On the Evolutionary Selection of Sets of Nash Equilibria

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Abstract

It is well established for evolutionary dynamics in asymmetric games that a pure strategy combination is asymptotically stable if and only if it is a strict Nash equilibrium. We use an extension of the notion of a strict Nash equilibrium to sets of strategy combinations called ‘strict equilibrium set’ and show the following. For a large class of evolutionary dynamics, including all monotone regular selection dynamics, every asymptotically stable set of rest points that contains a pure strategy combination in each of its connected components is a strict equilibrium set. A converse statement holds for two-person games, for convex sets and for the standard replicator dynamic.

Keywords: evolutionary dynamics, replicator dynamic, regular selection dynamics, strict equilibrium set, Nash equilibrium component.

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1 Introduction

Under a wide range of evolutionary dynamics, a pure-strategy Nash equilibrium of an asymmetric normal-form game will be asymptotically stable if and only if it is a strict Nash equilibrium. This observation has been made repeatedly, for instance by Eshel and Akin [12], Samuelson and Zhang [30] and Ritzberger [29], see also Ritzberger and Vogelsberger [28].

The objective of this paper is to capture in a similar vein the dynamic stability properties of sets of Nash equilibria.

Many games have Nash equilibrium components that do not consist of isolated equilibria. If a strategy combination is contained in such a component then it will not be asymptotically stable because Nash equilibria are rest points. Accordingly, we will investigate which Nash equilibrium components or, more generally, which sets of Nash equilibria can be asymptotically stable under a wide range of evolutionary dynamics.

The sets of Nash equilibria that play an important role in our paper are called strict equilibrium sets. Strict equilibrium sets generalize strict Nash equilibria to sets of strategy combinations. Following Balkenborg [3], a set of Nash equilibria is called a strict equilibrium set (short, SE set) if for any element in the set, either a player loses strictly by deviating unilaterally or else his deviation leads to another Nash equilibrium in the set. For instance, the set of strategy combinations leading to the forward induction outcome for a twice repeated battle-of-the-sexes game (van Damme [38]) is a SE set. Further examples of SE sets are provided throughout the paper.

We first study the asymptotic stability of sets of Nash equilibria under the standard replicator dynamic (Taylor [36]). Here we obtain a very clear-cut characterization. A set of Nash equilibria is an asymptotically stable set if and only if it is a strict equilibrium set. The proof of asymptotic stability relies on an alternative characterization of SE sets that is motivated by Thomas’ [37] notion of an evolutionarily stable set.

However, while some theoretical results single out the standard replicator dynamic as a central learning and imitation dynamic (e.g. Börgers and Sarin [8], Gale et al. [14], Schlag [32]), the standard replicator dynamic is a knife-edge case in the sense that very similar dynamics may behave very differently. We are hence interested in extending our analysis to a broad class of evolutionary dynamics that share only some basic qualitative features with the standard

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1Loosely speaking, a strategy combination is asymptotically stable if all trajectories starting nearby stay close and converge to it in the long run.
replicator dynamic.

The more general evolutionary dynamics we consider are dynamics that “reinforce best replies”. By this we mean that the dynamic is a regular selection dynamic – as defined in Samuelson and Zhang [30] – and that it satisfies the list of intuitive properties discussed in Section 4, which link growth rates and payoffs. For instance, best replies must have non-negative growth rates. Our definition encompasses all sign preserving (Nachbar [26]) and monotone dynamics (Samuelson and Zhang [30]).

Given our results on the standard replicator dynamic, only SE sets can be asymptotically stable under all dynamics that reinforce best replies. However, other sets of Nash equilibria can be asymptotically stable under some dynamics that reinforce best replies. For instance, it is well known that the interior mixed strategy Nash equilibrium in Matching Pennies is asymptotically stable under the adjusted replicator dynamic (see, e.g. Maynard Smith [22], Appendix J). We restrict attention to sets that contain a pure strategy combination in each connected component, a property shared by SE sets. We show that a set of Nash equilibria with this property is a SE set if it is asymptotically stable under some dynamic reinforcing best replies.

Finally we consider the converse and ask whether a SE set is asymptotically stable under any dynamic reinforcing best replies. We can currently prove this only if the SE set satisfies one additional restriction, namely, each strategy combination in the SE set must be a best reply to all best replies against it. This condition is always satisfied for SE sets in two-player games or for SE sets that are convex.

For two-player games our results imply the following tight characterization of asymptotic stability in purely static, game theoretic terms. A set containing a pure strategy combination in each connected component is an asymptotically stable set of rest points for a given dynamic reinforcing best replies if and only if it is a SE set.

In a related study, Ritzberger and Weibull [29] show for a slightly different class of evolutionary dynamics that a face is asymptotically stable if and only if it is closed under better replies. (There is no condition on how the dynamic behaves within the set.) To be closed under better replies means that after replacing the strategy of one or several players by a pure better reply to the current strategy combination the resulting strategy combination also belongs to the face. Our findings complement their results. We focus on asymptotically stable sets of Nash equilibria containing a pure strategy combination in each connected component while they focus on asymptotically stable faces.
To highlight the differences between the two approaches, note that a SE set need not exist, while there is always a face that is closed under better replies as the entire set of mixed strategy combinations trivially has this property. Like strict Nash equilibria, sets closed under better replies are robust to payoff perturbations while SE sets are not. As an advantage of our approach, sometimes SE sets have predictive power when sets closed under better replies do not, as in the universal Shapley example due to Hofbauer and Swinkels [19] discussed in the Appendix. In this example the SE set is a cycle whereas the unique set closed under better replies consists of the entire set of strategy combinations. When SE sets are convex our approaches turn out to be very similar. This is because a convex SE set is a face and hence a set that is closed under better replies.

This paper is directly connected to the literature on set-wise solution concepts for evolutionary games. As shown in more detail in Balkenborg [3], SE sets are the multi-population counterpart of evolutionarily stable sets defined for single population contests by Thomas [37] (see also Balkenborg and Schlag [5]). However, the use of explicit dynamics differentiates this paper from other investigations into the evolutionary stability of sets (e.g. Sobel [34] or Swinkels [35]) that remain in a static framework.

Our characterization result for the standard replicator dynamic complements recent research on strategic stability. Applying Corollary 1 of Demichelis and Ritzberger [11], it follows that a connected SE set contains a strategically stable set if its Euler characteristic is non-zero.

The rest of the paper is organized as follows. SE sets are defined in Section 2 and their basic properties are studied. In Section 3 the connection between SE sets and the standard replicator dynamic is made using the concept of a direct evolutionarily stable set. Section 4 introduces the properties defining a dynamic that reinforces best replies. Its two subsections investigate the relationship between asymptotically stable sets of rest points and SE sets. Section 5 contains the conclusions. The Appendix provides examples showing that our geometric characterization of SE sets given in Section 2 is tight. In particular, we give examples with two or three players where the SE set is a cycle.

2 Strict Equilibrium Sets

We start by recalling some basic game-theoretic terminology and notation. We closely follow Ritzberger and Weibull [29].
Consider a normal form game $\Gamma$ with finitely many players and strategies. Let $N := \{1, 2, \ldots, n\}$ be the set of $n \geq 2$ players and let $S_i = \{s_i^k, k = 1, 2, \ldots, K_i\}$ be the set of pure strategies of player $i \in N$. The set of pure strategy combinations is $S := \times_{i \in N} S_i$ with generic element $s = (s_1, s_2, \ldots, s_n)$. The set of mixed strategies of player $i$ is the $(K_i - 1)$-dimensional unit simplex $\Delta_i := \{\sigma_i \in \mathbb{R}^{K_i}_+ | \sum_{k=1}^{K_i} \sigma_i^k = 1\}$. The set of mixed strategy combinations is $\Delta := \times_{i \in N} \Delta_i$ with generic element $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. We will often ignore the order when describing an $n$-tuple provided it is clear from the indexation. For instance, we will write $(\sigma_{-i}, \tau_i)$ for the strategy combination played when player $i$ uses strategy $\tau_i$ while each player $j \neq i$ uses $\sigma_j$.

The support of a mixed strategy $\sigma_i \in \Delta_i$ is denoted by $\text{supp}(\sigma_i) := \{s_i^k \in S_i | \sigma_i^k > 0\}$. As usual, pure strategies $s_i^k$ are identified with the corresponding unit vectors $e_i^k \in \Delta_i$. In particular, $\text{supp}(\sigma_i)$ is identified with the set $\{e_i^k \in \Delta_i | \sigma_i^k > 0\}$. The support of a mixed strategy combination $\sigma \in \Delta$ is the Cartesian product $\text{supp}(\sigma) := \times_{i \in N} \text{supp}(\sigma_i)$. Every subset $T_i \subseteq S_i$ of a player’s pure strategy set spans a face $\mathfrak{F}(T_i) := \{\sigma_i \in \Delta_i | \text{supp}(\sigma_i) \subseteq T_i\}$ of his mixed strategy simplex $\Delta_i$. The faces of the convex polyhedron $\Delta$ are the sets $\mathfrak{F}(T) := \times_{i \in N} \mathfrak{F}(T_i)$ spanned by a Cartesian product of pure strategy sets $T = \times_{i \in N} T_i, T_i \subseteq S_i$ for $i \in N$. Every mixed strategy $\sigma_i \in \Delta_i$ generates a face $\mathfrak{F}(\sigma_i) := \mathfrak{F}(\text{supp}(\sigma_i)) \subseteq \Delta_i$ and every mixed strategy combination $\sigma \in \Delta$ generates a face $\mathfrak{F}(\sigma) := \mathfrak{F}(\text{supp}(\sigma)) \subseteq \Delta$.

