BEE1024 – Mathematics for Economists **Week 8**	Juliette Stephenson Department of Economics
Determinants	University of Exeter

Determinants

extend results for 2×2-matrices

$$A\vec{x} = \vec{b} \Leftrightarrow \vec{x} = A^{-1}\vec{b} \text{ where } A^{-1} = \frac{1}{\det A} \operatorname{ad} A$$

for arbitrary square matrices, in particular for 3×3-matrices

Overview/Recap

linear simultaneous system of two *equations* in two unknowns:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

or

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{b} = \begin{bmatrix} e \\ f \end{bmatrix}$$

is 2×2-*matrix* of coefficients
$\vec{x}$ the 2×1-*column vector* of unknowns
$\vec{b}$ the 2×1-column vector of constants.

If *determinant* $\det A \neq 0$:

$$\vec{x} = A^{-1}\vec{b}$$

whereby A^{-1} is the *inverse*, i.e., the unique 2×2-matrix with the property $A^{-1}A = AA^{-1} = \mathrm{Id}_2$.
Hereby

$$\mathrm{Id}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the *identity matrix*
$(\mathrm{Id}_2)\, B = B\, (\mathrm{Id}_2) = B$ holds for every 2×2-matrix B.

The inverse A^{-1} can be calculated as

$$A^{-1} = \frac{1}{\det A} \operatorname{ad} A$$

and one has hence

$$\vec{x} = \frac{1}{\det A} (\operatorname{ad} A)\, \vec{b}.$$

Hereby the determinant $\det A$ of A is the number

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

and the *adjoint* $\operatorname{ad} A$ is the matrix

$$\operatorname{ad} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

extend to $n \times n$-square matrices A,

$$\mathrm{Id}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

first 3×3-matrices

The adjoint of a 3×3-matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is calculated in three steps:

1. First we must find the so-called "2×2-*minors*" and place them in a 3×3-matrix M. This is the most calculational intensive step. We must calculate the determinants of nine 2×2-matrices constructed out of A.
2. By alternating the signs of the matrix M according to the "*chess-board rule*" we obtain the 3×3-*matrix of cofactors* C.
3. The adjoint $\mathrm{ad}\, A$ is then the *transposed* matrix of cofactors C'.

Step 1: The matrix of 2×2-minors M

For instance, find entry m_{23} of M for second row of the third column as follows:

delete second row and third column

$$\begin{bmatrix} a_{11} & a_{12} & * \\ * & * & * \\ a_{31} & a_{32} & * \end{bmatrix}$$

a 2×2-matrix remains:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

m_{23} is *determinant* of this 2×2-matrix

$$m_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

The matrix of all 2×2-minors of A is

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

Example 1 For the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

we obtain the matrix of 2×2-minors

$$M = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 3 \\ -1 & 1 & 5 \end{bmatrix}$$

This was the hard bit.

Step 2: The matrix of cofactors C

The *matrix of cofactors* C: alter the signs of M according to the "*chess board rule*"

$$\begin{matrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

$+$ and $-$ are arranged here as on a chessboard, with the top-left entry being $+$.

"$+$" means: Do not switch sign in M (multiply with $+1$).

"$-$" means: Switch sign in M (multiply with -1).

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & -m_{12} & m_{13} \\ -m_{21} & m_{22} & -m_{23} \\ m_{31} & -m_{32} & m_{33} \end{bmatrix}$$

Briefly, the cofactor c_{ij} for row i, column j is given by

$$c_{ij} = (-1)^{i+j}\, m_{ij}$$

because $(-1)^{i+j}$ gives the sign in the chessboard.

Example 2 From the above matrix M of 2×2-minors we obtain the matrix of cofactors

$$C = \begin{bmatrix} 1 & -1 & -1 \\ -(-1) & 1 & -3 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -3 \\ -1 & -1 & 5 \end{bmatrix}$$

Step 3: The adjoint

The *adjoint*: transposed of the matrix of cofactors: $\text{ad}\, A = C'$

$$\text{ad}\, A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Example 3 For the above example the adjoint is

$$\text{ad}\, A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -3 \\ -1 & -1 & 5 \end{bmatrix}' = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -3 & 5 \end{bmatrix}$$

Remark 1 This is consistent with rule for finding the adjoint of a 2×2-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Define the determinant of a 1×1-matrix $[a]$ as $\det [a] = a$

Step 1: Matrix of 1×1-minors:

$$M = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

Step 2: matrix of cofactors

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Step 3: The adjoint

$$\text{ad}\, A = C' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant of a 3×3-matrix

Proposition 1 *Multiplying a row of the matrix A with the corresponding column of the adjoint yields the determinant $\det A$. Multiplying a row of the adjoint with the corresponding column of the matrix A also yields the determinant.*

We had

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad \operatorname{ad} A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -3 & 5 \end{bmatrix}$$

Multiplying the second row of A with the second column of the adjoint yields

$$\det A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = 3 + 2 - 3 = 2$$

Multiplying the second row of the adjoint with the second column of A yields

$$\det A = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1 + 2 - 1 = 2$$

In fact, the previous proposition is just an application of the rule

$$(\operatorname{ad} A)\, A = A\, (\operatorname{ad} A) = (\det A)\, \mathrm{Id}_3$$

Verify this:

$$A\,(\operatorname{ad} A) = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -3 & 5 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(\operatorname{ad} A)\, A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the above results it is easy to solve the linear simultaneous system of 3 equations in 3 unknowns

$$\begin{aligned} x - y &= 1 \\ 3x + 2y + z &= 1 \\ 2x + y + z &= 1 \end{aligned}$$

