Cournot Oligopoly with $n$ firms

firm $i$'s output: $q_i$

total output: $q = q_1 + q_2 + \cdots + q_n$

opponent's output: $q_{-i} = q - q_i = \sum_{j \neq i} q_j$

constant marginal costs of firm $i$: $c_i$

inverse demand function: $p(q)$

firm $i$'s profit:

$$\Pi_i(q_{-i}, q_i) = p(q) \times q_i - c_i \times q_i = (p(q_{-i} + q_i) - c_i) q_i$$

FOC for profit maximum given $q_{-i}$:

$$\frac{\partial \Pi_i}{\partial q_i} = \frac{\partial p}{\partial q_i} \times q_i + p - c_i = 0$$

Solution defines reaction curve $q_i = r_i(q_{-i})$ which is often decreasing in $q_{-i}$.

Linear case: $p = A - Bq = A - B(q_{-i} + q_i)$

\[
\begin{align*}
\frac{\partial p}{\partial q_i} &= -B \\
-Bq_i + (A - B(q_{-i} + q_i)) - c_i &= 0 \\
2Bq_i &= A - c_i - Bq_{-i}
\end{align*}
\]

Reaction function

$$q_i = r_i(q_{-i}) = \frac{A - c}{2B} - \frac{1}{2}q_{-i}$$
Cournot-Nash equilibrium:

1. Every firm maximizes profit given her expectation of $q_{-i}$.
2. The expectation is correct.

This yields the simultaneous system of equations

$$q_i = r_i(q_{-i})$$

for all $i = 1, \ldots, n$. In the linear case the FOC yields, since $q_i + q_{-i} = q$

$$-Bq_1 + (A - Bq) - c_1 = 0$$
$$-Bq_2 + (A - Bq) - c_2 = 0$$
$$\vdots$$
$$-Bq_n + (A - Bq) - c_n = 0$$

Summation yields

$$-Bq + n(A - Bq) - n\bar{c} = 0$$

where

$$\bar{c} = \frac{c_1 + c_2 + \cdots + c_n}{n}$$

is the average marginal cost in the market.

Thus we can deduce the total quantity produced and the price in the market

$$\begin{align*}
(n + 1) B q &= n (A - \tilde{c}) \\
q &= \frac{n}{n + 1} \frac{A - \tilde{c}}{B} \\
p &= A - B q = \frac{1}{n + 1} A + \frac{n}{n + 1} \tilde{c} \to \tilde{c} \text{ for } n \to \infty
\end{align*}$$
Each firm produces in the $n$-firm oligopoly

$$q_i^n = \frac{A - Bq_i - c_i}{B} = \frac{A - c_i}{B} - \frac{n}{n+1} \frac{A - \bar{c}}{B} = \frac{1}{n+1} \frac{A}{B} + \frac{n(\bar{c} - c_i) - c_i}{(n+1)B}.$$

Let us now, for simplicity, assume that firms have identical marginal costs $c_i = \bar{c} = c$. Then

$$p = \frac{1}{n+1} A + \frac{n}{n+1} c \to c \text{ as } n \to \infty$$

$$q_i^n = \frac{1}{n+1} \frac{A - c}{B} \to 0 \text{ as } n \to \infty$$

$$\Pi_i^n = (p - c) q_i^n = \left( \frac{1}{n+1} A + \frac{n}{n+1} c - c \right) \frac{1}{n+1} \frac{A - c}{B} = \frac{1}{(n+1)^2} \frac{(A - c)^2}{B}$$

$$n \Pi_i^n = \frac{n}{(n+1)^2} \frac{(A - c)^2}{B} \to 0 \text{ as } n \to \infty$$

The total profit in the industry decreases with every additional firm entering the market since for all $n > 1$

$$(n-1) \Pi_i^{n-1} > n \Pi_i^n$$

$$\iff \frac{n-1}{(n)^2} > \frac{n}{(n+1)^2}$$

$$\iff (n-1)(n+1)^2 > n^3$$

$$\iff (n^2 - 1)(n+1) > n^3$$

$$\iff n^3 - n + n^2 - 1 > n^3$$

$$\iff n^2 > n - 1$$

which is true since $n^2 > n$ for all $n > 1$.

In particular, it always pays for the firms to form a cartel and share the monopolist profit since $n \Pi_i^n < \Pi_i^1$.

2 Stackelberg Equilibrium

Two firms with marginal costs 1. Different timing: Firm 1 moves first, firm 2 observes the move and then adapts.