The mapping $u : S \rightarrow \mathbb{R}^n$ assigns to each pure strategy combination the vector of payoffs for the players. The multilinear expected payoff function $U : \Delta \rightarrow \mathbb{R}^n$ and the mixed best reply correspondence $\tilde{\beta} = \times_{i \in N} \tilde{\beta}_i : \Delta \rightarrow \Delta$ are defined as usual. $\sigma \in \Delta$ is a Nash equilibrium when $\sigma \in \tilde{\beta}(\sigma)$, it is a strict Nash equilibrium when $\{\sigma\} = \tilde{\beta}(\sigma)$. The pure best reply correspondence $\beta = \times_{i \in N} \beta_i : \Delta \rightarrow S$ is given by $\beta(\sigma) := \tilde{\beta}(\sigma) \cap S$. We recall that $\rho \in \tilde{\beta}(\sigma)$ if and only if $\text{supp}(\rho) \subseteq \beta(\sigma)$.

A set of strategy combinations is connected if it cannot be written as a union of two non-empty relatively open subsets. A connected component of a set of mixed strategy combinations is a maximally connected subset. A Nash equilibrium component of a normal form game is a connected component of the set of Nash equilibria.

The following concept introduced in Balkenborg [3] is central to this paper. It extends the property of being strict from an equilibrium point to an equilibrium.

**Definition 1** A non-empty subset $G \subseteq \Delta$ is a strict equilibrium set (SE set) if for all
\( \sigma \in G \) and all \( \tau_i \in \Delta_i \) the inequality

\[
U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma)
\]

holds whereby equality implies \((\sigma_{-i}, \tau_i) \in G\).\(^2\)

The following example, discussed further at the end of Subsection 4.2, illustrates the notion of a SE set. In the game in Figure 1 player 1 chooses one of the rows \(U\) or \(D\), player 2 one of the columns \(L\) or \(R\) and player 3 one of the matrices \(F\) or \(B\). Let \(x_1 = \text{prob}(D)\), \(x_2 = \text{prob}(R)\) and \(x_3 = \text{prob}(B)\).

![Figure 1: An example of an SE set (shaded).](image)

The set \(G = \{x_1 = 1\} \cup \{x_3 = 1\}\) is a SE set of the game. It is shaded in the graphic.

The following observations are immediate. A singleton set \(\{\sigma\}\) is a SE set if and only if \(\sigma\) is a strict Nash equilibrium. A SE set consists of Nash equilibria. Every union and every non-empty intersection of SE sets is a SE set.

SE sets may or may not exist. Examples of games where a strict equilibrium set exists are the weighted potential games introduced by Monderer and Shapley [25] where the set of strategy

\(^2\)Note that we could equivalently require that equality holds if and only if \((\sigma_{-i}, \tau_i) \in G\). This is so because if \(G\) is a SE set and \(\sigma, \tau_i\) are as in the definition then \(U_i(\sigma_{-i}, \tau_i) = U_i(\sigma)\) if and only if \((\sigma_{-i}, \tau_i) \in G\).
combinations maximizing the potential is a SE set. On the other hand, there is no SE set in Matching Pennies.

In contrast to strict Nash equilibria, SE sets do not have to be robust under payoff perturbations.\(^3\) This is demonstrated in the Appendix using an example due to Hofbauer and Swinkels [19].

The examples above show that SE sets do not have to be convex. Examples in the appendix show that they can form cycles, even in two-player games. Nonetheless, SE sets have a very special geometric structure which we exploit in our proofs. The results in this section are concerned with this structure.

**Lemma 1** A strict equilibrium set \( G \) contains a strategy combination \( \sigma \) if and only if \( G \) contains the support of \( \sigma \).

**Proof.** Fix a strategy combination \( \sigma \in \Delta \). Consider for \( 0 \leq i \leq n \) the statements
\[
(S:i) \text{ "The set } (\times_{1 \leq j \leq i} \text{supp}(\sigma_j)) \times (\times_{i+1 \leq j \leq n} \{\sigma_j\}) \text{ is contained in the SE set } G."
\]

Notice that statement \((S:0)\) simply reformulates that \( \sigma \in G \) and that statement \((S:n)\) states that \( G \) contains the support of \( \sigma \). Thus, we can complete the proof by showing for all \( i \in \mathbb{N} \) that \((S:i-1)\) implies \((S:i)\) and vice versa.

First we will show that \((S:i-1)\) implies \((S:i)\). Assume that \((S:i-1)\) holds. Consider any \((r_1, \cdots, r_{i-1}) \in \times_{1 \leq j \leq i-1} \text{supp}(\sigma_j)\). Let \( \rho = (r_1, \cdots, r_{i-1}, \sigma_i, \sigma_{i+1}, \cdots, \sigma_n) \). Since \((S:i-1)\) holds, \( \rho \) belongs to the SE set. Since \( \rho \) is a Nash equilibrium we have \( U_i(\rho_{-i}, s_i) = U_i(\rho_{-i}, \sigma_i) \) for all \( s_i \in \text{supp}(\sigma_i) \). The definition of a SE set implies that all strategy combinations \((\rho_{-i}, s_i), s_i \in \text{supp}(\sigma_i)\) belong to the SE set. This shows that \((S:i)\) holds.

Now we will show the converse, namely that \((S:i)\) implies \((S:i-1)\). Consider again any \((r_1, \cdots, r_{i-1}) \in \times_{1 \leq j \leq i-1} \text{supp}(\sigma_j)\) and let \( \rho = (r_1, \cdots, r_{i-1}, \sigma_i, \sigma_{i+1}, \cdots, \sigma_n) \). Fix some \( s_i \in \text{supp}(\sigma_i) \). Since \((S:i)\) holds we have that \((\rho_{-i}, s_i) \in G\) and hence \( U_i(\rho_{-i}, s_i) \geq U_i(\rho_{-i}, r_i) \) for all \( r_i \in \text{supp}(\sigma_i) \). Combining this with the analogous argument with the roles of \( r_i \) and \( s_i \) interchanged shows that \( U_i(\rho_{-i}, s_i) = U_i(\rho_{-i}, r_i) \) holds for all \( r_i \in \text{supp}(\sigma_i) \). Thus, \( U_i(\rho_{-i}, s_i) = U_i(\rho_{-i}, \sigma_i) \) and hence \((\rho_{-i}, \sigma_i)\) belongs to the SE set. This shows that \((S:i-1)\) holds.

**Proposition 1** i) Every SE set contains a pure strategy combination.

ii) Every SE set is a closed union of faces.

\(^3\)We thank an anonymous referee for pointing this out to us.
iii) A SE set is convex if and only if it is a face. If a face \( F \) is a SE set then \( \tilde{\beta}(\sigma) = F \) for all \( \sigma \in \tilde{F} \).

iv) Every SE set is a finite union of connected SE sets.

v) A SE set is connected if and only if it is a minimal SE set in the sense that it does not properly contain a SE set.

**Proof.** Statements (i) and, since there are only finitely many faces, (ii) are obvious given Lemma 1.

iii) Let \( G \) be a SE set and hence by (ii) a union of faces of \( \Delta \). Assume \( G \) is convex. Then there exists a maximal face \( \tilde{F}(\sigma) \) contained in \( G \), i.e. a face contained in \( G \) that is not a proper subset of another face contained in \( G \). Assume that \( \tau \in G \). Because \( G \) is convex it contains \( \frac{1}{2}\sigma + \frac{1}{2}\tau \). The lemma implies then that \( G \) contains the face \( \tilde{F}\left(\frac{1}{2}\sigma + \frac{1}{2}\tau\right) \). Since \( \tilde{F}(\sigma) \) is a maximal face contained in \( G \) we must have \( \tilde{F}(\sigma) = \tilde{F}\left(\frac{1}{2}\sigma + \frac{1}{2}\tau\right) \) and therefore \( \tau \in \tilde{F}(\sigma) \). It follows that \( G \) is a subset of \( \tilde{F}(\sigma) \) and hence that \( \tilde{F}(\sigma) = G \).

