if and only if \( R \) tightly absorbs each sufficiently small convex open neighborhood of itself. In particular, each component \( R_i \) is sparse.

**Proposition A.4.** Every absorbing set contains a persistent set.

**Proof.** Let \( R \) be an absorbing set. Using Lemma A.2 a) we can find a product set \( R^0_{1, \mu} R \) that tightly absorbs some convex open neighborhood \( U \) of \( R \). \( R^0 \) is finite since each of its components is sparse. If \( R^0 \) is not a persistent set, then it will properly contain an absorbing set. If the latter is not persistent, it will again properly contain some absorbing set etc. Since \( R^0 \) is finite, we will find after finitely many steps an absorbing set that does not properly contain an absorbing set, i.e., a persistent set. /
We have now all the tools to give the

**Proof of Proposition 3.6.** We only give the proof for persistency. The proof for tight absorbing sets/retracts is similar.

(i) implies (ii): Let \( R \) be a persistent retract. By Proposition A.4 \( R \) contains a persistent set \( R_0 \) \( \mu \) \( R \). Then \( \text{conv}(R_0) \mu R \) is an absorbing retract and therefore \( R = \text{conv}(R_0) \).

(ii) implies (iii): Let \( R = \text{conv}(R_0) \), where \( R_0 \) is a persistent set. Since each \( R_0 \) is a sparse, it consists of finitely many strategies lying on disjoint faces of \( \zeta(A_i) \). We obtain \( \text{ext}(R_i) = \text{ext}(\text{conv}(R_0)) = R_0 \) for each player \( i \) and hence \( \text{ext}(R) = R_0 \). Moreover, \( R \) is closed and convex.

(iii) implies (i): Since \( R \) is convex, \( \text{ext}(R) = R_0 \) and \( R_0 \) is finite, we obtain \( R = \text{conv}(R_0) \). An absorbing retract \( R_0 \) contained in \( R \) must contain a persistent set by Proposition A.3. But this persistent set must be \( R \). Hence \( \text{conv}(R_0) \mu R_0 \mu R = \text{conv}(R) \) since \( \text{conv}(R_0) \) and \( R \) are closed. Therefore \( R \) is a persistent retract.

/ **References**