Namely,

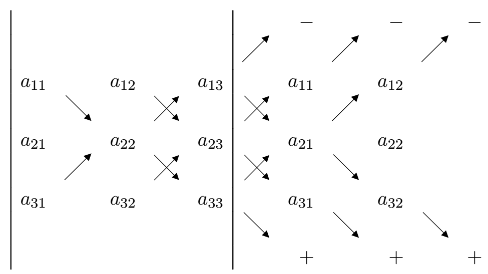
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

The Sarrus rule

An explicit formula for the determinant of a 3×3-matrix A is

$$\begin{aligned} \det(A) = \; & a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} \\ & + a_{21}a_{13}a_{32} - a_{21}a_{12}a_{33} \\ & + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \end{aligned}$$

Sarrus' allows us to memorize this formula:



Example 4

$$\left|\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & -1 \\ 3 & 2 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 & 1 \end{array}\right| = \begin{array}{lll} 1\times 2\times 1 & +(-1)\times 1\times 2 & +0\times 3\times 1 \\ -2\times 2\times 0 & -1\times 1\times 1 & -1\times 3\times(-1) \end{array}$$

$$= 2-2+0-0-1+3=2$$

How the story continues

proceed by "induction":

Suppose we can calculate the determinants of $(n-1)\times(n-1)$-square matrices.

Then we can find the adjoint, the determinant and the inverse of a $n \times n$-matrix A as follows:

We delete in each of the n^2 possible ways a row and a column of A and calculate the determinant of the remaining $(n-1)\times(n-1)$-square matrix.

- The resulting n^2 $(n-1)\times(n-1)$-minors are put into a matrix M.
- We adjust the signs of the matrix M according to the chess-board rule and obtain the matrix of cofactors C.
- We transpose C to obtain the adjoint $\mathrm{ad}\, A$.
- We multiply, say, the first row of A with the first column of

the adjoint and obtain the determinant $\det A$.

- The inverse is

$$A^{-1} = \frac{1}{\det A} \operatorname{ad} A$$

Proposition 2 *For an $n \times n$-matrix A*

$$\begin{aligned}\det A = {} & (-1)^{i+1} a_{i1} m_{i1} + (-1)^{i+2} a_{i2} m_{i2} + \dots \\ & + (-1)^{i+j} a_{ij} m_{ij} + \dots + (-1)^{i+n} a_{in} m_{in}\end{aligned}$$

where m_{ij} is the determinant of the matrix with row i and column j deleted.

Because a_{ij} is multiplied with $c_{ij} = (-1)^{i+j} m_{ij}$ in this formula one refers to the latter as the COFACTOR.

Example 5 We apply the above formular for row four to calculate the determinant. Then we apply it to row 2 in the 3×3-matrices.

$$\begin{vmatrix} 1 & 1 & 4 & 0 \\ 2 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix}$$

$$= -2 \times \begin{vmatrix} 1 & 4 & 0 \\ 2 & 0 & 4 \\ 2 & 1 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 4 & 0 \\ 2 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & 4 \\ 1 & 2 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= -2 \times \left(-2 \times \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 4 \times \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \right)$$
$$+3 \times \left(-2 \times \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 4 \times \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} \right)$$
$$= 16 - 56 - 24 + 36 = -28$$

Further rules for determinants

- If a multiple of one row is added to another row the determinant is not changed.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

  last row on the right: last row of first matrix, subtract twice first row

- If all entries in a row (or column) are multiplied with the same factor, the determinant multiplies with this factor.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- Transposition does not change the determinant.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{vmatrix}$$

- If two rows (or columns) are interchanged the determinant changes sign.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

- If a square matrix consist of two square matrices with zeroes below and to the left, its determinant is the product of the determinants of the block.

$$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = |A|\,|B|$$

Hereby the whole matrix is, say, an $n \times n$-matrix, the block A is a $k \times k$-matrix with $0 < k < n$, B is an $(n-k) \times (n-k)$-matrix, C is an $k \times (n-k)$-matrix and the block “0” is a $(n-k) \times k$-matrix.

$$\begin{vmatrix} 2 & 3 & 2 & 3 & 4 \\ 1 & 3 & 4 & 3 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} \begin{vmatrix} 2 & 2 & 2 \\ 3 & 1 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 3 \times 0 = 0$$

- determinant of an upper (or lower) diagonal matrix is the product of the entries on the diagonal

$$\begin{vmatrix} 1 & 3 & 3 & 3 & 3 \\ 0 & 2 & 3 & 3 & 3 \\ 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

- determinant is zero if and only if the rows (or columns) of the matrix are *linearly dependent*!

  The rows $\vec{a}'_1,\cdots,\vec{a}'_n$ of the matrix are called linearly dependent if one can find scalars $\alpha_1, \cdots, \alpha_n$, not all of them zero, such that

$$\alpha_1\vec{a}'_1 + \alpha_2\vec{a}'_2 + \cdots + \alpha_n\vec{a}'_n = \vec{0}.$$

  For instance,

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 6 & 5 & 4 \end{vmatrix} = 0$$

  because $\vec{a}'_4 = \vec{a}'_1 + \vec{a}'_2 + \vec{a}'_3$.

- When two columns or two rows are *identical* the determinant is zero.

$$\begin{vmatrix} 1 & 1 & 4 & 4 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 4 & 4 & 1 & 1 \end{vmatrix} = 0$$

- deteminant is zero if one column or row is zero

$$\begin{vmatrix} 1 & 0 & 4 & 4 \\ 2 & 0 & 3 & 3 \\ 3 & 0 & 2 & 2 \\ 4 & 0 & 1 & 1 \end{vmatrix} = 0$$