Now suppose that the face \( \tilde{F} = \times_{i \in \mathcal{N}} \tilde{F}_i \) is a SE set. Let \( \sigma \in \tilde{F} \) and \( i \in \mathcal{N} \). The definition of a SE set implies for every \( \tau_i \in \tilde{\beta}_i(\sigma) \) that \( \tau_i \in \tilde{F}_i \) and hence \( \tilde{\beta}(\sigma) \subseteq \tilde{F} \). Conversely, if \( \tau_i \in \tilde{F}_i \) then both \( \sigma \) and \( (\sigma_{-i}, \tau_i) \) are Nash equilibria as elements of \( \tilde{F} \) and therefore \( \tau_i \in \tilde{\beta}_i(\sigma) \) holds. This proves \( \tilde{F} \subseteq \tilde{\beta}(\sigma) \) and hence (iii).

iv) As a finite union of faces a SE set \( G \) has finitely many connected components. Consider such a connected component \( G' \) of \( G \). Let \( \sigma \in G' \) and let \( \tau_i \) be a strategy of player \( i \in \mathcal{N} \). Then \( U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma) \). Suppose \( U_i(\sigma_{-i}, \tau_i) = U_i(\sigma) \). Setting \( \tau_i^\alpha = (1 - \alpha)\sigma_i + \alpha\tau_i \) we find that \( U_i(\sigma_{-i}, \tau_i^\alpha) = U_i(\sigma) \) holds for all \( 0 \leq \alpha \leq 1 \). Hence \( \{(\sigma_{-i}, \tau_i^\alpha)\}_{0 \leq \alpha \leq 1} \) is contained in \( G \). Together with the fact that \( G' \) is a connected component of \( G \) and that \( \sigma \in G' \) we obtain that \( \{(\sigma_{-i}, \tau_i^\alpha)\}_{0 \leq \alpha \leq 1} \) is contained in \( G' \). In particular \( (\sigma_{-i}, \tau_i) \) belongs to \( G' \) which proves that \( G' \) is a SE set. Hence (iv) holds.

v) Given (iv) a minimal SE set must be connected. To prove the converse, assume that a SE set \( G' \) is properly contained in a connected SE set \( G \). Since \( G \) is a connected union of faces it must contain a face \( \tilde{F} = \times_{i \in \mathcal{N}} \tilde{F}(T_i) \) \( (T_i \subseteq S_i) \) that intersects \( G' \), but is not contained in \( G' \). Since \( G' \) is also a union of faces, there exists a face \( \tilde{F}' = \times_{i \in \mathcal{N}} \tilde{F}(T'_i) \) that is maximal with respect to the property of being contained in the relative boundary of \( \tilde{F} \) and being contained in \( G' \). For some player \( i \in \mathcal{N} \) there is a pure strategy \( r_i \in T_i \setminus T'_i \). Hence there must exist a pure strategy combination \( s \in \times_{i \in \mathcal{N}} \tilde{F}(T'_i) \subseteq G' \) with \( (s_{-i}, r_i) \notin G' \) since otherwise a face larger than \( \tilde{F}' \) in the boundary of \( \tilde{F} \) would be contained in \( G' \). We have \( U_i(s) = U_i(s_{-i}, r_i) \) since both \( s \) and \( (s_{-i}, r_i) \)
are contained in $G$. However, since $s \in G'$ and $(s_{-i}, r_i) \notin G'$ this contradicts the assumption that $G'$ is a SE set. This concludes the proof of (v). ■

It may be worthwhile to point out that a strong solution in the sense of Nash [27] is a convex strict equilibrium set. Conversely, every convex strict equilibrium set is a weak solution in the sense of Nash [27].

For an arbitrary subset $T \subseteq S$ of pure strategy combinations we call

$$G(T) := \{ \sigma \in \Delta \mid \text{supp}(\sigma) \subseteq T \}$$

the $n$-convex hull of $T$.\(^4\) Thus $G(T)$ is the (finite) union of all faces $\mathcal{G}(\sigma)$ with $\text{supp}(\sigma) \subseteq T$.\(^5\)

**Proposition 2** Let $G$ be a non-empty set of mixed strategy combinations and let $T$ be the set of pure strategy combinations contained in $G$. Then $G$ is a SE set if and only if

i) $G$ is the $n$-convex hull of $T$, and

ii) for all $s \in T$ and all $t_i \in S_i$ the inequality

$$U_i(s_{-i}, t_i) \leq U_i(s)$$

holds whereby equality implies $(s_{-i}, t_i) \in T$.

The necessity of condition (i) above is simply a reformulation of Lemma 1 in the new terminology. Condition (ii) simplifies the investigation of SE sets as it states that it suffices to check the SE set conditions for pure – and hence finitely many – strategy combinations only.

**Proof.** Given Lemma 1 clearly the “only if” conditions of the proposition hold when $G$ is a SE set. Suppose, conversely, that the “only if” conditions of the proposition hold for $T$. Let $\tau_i \in \Delta_i$ and $\sigma \in G(T)$, i.e. $\text{supp}(\sigma) \subseteq T$. The multi-linearity of the expected payoff function then implies $U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma)$ because (1) applies to every $s \in \text{supp}(\sigma)$ and $r_i \in \text{supp}(\tau_i)$. $U_i(\sigma_i, \tau_i) = U_i(\sigma)$ implies equality in (1) and thereby $(s_{-i}, t_i) \in T$ for all $s \in \text{supp}(\sigma)$ and $r_i \in \text{supp}(\tau_i)$, which implies $(\sigma_{-i}, \tau_i) \in G(T)$. Thus $G(T)$ is a SE set. ■

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\(^4\)Generalizing the notion of a biconvex set due to Aumann and Hart [2] we call a subset $R \subseteq \Delta$ $n$-convex if it contains with any $\sigma$, $(\sigma_{-i}, \rho_i) \in R$ the strategy combinations $(\sigma_{-i}, (1 - \alpha)\sigma_i + \alpha\rho_i)$ for all $0 \leq \alpha \leq 1$. It is not difficult to show that the set $G(T)$ defined in the text is the smallest $n$-convex set containing $T$.

\(^5\)Notice that not any union of faces is the $n$-convex hull of a set of pure strategy combinations. A counterexample is provided by the relative boundary of the set of mixed strategy combinations, i.e., by the union of its proper faces.
Thus any SE set is the $n$-convex hull of pure strategy combinations. The examples constructed in the Appendix show, conversely, that any $n$-convex hull of pure strategy combinations is a SE set for a game with suitably chosen payoff functions.

3 Evolutionary Stability and the Standard Replicator Dynamic

Before we can address the dynamic properties of SE sets we establish the equivalence of the notion of SE sets to a set-valued concept of evolutionary stability adapted from Thomas [37]. The results in this section were first proved in Balkenborg [3], where our concept defined below is compared to Thomas’ definition for evolutionary conflicts within a single population. The term “direct” is borrowed from Selten [33] to emphasize this difference.

Definition 2 A closed, non-empty subset $G \subseteq \Delta$ is a direct evolutionarily stable set (direct ES set) if every strategy combination $\sigma$ in $G$ has a neighborhood $V(\sigma)$ such that

$$\sum_{i \in N} U_i(\rho) \leq \sum_{i \in N} U_i(\rho_{-i}, \sigma_i)$$

holds for all $\rho \in V(\sigma)$ whereby equality implies $\rho \in G$.

Proposition 3 A set of strategy combinations is a direct ES set if and only if it is a SE set.

Proof. Let $\sigma$ be a strategy combination in a direct ES set $G$ and let $V(\sigma)$ be a neighborhood of $\sigma$ as described in the definition. Let $\tau_i$ be any strategy of player $i$, let $\varepsilon \in (0,1)$ and let $\rho = (\sigma_{-i}, (1-\varepsilon) \sigma_i + \varepsilon \tau_i)$ where $\varepsilon$ is sufficiently small such that $\rho \in V(\sigma)$. Then inequality (2) must hold for this particular $\rho$. However, since $\rho = (\rho_{-j}, \sigma_j)$ for all $j \neq i$, this inequality simplifies to

$$U_i(\sigma_{-i}, (1-\varepsilon) \sigma_i + \varepsilon \tau_i) \leq U_i(\sigma)$$

or

$$U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma).$$

i.e. $\sigma$ is a Nash equilibrium. Moreover, equality in (3) implies $(\sigma_{-i}, (1-\varepsilon) \sigma_i + \varepsilon \tau_i) \in G$.\footnote{This argument is borrowed from Selten’s [33] proof that a direct ESS is a pure strategy.} In order to conclude that $(\sigma_{-i}, \tau_i)$ is itself in $G$ we must use in addition that $G$ is a closed set. If we have equality in (3) we obtain $U_i(\sigma_{-i}, \tau_i) = U_i(\sigma_{-i}, \rho_i^* \delta)$ for all $\rho_i^* = (1-\alpha) \sigma_i + \alpha \tau_i$, $0 \leq \alpha \leq 1$. Let $\delta$ be the supremum of all $\alpha$ with $(\sigma_{-i}, \rho_i^* \delta) \in G$. Closedness implies $(\sigma_{-i}, \rho_i^* \delta) \in G$ and hence $U_i(\sigma_{-i}, \tau_i) = U_i(\sigma_{-i}, \rho_i^* \delta)$ implies $(\sigma_{-i}, (1-\varepsilon) \rho_i^* + \varepsilon \tau_i) \in G$ for all small $\varepsilon > 0$. This
contradicts the definition of δ unless δ = 1. Therefore (σ_−i, τ_i) ∈ G. Consequently, every direct ES set is a SE set.

In order to prove the converse it is not enough to consider strategy combinations ρ that coincide with an element σ of the SE set except for the strategy of a single player. For arbitrary ρ close to σ we use multi-linearity to obtain a Taylor expansion of the difference \( \sum_{i \in N} (U_i(\rho) - U_i(\rho_−i, \sigma_i)) \) around σ. We show that the lowest nonzero terms in the expansion correspond to the payoffs losses of players who deviate unilaterally from strategy combinations in the SE set. These are hence negative and it turns out that higher-order terms can be ignored if ρ is sufficiently close to σ.