If a rational firm 2 observes the quantity $q_1$ it will choose the quantity

$$q_2 = r_2 (q_1) = \frac{A - c}{2B} - \frac{1}{2} q_1$$

Total output is

$$q_1 + q_2 = \frac{A - c}{2B} + \frac{1}{2} q_1$$

and the price will be

$$p = A - B(q_1 + q_2) = A - \frac{A - c}{2} - \frac{B}{2} q_1 = \frac{A + c - Bq_1}{2}$$

Anticipating this, firm 1 expects to make the profit

$$\Pi_1(q_1, r_1(q_2)) = \left( \frac{A + c - Bq_1}{2} - c \right) \times q_1 = \frac{A - c - Bq_1}{2} \times q_1$$

which is maximized for

$$q_1 = \frac{A - c}{2B}$$
yielding the price
\[ p = \frac{A + c - B \frac{A - c}{2B}}{2} = \frac{A - c}{4} \]
and the profit
\[ \Pi_1 = \frac{1}{8} \frac{(A - c)^2}{B} \]
Firm 2 produces
\[ q_2 = \frac{A - c}{2B} - \frac{1}{2} q_1 = \frac{A - c}{4B} \]
and makes the profit
\[ \Pi_2 = \frac{1}{2} \Pi_1 = \frac{1}{16} \frac{(A - c)^2}{B} \]
Notice that this would not be a Nash equilibrium if firm 2 could not observe the quantity choice because firm 2 reacts optimally while firm 1 should produce
\[ q_1 = r_1 (q_2) = \frac{A - c}{2B} - \frac{1}{2} q_2 = \frac{A - c}{2B} - \frac{A - c}{8B} = \frac{3A - c}{8} \]
Total quantity would be \( \frac{3A - c}{8} \) and the the price would reduce to
\[ p = A - \frac{5}{8} (A - c) = \frac{3A + 5c}{8} \]
\[ \Pi_1 = \left( \frac{3A + 5c}{8} - c \right) \left( \frac{3A - c}{4B} \right) = \frac{9}{8} \frac{(A - c)^2}{B} > \frac{1}{8} \frac{(A - c)^2}{B} \]
The leader produces in the Stackelberg equilibrium twice as much than the follower and makes twice the profit. In the Cournot duopoly the payoff \( \Pi_i = \frac{1}{8} \frac{(A - c)^2}{B} \) which is in between the profit of the leader and the follower.

3 Bertrand competition with differentiated products

The two firms have the demand functions
\[ Q_1 = 100 - 2P_1 + P_2 \]
\[ Q_2 = 100 - 2P_2 + P_1 \]
and constant marginal costs \( c = 5 \). The profit function for firm \( i \) is
\[ \Pi_i (p_1, p_2) = (P_i - c) Q_i = (P_i - 5) (100 - 2P_i + P_j) \]
where \( j = 3 - i \). The first order condition for a profit optimum (taking the other firm’s price as given) is
\[ \frac{\partial \Pi_i}{\partial P_i} = (+1) \times (100 - 2P_i + P_j) + (P_i - 5) \times (-2) = 110 - 4P_i + P_j = 0, \ i = 1, 2 \]
The solution to this system of equations is \( P_1 = P_2 = \frac{110}{4} = 36.25 \). Each firm produces \( \frac{110}{4} = 27.5 \) units and makes the profit \( 73.75 \times 36.25 \approx 2688 \times 2 \) is made. Together they make the profit 5376. If they would form a cartel they could make the profit \( \Pi_1 (p_1, p_2) + \Pi_2 (p_1, p_2) \). Maximizing joint profit leads to the two first order conditions
\[ \frac{\partial (\Pi_1 + \Pi_2)}{\partial P_i} = 110 - 4P_i + P_j + (P_i - 5) = 105 - 3P_i + P_j = 0, \ i = 1, 2 \]
which have the solution \( P_1 = P_2 = 52.5 \). Of each commodity 57.5 units are produced and the total profit is \( 2 \times (47.5) \times (57.5) = 5462.5 \), which is obviously higher than in competition.
4 Bertrand “competition” with perfect complements.

Two price-setting firms produce with constant marginal costs \( c = 3 \) produce goods which are perfect complements. Consumers therefore buy equal amounts from both firms. The total amount they by of each commodity is

\[
Q = Q(P_1, P_2) = 15 - (P_1 + P_2)
\]

The profit of firm \( i = 1 \) or \( i = 2 \) is

\[
\Pi_i(P_1, P_2) = (P_i - 3)Q = (P_i - 3)(15 - (P_1 + P_2))
\]

The first-order condition for a profit maximum is

\[
\frac{\partial \Pi_i}{\partial P_i} = 15 - (P_1 + P_2) - (P_i - 3) = 18 - 2P_i - P_j = 0
\]

where \( j = 3 - i \). By symmetry, \( P_1 = P_2 \) in equilibrium, so \( 3P_1 = 18 \) or \( P_1 = P_2 = 6 \). It follows that \( Q = 15 - 12 = 3 \) pairs are sold at the price 6. Each firm makes the profit \((6 - 3) \times 3 = 9\) and the total profit in the industry is 18.

If a monopolist takes over both plants and takes the price \( 2P \) per pair his profit is

\[
\Pi(P) = (2P - 6)(15 - 2P)
\]

which is maximized for \( 2P = \frac{15 + 6}{2} = 10.5 \) where \( 15 - 10.5 = 4.5 \) pairs are demanded. Consumer surplus is up in the monopoly because they get more at a lower price. Producer surplus goes up because the monopolist’s profit is \( 4.5^2 = 20.25 > 18 \).