More specifically, fix a strategy combination σ in the SE set G. It is well-known and immediately verified that the sets

\[
V_\varepsilon(\sigma) = \{ \rho \in \Delta \mid \rho_i = (1 - \varepsilon_i) \sigma_i + \varepsilon_i \tau_i \text{ for some } \tau_i \in \Delta_i \text{ and } 0 \leq \varepsilon_i \leq \varepsilon \text{ for all } i \in N \}
\]

form for 0 < \varepsilon < 1 a basis of neighborhoods of σ in Δ since Δ is a convex polyhedron.\(^7\) For a subset of players \( I \subseteq N \) we use the following notation. The set of strategy combinations for the players in I is denoted by \( S^I = \times_{i \in I} S_i \). For \( x \in \mathbb{R}_+^n \) the number \( x_I = \prod_{i \in I} x_i \) is obtained by multiplying all \( x_i, i \in I \). Similarly, \( \tau(s_I) = \prod_{i \in I} \tau_i(s_i) \) is the product of all probabilities \( \tau_i(s_i), i \in I \). \( (\sigma_{N \setminus I}, \tau_I) \) denotes the strategy combination, where the players in I use their strategies in τ, while the others use their strategies in σ. For \( \rho = ((1 - \varepsilon_i) \sigma_i + \varepsilon_i \tau_i)_{i \in N} \in V_\varepsilon(\sigma) \) we obtain the multilinear expansion

\[
\sum_{i \in N} (U_i(\rho) - U_i(\rho_−i, \sigma_i)) = \sum_{I \subseteq N} \sum_{\emptyset \neq J \subseteq N \setminus \{i\}} \varepsilon_i (1 - \varepsilon)_{N \setminus (J \cup \{i\})} \varepsilon_J \left[U_i(\sigma_{N \setminus (J \cup \{i\})}, \tau_{J \cup \{i\}}) - U_i(\sigma_{N \setminus J \cup \{i\}}, \tau_J)\right]
\]

\[
= \sum_{\emptyset \neq I \subseteq N} (1 - \varepsilon)_{N \setminus I} \varepsilon_I \left[\sum_{i \in I} (U_i(\sigma_{N \setminus I}, \tau_I) - U_i(\sigma_{N \setminus (I \cup \{i\})}, \tau_{I \cup \{i\}}))\right]
\]

\[
= \sum_{\emptyset \neq I \subseteq N} \sum_{s_I \in S^I} (1 - \varepsilon)_{N \setminus I} \varepsilon_I \tau(s_I) \left[\sum_{i \in I} (U_i(\sigma_{N \setminus I}, s_I) - U_i(\sigma_{N \setminus (I \cup \{i\})}, s_{I \cup \{i\}}))\right]
\]

For \( \emptyset \neq I \subseteq N \) we call \( s_I \in S^I \) trivial if \( U_i(\sigma_{N \setminus I}, s_I) = U_i(\sigma_{N \setminus (I \cup \{i\})}, s_{I \cup \{i\}}) \) holds for all \( i \in I \). We define recursively for any set of players \( \emptyset \neq I \subseteq N \) the sets \( S^0, S^I, S^+ \subseteq S^I \) as follows. The strategy combination \( s_I \in S^I \) belongs to \( S^0 \) if \( s_I \) is trivial and if either \( I = \{i\} \) for

\(^7\)For arbitrary closed convex sets a corresponding statement would not necessarily be true.
some \( i \in \mathcal{N} \) or if \( s_{I \setminus \{i\}} \in S_{0}^{I \setminus \{i\}} \) for all \( i \in I \). It belongs to \( S_{-}^{I} \) if \( s_{I} \) is not trivial and if either \( I = \{i\} \) for some \( i \in \mathcal{N} \) or if \( s_{I \setminus \{i\}} \in S_{0}^{I \setminus \{i\}} \) for all \( i \in I \). Otherwise \( s_{I} \in S_{+}^{I} \) belongs to \( S_{+}^{I} \).

Since \( \sigma \in G \) it follows by induction for all \( I \subseteq \mathcal{N} \) that \( (\sigma_{\mathcal{N} \setminus I}, s_{I}) \in G \) for all \( s_{I} \in S_{0}^{I} \). For \( s_{I} \in S_{+}^{I} \) we hence obtain \( U_{i}(\sigma_{\mathcal{N} \setminus I}, s_{I}) \leq U_{i}(\sigma_{\mathcal{N} \setminus (I \setminus \{i\})}, s_{I \setminus \{i\}}) \) for all \( i \in I \), whereby a strict inequality must hold for at least one (and actually, all) \( i \in I \) because \( s_{I} \) is not trivial. Because there are only finitely many strategy combinations of this type we can find a strictly positive number \( A \) such that

\[
\sum_{i \in I} \left( U_{i}(\sigma_{\mathcal{N} \setminus I}, s_{I}) - U_{i}(\sigma_{\mathcal{N} \setminus (I \setminus \{i\})}, s_{I \setminus \{i\}}) \right) < -A
\]

holds for all \( \emptyset \neq I \subseteq \mathcal{N} \), \( s_{I} \in S_{+}^{I} \). Similarly, there exists a positive constant \( C \) such that

\[
\sum_{i \in I} \left( U_{i}(\sigma_{\mathcal{N} \setminus I}, s_{I}) - U_{i}(\sigma_{\mathcal{N} \setminus (I \setminus \{i\})}, s_{I \setminus \{i\}}) \right) < C
\]

holds for all \( \emptyset \neq I \subseteq \mathcal{N} \), \( s_{I} \in S_{-}^{I} \). Moreover, our construction implies for all \( \emptyset \neq I \subseteq \mathcal{N} \) and \( s_{I} \in S_{-}^{I} \) that there exists a proper, non-empty subset \( J \) of \( I \) such that \( s_{I} = (s_{J}, s_{I \setminus J}) \) with \( s_{J} \in S_{-}^{I} \). Hence we obtain the following upper bound for the expression in (6):

\[
\sum_{\emptyset \neq I \subseteq \mathcal{N}} \sum_{s_{I} \in S_{-}^{I}} \varepsilon_{I} \tau(s_{I}) \left[ -A(1 - \varepsilon)^{n - \#I} + \varepsilon WC \right]
\]

where \( W \) is an appropriate positive constant. We can now choose \( \varepsilon > 0 \) so small that the terms in square brackets (which do depend on the choice of \( \sigma \) but not on the choice of \( \tau \)) are strictly negative. It follows then that (4) is not positive for any \( \tau \). Moreover, if there exists \( \emptyset \neq I \subseteq \mathcal{N} \) and \( s_{I} \in S_{-}^{I} \) such that \( \varepsilon_{I} \tau(s_{I}) > 0 \) then (4) is strictly negative. Suppose hence that \( \varepsilon_{I} \tau(s_{I}) = 0 \) for all \( \emptyset \neq I \subseteq \mathcal{N} \) and \( s_{I} \in S_{-}^{I} \). We must then also have \( \varepsilon_{J} \tau(s_{J}) = 0 \) for all \( \emptyset \neq I \subseteq \mathcal{N} \) and \( s_{J} \in S_{-}^{I} \) since each such \( s_{I} \) takes the form \( (s_{J}, s_{I \setminus J}) \) with \( s_{J} \in S_{-}^{I} \). Thus all terms in the sum of (6) and of (5) are zero. It then follows by induction that all strategy combinations \( (\sigma_{\mathcal{N} \setminus I}, \tau_{I}), I \subseteq \mathcal{N} \), are in the SE set. By Lemma 1, \( G \) contains the support of all these strategy combinations and hence also the support of \( \rho \). Again by Lemma 1, \( \rho \) belongs to the SE set. Thus \( G \) is a direct ES set.

Now we turn to dynamics. Throughout the paper we use the standard evolutionary dynamic setting for asymmetric games, as in Samuelson and Zhang [30]. There is an infinite population associated with each player in the game. Members of each population only play pure strategies.\(^8\)
Average play in each population changes in continuous time. In the following we first consider a specific dynamic known as the standard replicator dynamic (Taylor [36]) which is defined by the following system of ordinary differential equations:\footnote{Under the adjusted replicator dynamic (see Maynard Smith [22]) the right hand side is divided by $U_i(\sigma)$ (and payoffs are assumed to be strictly positive).}

$$\dot{\sigma}_i^k = \sigma_i^k \left[ U_i \left( \sigma_{-i}^k, s_i^k \right) - U_i(\sigma) \right], \quad \forall k = 1, \ldots, K_i, \forall i \in \mathcal{N}.$$ 

Börgers and Sarin [8] show how the standard replicator dynamic arises from a model of learning and Schlag [32] shows how it is derived from an “optimal” imitation rule.

Most of the following concepts are standard for dynamical systems. The strategy combination $\sigma \in \Delta$ is a rest point if $\dot{\sigma}_i^k = 0$ for all $i \in \mathcal{N}$ and $1 \leq k \leq K_i$. A set $F \subseteq \Delta$ is forward invariant if all trajectories starting in this set remain in this set. A set $U \subseteq \Delta$ is a neighborhood of a non-empty set $M \subseteq \Delta$ if $U$ contains an open set which contains the closure of $M$.\footnote{In the literature neighborhoods tend to be defined only for closed sets. We need a more general definition and choose one in which not all open sets are neighborhoods of themselves.} The set $M$ is (Lyapunov) stable if every neighborhood $U$ of $M$ contains a neighborhood $V$ of $M$ such that any trajectory starting in $V$ never leaves $U$. The strategy combination $\rho$ is an $\omega$-limit point of a trajectory $\{\sigma_t^i\}_{t \geq 0}$ if every neighborhood of $\rho$ contains points $\sigma^t$ of the trajectory with arbitrarily large $t$. The $\omega$-limit set of a trajectory is the set of all its $\omega$-limit points. The set $M$ is an attractor if it has a neighborhood $U$ such that the $\omega$-limit sets of trajectories starting in $U$ are contained in $M$. A stable attractor is called asymptotically stable. A single strategy combination $\sigma \in \Delta$ is (asymptotically) stable if the singleton set $\{\sigma\}$ is (asymptotically) stable.

In this paper we are primarily interested in asymptotically stable sets of rest points. Such sets are isolated sets of rest points in the sense that they have neighborhoods that do not contain other rest points. In particular, these sets are closed since the dynamic is continuous. It follows that if an asymptotically stable set of rest points has only finitely many connected components, then each of them is isolated and itself asymptotically stable.

The relevance of direct ES sets for the standard replicator dynamic becomes apparent if we use Lyapunov functions similar to those considered in Zeeman [41]. For a fixed mixed strategy combination $\sigma \in \Delta$ we consider the function

$$L^\sigma(\rho) = \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \text{supp}(\sigma_i)} \sigma^k_i \ln \left( \rho^k_i \right)$$
defined for any mixed strategy combination $\rho \in \Delta$ with $\text{supp}(\sigma) \subseteq \text{supp}(\rho)$.\(^{11}\)

It is a well-known straightforward exercise to show that $L^\sigma$ is strictly concave with a unique maximum at $\sigma$. Hence the sets $V_\delta (\sigma) = \{ \rho \in \Delta \mid L^\sigma (\rho) \geq L^\sigma (\sigma) - \delta \}$ ($\delta > 0$) form a basis of neighborhoods of $\sigma$. Let $(\rho^t)_{t \geq 0}$ be the trajectory under the standard replicator dynamic where $\text{supp}(\sigma) \subseteq \text{supp}(\rho^0)$. Then

$$
\frac{dL^\sigma (\rho^t)}{dt} = \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \text{supp}(\sigma_i)} \frac{\partial L^\sigma}{\partial \rho^{k_i}_i} \frac{d (\rho^{k_i}_i)^k}{dt} = \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \text{supp}(\sigma_i)} \frac{\sigma_i^k}{(\rho^t)^k_i} \left[ (\rho^t)^k_i \left( U_i (\rho^t_{i-1}, s_i^k) - U_i (\rho^t) \right) \right] = \sum_{i \in \mathcal{N}} \left( U_i (\rho^t_{i-1}, \sigma_i) - U_i (\rho^t) \right).
$$

Assume that $\sigma$ is contained in a direct ES set $G$. If $\delta > 0$ is sufficiently small then the neighborhood $V_\delta (\sigma)$ is forward invariant under the standard replicator dynamic and $\sigma$ is a rest point. So $\sigma$ is stable. Since $G$ is compact, the entire set $G$ is also stable. Moreover, $L^\sigma$ increases along any trajectory starting sufficiently close to $\sigma$ with $\dot{L}^\sigma (\hat{\rho}) = 0$ for any $\omega$-limit point $\hat{\rho}$ (Lemma 2.6.1 in Hofbauer and Sigmund [18]). Hence $\hat{\rho} \in G$ by the definition of a direct ES set. Invoking compactness of $G$ we conclude that $G$ is an attractor. To summarize, we have shown:

**Lemma 2** For the standard replicator dynamic, a direct ES set is an asymptotically stable set of stable rest points.

This leads us to our main result concerning the standard replicator dynamic.

**Theorem 1** A non-empty set of strategies is a SE set if and only if it is an asymptotically stable set of rest points under the standard replicator dynamic. Moreover, for this dynamic every element of a SE set is stable.

Proposition 3 together with Lemma 2 implies the “only if” and “moreover” statements of this result. The remaining proof is presented at the end of Section 4.1.

Theorem 1 allows us to refine our understanding of SE sets given in Proposition 3. Since Nash equilibria are rest points of the standard replicator dynamic and since SE sets are asymptotically stable there can be no rest points arbitrarily close to a SE set. Thus we obtain the following result.

\(^{11}\) $L^\sigma (\rho)$ is a linear transformation of the Kullback-Leibler norm (see [21]).
Corollary 1 A SE set is a finite union of Nash equilibrium components. 

As an application of Corollary 1 in Demichelis and Ritzberger [11], we can relate, using Proposition 1 and Theorem 1, SE sets to strategic stability.

Corollary 2 A SE set with non-zero Euler characteristic contains a strategically stable set in the sense of Mertens [24].

4 Dynamics That Reinforce Best Replies

In the rest of the paper we consider more general evolutionary dynamics. The terminology and notation again follow closely Ritzberger and Weibull [29].

Like Samuelson and Zhang [30] we restrict attention to regular selection dynamics. These dynamics preserve two features of the standard replicator dynamic. Namely, no new strategies emerge under this selection dynamic and the evolution of play in each population is described by continuous growth rate functions. While the restriction to regular selection dynamics is natural for models of evolution, other processes which assume more rationality of the agents, like the best response dynamic (see Gilboa and Matsui [15]), fictitious play or perturbed versions of these dynamics (see, e.g., Fudenberg and Levine [13] and Hopkins [20]) are ruled out. Formally, a regular selection dynamic on $\Delta$ is a system of ordinary differential equations

$$
\dot{\sigma}_k^i = f_k^i (\sigma) \sigma_k^i, \quad \forall k = 1, \ldots, K_i, \forall i \in N,
$$

such that (i) the growth rate $f_k^i : \Delta \to \mathbb{R}$ of the pure strategy $s_k^i$ is a continuous function, (ii) $f_k^i (\sigma) \sigma_k^i$ is Lipschitz continuous, and (iii) $\sum_{k'} f_k^{i'} (\sigma) \sigma_{k'}^{i'} = 0$ for all $\sigma \in \Delta$, $i \in N$ and $k = 1, \ldots, K_i$. For every initial starting point $\sigma^0 \in \Delta$ this system of differential equations has a unique solution $\{\sigma^t\}_{t \geq 0} \subset \Delta$ starting in $\sigma^0$. The trajectory satisfies $\text{supp} (\sigma^t) = \text{supp} (\sigma^0)$ for all $t \geq 0$. In particular, the faces of $\Delta$ are invariant under the dynamic. We call the dynamic differentiable if $f_k^i (\sigma) \sigma_k^i$ is a differentiable function of $\sigma$ for all $i$ and $k$.

The results in the remainder of this paper hold for any regular selection dynamic that shares some or all of the following properties with the replicator dynamic. Dynamics that satisfy all four properties and to which hence all our results apply are given a special name.

Definition 3 A dynamic REINFORCES BEST REPLIES if it is a regular selection dynamic that satisfies the following conditions for any player $i \in N$, any strategy combination $\sigma \in \Delta$ and any pure strategy $s_k^i \in S_k^i$. 

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A) The inequality $f^k_i(\sigma) \geq 0$ holds whenever $s^k_i \in \beta_i(\sigma)$.

B) The inequality $f^k_i(\sigma) > 0$ holds whenever $s^k_i \in \beta_i(\sigma)$ and $\sigma_i \notin \tilde{\beta}_i(\sigma)$.

C) The inequality $f^k_i(\sigma) < 0$ holds whenever $s^k_i \notin \beta_i(\sigma)$ and $\sigma_i \in \tilde{\beta}_i(\sigma)$.

D) If $U_i(\sigma_{-i}, s^k_i) \neq U_i(\sigma)$ and $s^k_i \in \text{supp}(\sigma_i)$ then $\sigma$ is not a rest point.

Property (A) states that best replies are not selected against. Since $\sum_{s^k_i \in \text{supp}(\sigma_i)} f^k_i(\sigma) \sigma^k_i = 0$ holds for all $\sigma \in \Delta$ and $i \in N$ it implies that each Nash equilibrium is a rest point. Property (B) means literally that best replies are reinforced (i.e. have strictly positive growth rates) unless all strategies currently played are already best replies.\footnote{Notice that Property (B) does not imply Property (A). Consider the $2 \times 2$ game where all payoffs are identically zero. The conditions in Property (B) are vacuous so any regular selection dynamic satisfies Property (B). On the other hand, Property (A) implies that all strategy combinations are rest points.} In accordance with the literature, we impose restrictions on the growth rates instead of on the dynamic itself. Assuming strictly positive growth rates of a pure strategy that is not being played has implications on nearby states in which it is played.

**Lemma 3** The following holds for every regular selection dynamic satisfying Property (B).

a) Any pure strategy combination belonging to a stable set of rest points is a Nash equilibrium.

b) If the dynamic is also differentiable, then any strategy combination belonging to a stable set of rest points is a Nash equilibrium.

**Proof.** Suppose $\sigma$ belongs to a stable set of rest points, but is not a Nash equilibrium. Since $\sigma$ is not a Nash equilibrium there exists by Property (B) a pure best reply $s^k_i$ with $f^k_i(\sigma) > 0$. Since $f^k_i(\sigma) \sigma^k_i = 0$ holds at a rest point, $s^k_i$ is not in the support of $\sigma_i$.

a) If $\sigma = s$ is a pure strategy combination then the strategy combinations

$\sigma^\alpha = (s_{-i}, (1 - \alpha)s_i + \alpha s^k_i) \, (0 \leq \alpha \leq 1)$ form a one-dimensional face of $\Delta$ which is invariant under the dynamic. By continuity, $f^k_i(\sigma^\alpha) > 0$ holds for all sufficiently small $\alpha$. Consequently, there is a $\beta > 0$ such that any trajectory starting in $\sigma^\alpha$ with $\alpha > 0$ sufficiently small will reach $\sigma^\beta$ in finite time. The set $V = \{ \rho \in \Delta \, | \, f^k_i(\rho) \rho^k_i < \frac{1}{2} f^k_i(\sigma^\beta) \beta \}$ is a neighborhood of the set of all rest points that does not contain $\sigma^\beta$. Since trajectories starting arbitrarily close to $s$ leave this neighborhood, $s$ cannot belong to a stable set of rest points.

b) We can assume without loss of generality that $s^k_j \in \text{supp}(\sigma_j)$ for all $j \in N$. The
linearization of the vector field around $\sigma$ is then described by the matrix

$$ A = \begin{pmatrix} \frac{\partial f^k_i \rho^k_i}{\partial \rho^l_j} |_{\rho=\sigma} \end{pmatrix}_{i,j \in \mathcal{N}, l,k \in \{1,\ldots,K_i-1\}}. $$

The product rule yields

$$ \frac{\partial f^k_i \rho^k_i}{\partial \rho^l_j} |_{\rho=\sigma} = \begin{cases} \left( \rho^k_i \frac{\partial f^k_i}{\partial \rho^l_j} \right) |_{\rho=\sigma} = 0, & \text{for } j \neq i \text{ or } l \neq k, \\ \left( f^k_i (\rho) + \rho^k_i \frac{\partial f^k_i}{\partial \rho^l_j} \right) |_{\rho=\sigma} = f^k_i (\sigma), & \text{for } j = i \text{ and } l = k. \end{cases} $$

The matrix $A$ hence has a row where the diagonal element is $f^k_i (\sigma) > 0$ and where all other entries are zero. This implies that the linearization of the vector field around $\sigma$ has a positive Eigenvalue. It is well known that $\sigma$ cannot be a stable rest point under this condition (see e.g. Hirsch and Smale [16]).

The proof given by Hirsch and Smale [16] in fact shows more. In their proof the state space is transformed so that the rest point under consideration is the origin $x = 0$. Then a closed neighborhood $U$ of the rest point and a closed cone $C$ are constructed with the following three properties. i) There are no rest points other than $x = 0$ in $C \cap U$. ii) Trajectories starting in $C \cap U$ remain in $C$ as long as they remain in $U$. iii) Trajectories which start in $C \cap U$ but not at the rest point $x = 0$ leave the neighborhood $U$ in finite time. Notice that (i) follows from part (a) of the lemma on page 188 and that (iii) emerges from the discussion following the lemma on pages 189 – 190.

Now let $D$ be the boundary of $U \cap C$, which is a closed set containing no rest points. Its complement $N$ is an open neighborhood of the set of all rest points. By construction there are trajectories starting arbitrarily close to the rest point $x = 0$ in $C \cap U$ which enter $D$ and hence leave $N$. This implies that $x = 0$ cannot belong to a stable set of rest points.

Thus, Property (B) is added to ensure that a stable set of rest points consists only of Nash equilibria. Property (C) is the counterpart to Property (B) which applies only when a player plays a best reply to the current strategy combination. It requires that non best replies are not reinforced and helps to stabilize Nash equilibria against non-best responses. Property C is used in the next section.

Properties (A) and (B) do not impose any restrictions on the dynamic at points where no player is choosing a pure best response, for instance when all players use only strictly dominated strategies. Property (D) requires that some selection is still at work in such situations. It states
that the selection process does not come to a halt as long as not all pure strategies currently used are equally good. Property (D) implies that all rest points are “partial equilibria” in the sense that the Nash equilibrium condition $U_i(\sigma) \geq U_i(\sigma_{-i}, s_i^k)$ (effectively “=” ) is satisfied with respect to all strategies $s_i^k$ in the support of $\sigma_i$, $i \in \mathcal{N}$. These “partial equilibria” are precisely the rest points of the standard replicator dynamics. Notice that partial equilibria are described in non-dynamic, purely game theoretic terms.

If a dynamic satisfies Property (A) then all rest points are Nash equilibria. If it satisfies Property (D) then all rest points are also rest points of the standard replicator dynamic. Since a SE set is an isolated set of rest points for the standard replicator dynamic (Theorem 1) we conclude.

**Lemma 4** Any SE set is an isolated set of rest points under a regular selection dynamic satisfying Properties (A) and (D).

Sign preserving (Nachbar [26]) and monotone (Samuelson and Zhang [30]) dynamics are regular selection dynamics frequently discussed in the literature. A dynamic is *monotone* if

$$\text{sign} \left( f_i^k(\sigma) - f_i^l(\sigma) \right) = \text{sign} \left( U_i(\sigma_{-i}, s_i^k) - U_i(\sigma_{-i}, s_i^l) \right)$$

holds for all $\sigma \in \Delta$, $i \in \mathcal{N}$, $s_i^k, s_i^l \in \mathcal{S}_i$. Nachbar [26] calls a dynamic *sign preserving* if

$$\text{sign} f_i^k(\sigma) = \text{sign} \left( U_i(\sigma_{-i}, s_i^k) - U_i(\sigma) \right)$$

holds for all $\sigma \in \Delta$, $i \in \mathcal{N}$, $s_i^k \in \mathcal{S}_i$. (Weibull [40] speaks of a “payoff positive dynamics”.) It is verified immediately that both of these dynamics reinforce best replies. This is however not true for the dynamics Ritzberger and Weibull [29] call sign preserving. They only require that $f_i^k(\sigma) < 0$ holds if and only if $U_i(\sigma_{-i}, s_i^k) < U_i(\sigma)$. This weaker notion does not imply our Property (B).

The standard and the adjusted replicator dynamic as well as the sophisticated imitation dynamics in Hofbauer and Schlag [17] are monotone and sign preserving.

### 4.1 When are Asymptotically Stable Sets SE Sets?

Recall that an asymptotically stable set of rest points is isolated.

**Proposition 4** Consider a regular selection dynamic that satisfies Properties (A) and (B) and is either differentiable or satisfies Property (D). Then an isolated and stable set of rest points that contains a pure strategy combination in each of its connected components is a SE set.

Note that the statement is not correct without the restriction to sets that contain a pure strategy combination in each connected component. Matching Pennies is a counterexample.
Its unique Nash equilibrium is interior and is hence not contained in a SE. However, it is asymptotically stable under the adjusted replicator dynamic (see Maynard Smith [22], Appendix J, or Weibull [40], p. 199ff) and under the sophisticated imitation dynamics in Hofbauer and Schlag [17].

**Proof.** Let $T$ be the non-empty set of all pure strategy combinations in a set of rest points $R$ with the required properties. We will first apply Proposition 2 to show that $\mathcal{G}(T)$ is a SE set. By Lemma 3 (a) $T$ consists of Nash equilibria. Hence it remains to be shown for all $s \in T$ and $s_i^k \in S_i$ that $U_i(s_{-i}, s_i^k) = U_i(s)$ implies $(s_{-i}, s_i^k) \in T$. If this equality holds then both $s_i$ and $s_i^k$ are best replies to $s$. Property (A) implies that all strategy combinations $\sigma^\alpha \ (0 \leq \alpha \leq 1)$ defined as in the proof of Lemma 3 (a) are rest points. Since $R$ is an isolated set of rest points it must contain all $\sigma^\alpha$ and in particular the strategy combination $\sigma^1 = (s_{-i}, s_i^k)$, which was to be proven.

Thus $\mathcal{G}(T)$ is a SE set, consists of Nash equilibria and hence of rest points. Since every connected component of $\mathcal{G}(T)$ intersects the isolated set of rest points $R$, $\mathcal{G}(T)$ is contained in $R$.

If the dynamic is differentiable we know from Lemma 3 (b) that $R$ consists of Nash equilibria. Because $\mathcal{G}(T)$ is a finite union of Nash equilibrium components (by Corollary 1) we can find a neighborhood of $\mathcal{G}(T)$ that contains no Nash equilibria outside $\mathcal{G}(T)$. Hence every connected component of $R$ that intersects $\mathcal{G}(T)$ is contained in $\mathcal{G}(T)$. Therefore $R$ is contained in $\mathcal{G}(T)$ and we conclude $\mathcal{G}(T) = R$.

If the dynamic satisfies Property (D) we can instead argue as follows. By Lemma, 4, $\mathcal{G}(T)$ is an isolated set of rest points. Since every connected component of $R$ intersects $\mathcal{G}(T)$ by assumption it must hence be contained in $\mathcal{G}(T)$. Again we obtain $R \subseteq \mathcal{G}(T)$ and hence $R = \mathcal{G}(T)$. 

We can now prove the “if” statement of Theorem 1.

**Proof.** Given Proposition 4 all we have to show is that every connected component of an asymptotically stable set of rest points under the standard replicator dynamic contains a pure strategy combination. We first show that each of its connected components is itself asymptotically stable. The rest points of the standard replicator dynamic are precisely the “partial equilibria” described earlier. These form a closed semi-algebraic set. A closed semi-algebraic set has a (finite) triangulation (see Bochnak Coste and Roy [7] Theorem 9.2.1) and is hence a finite union of closed, connected components. Given the definition of asymptotic stability it follows
that each of these components is itself asymptotically stable.

Next we show that any given connected asymptotically stable set $R$ contains a pure strategy combination. Consider a strategy combination $\sigma$ contained in this set with minimal support. Let $\mathcal{F}$ be the face spanned by the support of this strategy combination. The intersection $R \cap \mathcal{F}$ is asymptotically stable for the standard replicator dynamic restricted to $\mathcal{F}$, which is the standard replicator dynamic of the game restricted to this face. By minimality, $\sigma$ is a pure strategy combination or $R \cap \mathcal{F}$ is contained in the relative interior of $\mathcal{F}$. Proposition 6 in Ritzberger and Weibull [29] states that the latter cannot be true and hence $\sigma$ is a pure strategy combination.

4.2 When are SE Sets Asymptotically Stable?

For dynamics other than the standard replicator dynamic (see Theorem 1) we will now prove a converse to Proposition 4. We will, however, need an additional assumption, which is always satisfied in two-player games and whenever the SE set is convex.\footnote{As discussed in the Appendix, notice that there are SE sets of two person games that are not convex.} It also holds for the cyclical SE set of the three player game discussed in the Appendix. A counterexample at the end of this section illustrates the role of our additional assumption.

**Lemma 5** Assume that $G$ is a SE set in a two-player game or that $G$ is a convex SE set. Then $G$ satisfies the property:

Every strategy combination $\sigma$ in $G$ has a neighborhood $V(\sigma)$ such that

$$\sigma \in G \text{ and } \rho \in V(\sigma) \cap \bar{\beta}(\sigma) \text{ implies } \sigma \in \bar{\beta}(\rho). \quad (*)$$

**Proof.** Consider $\sigma \in G$ and $\rho \in \bar{\beta}(\sigma)$. Assume that there are only two players. Then $(\sigma_{-i}, \rho_i) \in G$ for $i = 1, 2$ and hence $\sigma \in \beta(\rho)$. Alternatively, assume that $G$ is convex. Then $\sigma \in \bar{\beta}(\rho)$ by Proposition 1 (iii). In both cases Property (*) holds for $V(\sigma) = \Delta$. 

**Proposition 5** Every SE set satisfying Property (*) is an asymptotically stable set of stable rest points for any regular selection dynamic satisfying Properties (A), (B) and (C).

Although the proof of Proposition 5 is complicated by the fact that we do not impose differentiability the intuition behind it is simple. Starting close to a SE set, all non-best replies die out sufficiently fast because of Property (C). Consequently, the local behavior of the dynamic
near an element $\sigma$ in the SE set is characterized by its behavior on the face $\tilde{\beta}(\sigma)$.\footnote{When the dynamic is differentiable $\tilde{\beta}(\sigma)$ can be shown to be a center manifold. If it is twice differentiable this part of the argument then follows directly from the reduction principle for center manifolds, see Arnold and Il’yashenko [1] Part I, §4.3.} For trajectories starting in $\tilde{\beta}(\sigma)$ close to $\sigma$, Properties (A) and (*) imply that the frequency of every pure strategy in the support of $\sigma$ cannot decrease over time. This causes $\sigma$ to be stable. Property (B) can then be used to ensure that trajectories converge to Nash equilibria near $\sigma$ which by Corollary 1 must be in the SE set.

**Proof.** Fix a strategy combination $\sigma$ in the SE set $G$. Then $\sigma$ is a rest point as it is a Nash equilibrium. Let $\mathfrak{F}(\sigma) = \times_{i \in N} \mathfrak{F}(\sigma_i) \subseteq G$ be the face generated by $\sigma$.

We define for $\rho \in \Delta$

$$\gamma(\rho) := \sum_{i \in N} \sum_{s_i \notin \beta_i(\sigma)} \rho_i^k$$

and

$$\dot{\gamma}(\rho) := \sum_{i \in N} \sum_{s_i \notin \beta_i(\sigma)} \rho_i^k f_i^k(\rho).$$

With these definitions $\frac{d\gamma(\rho)}{dt} = \dot{\gamma}(\rho)$ holds along every trajectory $\{\rho^t\}$. Since the dynamic satisfies Property (C) we have $f_i^k(\sigma) < 0$ for all $s_i \notin \beta_i(\sigma)$. Because each $f_i^k(\sigma)$ is continuous there exists a constant $c > 0$ such that $f_i^k(\rho) < -c$ holds for all $i \in N$ and all $s_i \notin \beta_i(\sigma)$ in a sufficiently small neighborhood of $\sigma$. Therefore

$$\dot{\gamma}(\rho) \leq -c\gamma(\rho) \leq 0$$

holds in such a neighborhood.

Since the dynamic is Lipschitz continuous we can find a constant $L > 0$ such that the inequality

$$\sum_{i \in N} \sum_{s_i \in S_i} |\rho_i^k f_i^k(\rho) - \rho_i^k f_i^k(\rho)| \leq L \sum_{i \in N} \sum_{s_i \in S_i} |\rho_i - \rho_i^k|$$

(8)

holds for all $\rho, \dot{\rho} \in \Delta$ sufficiently close to $\sigma$.

We now show that the sets

$$V_\delta(\sigma) = \left\{ \rho \in \Delta \left| \begin{array}{c}
\gamma(\rho) < \delta \\
\rho_i^k > \sigma_i^k + \frac{2L}{c} \gamma(\rho) - \frac{2L}{c} \delta \\
\text{for all } i \in N, s_i \in \text{supp}(\sigma_i) \end{array} \right. \right\}$$

 indexed by $\delta > 0$ form a basis of neighborhoods of $\sigma$. Notice that each $V_\delta(\sigma)$ is an open set containing $\sigma$ and hence a neighborhood of $\sigma$. The claim then follows once we show that every neighborhood $U_\varepsilon(\sigma) = \{\rho \in \Delta | |\rho_i^k - \sigma_i^k| < \varepsilon\}$ with $\varepsilon > 0$ contains a neighborhood $V_\delta(\sigma)$.
Given \( \rho \in V_\delta (\sigma) \) we have for each \( i \in N \) that \( \rho^i_k - \sigma^i_k > -\frac{2L_c}{c} \delta \) holds for all \( s^k_i \in \text{supp} (\sigma_i) \) and also for all \( s^k_i \notin \text{supp} (\sigma_i) \) since then \( \sigma^i_k = 0 \). This implies for all \( s^k_i \in S_i \),

\[
\rho^i_k - \sigma^i_k = \sum_{l \neq k} (\sigma^i_l - \rho^i_l) < (K_i - 1) \frac{2L_c}{c} \delta.
\]

Therefore \( V_\delta (\sigma) \subseteq U_\varepsilon (\sigma) \) whenever \( \delta \leq \frac{\varepsilon}{4\Pi(K - 1)} \) where \( K = \max_{i \in N} K_i \).

Next we establish a useful inequality. Represent each \( \rho_i \in \Delta_i \) as \( \rho_i = (1 - \lambda_i) \tau_i + \lambda_i \nu_i \) where \( \lambda_i \in [0, 1] \), \( \tau_i \in \tilde{\beta}_i (\sigma) \) and \( \text{supp} (\nu_i) \cap \beta_i (\sigma) = \emptyset \). For this representation

\[
\sum_{i \in N} \sum_{s^i_j \in S_i} |\rho^i_j - \tau^i_j| = \frac{2}{c} \sum_{i \in N} \lambda_i = 2 \gamma (\rho).
\]

Fix \( i \) and \( k \) such that \( s^k_i \in \text{supp} (\sigma_i) \). Assumption (*) implies for \( \tau = (\tau_i)_{i \in N} \in \tilde{\beta} (\sigma) \) that \( s^k_i \) is a best reply to \( \tau \). Hence \( f^k_i (\tau) \geq 0 \) holds since the dynamic satisfies Property (A). Inequality (8) and equation (9) yield

\[
-\rho^i_k f^k_i (\rho) \leq -\rho^i_k f^k_i (\rho) + \tau^k_i f^k_i (\tau) \leq L \sum_{j \in N} \sum_{s^j_l \in S_j} |\rho^j_l - \tau^j_l| = 2L \gamma (\rho).
\]

Together with (7) we obtain

\[
\rho^i_k f^k_i (\rho) \geq \frac{2L_c}{c} \hat{\gamma} (\rho)
\]

for all \( \rho \) sufficiently close to \( \sigma \).

Now we are ready to show that \( V_\delta (\sigma) \) is forward invariant for sufficiently small \( \delta > 0 \), which means that \( \sigma \) is Lyapunov stable. Let \( \delta > 0 \) be sufficiently small such that inequalities (7) and (8) hold for all \( \rho \in V_\delta (\sigma) \). Let \( \{ \rho (t) \}_{t \geq 0} \) be a trajectory with \( \rho (0) \in V_\delta (\sigma) \). Let \( T > 0 \) be the first time where the trajectory leaves \( V_\delta (\sigma) \). We must have \( \gamma (\rho (T)) < \delta \) since \( \hat{\gamma} (\rho) \leq 0 \) holds for all \( \rho \in V_\delta (\sigma) \). Moreover, for all \( s^k_i \in \text{supp} (\sigma_i), i \in N \),

\[
\rho^i_k (T) - \rho^i_k (0) = \int_0^T \rho^i_k (t) f^k_i (\rho (t)) dt \geq \frac{2L_c}{c} \int_0^T \hat{\gamma} (\rho (t)) dt = \frac{2L_c}{c} (\gamma (\rho (T)) - \gamma (\rho (0)))
\]

and hence

\[
\rho^i_k (T) \geq \rho^i_k (0) + \frac{2L_c}{c} \gamma (\rho (T)) - \frac{2L_c}{c} \gamma (\rho (0)) > \sigma^k_i + \frac{2L_c}{c} \gamma (\rho (T)) - \frac{2L_c}{c} \delta
\]

so that \( \rho (T) \in V_\delta (\sigma) \), a contradiction.

Finally, we have to show that \( G \) is asymptotically stable, which follows once we show for all sufficiently small \( \delta > 0 \) that the \( \omega \)-limit set of any trajectory \( \{ \rho (t) \}_{t \geq 0} \) starting in \( V_\delta (\sigma) \) is contained in \( G \). Let \( \hat{\rho} \in V_\delta (\sigma) \) be an \( \omega \)-limit point of a trajectory \( \{ \rho (t) \}_{t \geq 0} \) starting in \( V_{\delta/2} (\sigma) \).
By Corollary 1 we can assume that all Nash equilibria in $V_\delta(\sigma)$ belong to $G$. We can also assume that $\text{supp}(\sigma) \subseteq \text{supp}(\rho)$ for all $\rho \in V_\delta(\sigma)$. Then $\gamma(\rho(t))$ is non-increasing along the trajectory and hence (see Hofbauer and Sigmund [18] Theorem 2.6.1) $\dot{\gamma}(\hat{\rho}) = 0$. Because $\dot{\gamma}(\rho) < 0$ for all $\rho \in V_\delta(\sigma) \setminus \tilde{\beta}(\sigma)$ we conclude $\hat{\rho} \in \tilde{\beta}(\sigma)$. By (*) it follows that each $s^k_i \in \text{supp}(\sigma)$ is a best reply to $\hat{\rho}$. If $\hat{\rho}$ were not a Nash equilibrium we could find by Property (B) of the dynamic a player $i \in N$ such that $f^k_i(\hat{\rho}) > 0$ and hence $\hat{\rho}_i^k f^k_i(\hat{\rho}) > 0$ would hold for all $s^k_i \in \text{supp}(\sigma)$. Then $\rho^k_i(t)$ would be strictly increasing for sufficiently large $t$. Theorem 2.6.1 in Hofbauer and Sigmund [18] would yield the contradiction $\hat{\rho}_i^k f^k_i(\hat{\rho}) = 0$. Consequently $\hat{\rho}$ is a Nash equilibrium and therefore in $G$. 

Central to the proof above is the use of (*) to show that each strategy combination in a SE set is stable. We derive asymptotic stability of the set as a consequence. We do not know whether SE sets are still asymptotically stable under any dynamic that reinforces better replies if (*) is not satisfied. However, one can no longer expect stability of each strategy combination in the set under any dynamic that reinforces best replies. We show this here by using the example introduced earlier, after Definition 1. The SE set of this game does not satisfy property (*) because $(U, L, F)$ is a best reply to $(D, L, B) \in G$, but $(D, R, B)$ is the unique best reply to $(U, L, F)$. Consider the following dynamic that satisfies all our conditions and is actually aggregate monotonic in the sense of Samuelson and Zhang [30]:

$$\dot{x}_1 = x_1 (1 - x_1) (1 - x_3)^2$$
$$\dot{x}_2 = x_2 (1 - x_2) (1 - x_1) (1 - x_3)$$
$$\dot{x}_3 = x_3 (1 - x_3) (1 - x_1)^2$$

This dynamic is obtained from the standard replicator dynamic by squaring the terms $(1 - x_3)$ and, respectively, $(1 - x_1)$ in the equations for $\dot{x}_1$ and $\dot{x}_3$ so that the speed of a trajectory in the $x_1$ and $x_3$ direction is slowed down while the speed in the $x_2$ direction is unaltered. It is not difficult to see that $G$ is an asymptotically stable set of rest points. We claim that the points $(1, x_2, 1) \in G$ with $x_2 < 1$ are not stable.

By symmetry, the plane $\{x_1 = x_3\}$ is invariant. Consider any trajectory $x(t)$ starting in the relative interior of this plane. Then $\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_3(t) = 1$. Separation of variables shows that

$$x_2(t) = \frac{cx_1(t)}{1 - x_1(t) + cx_1(t)}$$
holds for an appropriate constant $c > 0$. Therefore, $\lim_{t \to \infty} x_2(t) = 1$. Thus we can find, as indicated in Figure 1, trajectories starting arbitrarily close to any $(1, x_2, 1)$ which converge to $(1, 1, 1)$.

Bringing together Propositions 4, 5 and Lemma 5 we obtain the following characterization result which is more restrictive than Theorem 1 but applies to a large class of dynamics.

**Theorem 2** Consider a dynamic reinforcing best replies and a non-empty set $G$ for which each of its connected components contains a pure strategy combination. Assume that the game involves only two players or that $G$ is convex. Then $G$ is an asymptotically stable set of rest points if and only if $G$ is a strict equilibrium set. Moreover, every element of a SE set is stable.

As a consequence of each element of a SE set being stable, we obtain that trajectories starting close will converge to an element of the set. Notice that without point-wise stability we could not exclude the possibility of endless cycling.

### 5 Conclusion

Conditions for meeting the requirements of a SE set are easily verified in specific examples which makes the results of this paper very applicable. For illustration we point out two evolutionary investigations where the existence of SE sets is now easily verified. The forward induction outcome for the twice repeated battle-of-the-sexes game (between two players) selected in van Damme [38] is a SE set and thus asymptotically stable under any dynamic that reinforces best replies. The set of efficient equilibria in the action commitment game between two players studied by van Damme and Hurkens [39] is also a SE set. Here the notion of a SE set refines their solution concept which also included inefficient outcomes.

SE sets have been investigated in cheap talk games by Schlag [31] and in repeated games by Balkenborg [4]. The former proves that the set of efficient strategy combinations is a SE set when the game has common interest. The latter shows for a repeated common interest game that a SE set contains only approximately efficient strategy combinations if the game is repeated sufficiently often.

Unfortunately SE sets often do not exist. In such cases, following Proposition 4, there will be no asymptotically stable set of rest points that contains a pure strategy in each component. In order to investigate whether other asymptotically stable sets of rest points exist, following
Theorem 1, more information about the specific dynamic is needed. Moreover, it may be necessary to impose weaker dynamic conditions on the sets predicted, e.g. to abandon the restriction to rest points as in Ritzberger and Weibull [29], or to consider asymptotic stability only with respect to interior trajectories as in Binmore and Samuelson [6], Cressman [9] and Cressman and Schlag [10].

We conjecture that SE sets are asymptotically stable even if Property (*) is not satisfied and hence that a SE set does not have to consist of stable rest points. However, our method of proof cannot be used to show this because it uses the stability of the rest points to show the asymptotic stability of the entire set. An additional interesting topic for future research is to analyze evolutionary dynamics close to a SE set when payoffs are perturbed.

Appendix: Games with SE sets - A Class of Examples

In Section 2 we have shown that every SE set is the $n$-convex hull of some set of pure strategy combinations $T$. Conversely we will now show that the $n$-convex hull $G(T)$ of any non-empty set of pure strategy combinations $T$ is the SE set of some game. The examples used to show this are games with identical interests where all payoffs are either zero or one.

More precisely, fix a set of players $\mathcal{N}$ and pure strategy sets $S_i$ for each player. Choose any $T \subseteq S$ with $T \neq \emptyset$. Consider the game with identical interests (i.e. a game with $U_i(s) = U_j(s)$ for all $s \in S$, $i, j \in \mathcal{N}$) defined for all $i \in \mathcal{N}$ by

$$U_i(s) = \begin{cases} 
1 & \text{for } s \in T \\
0 & \text{for } s \notin T 
\end{cases}.$$ 

From Proposition 2 it follows immediately that $G(T)$ is a SE set of this game.

In particular, a SE set can be a cycle. To see this, consider the case of three players with two strategies each ($S_i = \{s^1_i, s^2_i\}$ for $i = 1, 2, 3$) and let $T$ be the set of all pure strategy combinations where at least two players choose strategies with different index, so $T = S \setminus \{(s^1_1, s^2_i, s^3_3), (s^2_1, s^3_2, s^3_3)\}$. The space of mixed strategy combinations is a cube and the SE set $G(T)$ is a cycle consisting of six edges of the cube, as indicated in Figure 2 on the left where the cycle is indicated by bold lines.

The properties of this game are studied in Hofbauer and Swinkels [19] for a large class of dynamics including the standard replicator dynamic, fictitious play and the best reply dynamic. First they show that $G(T)$ is asymptotically stable for the dynamic. Then they perturb the pay-
offs such that the best response structure along the original SE set is now cyclical as illustrated in Figure 2 on the right. Hofbauer and Swinkels’ [19] main result is that trajectories starting close to these edges no longer converge in the perturbed game but instead will cycle indefinitely. Accordingly, they refer to the game as the “universal Shapley example”. Non-convergence in the perturbed game also means that there is no SE set of the perturbed game near to the original SE set. Consequently, SE sets need not be robust to payoff perturbations.

Notice that one can similarly construct a two-payer game where the SE set is a cycle. Namely, let \( S_i = \{ s^1_i, s^2_i, s^3_i \} \) for \( i = 1, 2 \) and let \( T = S \setminus \{ (s^1_1, s^1_2), (s^3_1, s^2_2), (s^3_2, s^3_2) \} \). Then \( G(T) \) is a cyclical SE set for the game with identical interests defined above.

References


