BEE1024 – Mathematics for Economists **Week 7**	Juliette Stephenson Department of Economics University of Exeter
Linear Algebra I	

Relevance:

– simultaneous systems of equations

– econometrics

Vectors and Matrices

Definition 1 *An* $m \times 1$-COLUMN VECTOR *is a column of* m *numbers*

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

surrounded by square brackets. We will denote such vectors by $\vec{a}$, $\vec{b}$, $\vec{x}$, $\vec{y}$ *etc. Unless otherwise specified, the term "vector" refers to a column vector.*

Definition 2 *A* $1 \times n$-ROW VECTOR *is a row of* n *numbers*

$$\begin{bmatrix} y_1 \ y_2 \ \cdots \ y_n \end{bmatrix}$$

surrounded by square brackets.

Definition 3 *A* $m \times n$-MATRIX *is is a block of numbers with* m *rows and* n *columns*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

enclosed in square brackets. We denote matrices by capital letters like A, B, X, Y, *etc. For indices* $1 \leq i \leq m$ *and* $1 \leq j \leq n$ *the notation* a_{ij}, b_{ij}, *etc., refers to the entry in the* i*-th row and the* j*-th column of the matrix. A matrix with an equal number of columns and rows is called a* SQUARE MATRIX*.*

Example 1

$$\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}$$

is a 1×4-row vector,

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 5.8 & 2 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a 3×4-matrix (3 rows, 4 columns).

Example 2 For a linear simultaneous system of equations like

$$\begin{aligned} 5x + 3y + 4z &= 12 \\ 7x + 2y + z &= 13 \\ 3x + 6y + 9z &= 2 \end{aligned}$$

one can form the matrices and vectors

$$A = \begin{bmatrix} 5 & 3 & 4 \\ 7 & 2 & 1 \\ 3 & 6 & 9 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 12 \\ 13 \\ 2 \end{bmatrix}.$$

One calls A the matrix of coefficients, $\vec{x}$ the vector of unknowns and $\vec{b}$ the vector of constants or the right-hand-side vector.

Example 3 two students with same A-level grades, one from a public- and one from a state school.
Who has better chances to obtain a good university degree?
Economists at Warwick:econometric analysis; data on
– university degree,
– school degree,
– type of school,
– other characteristics like male/female etc.
matrix:
– one row for each student
– one columns for each characteristic
estimated a linear relation

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n$$

Basic matrix operations

$A + B$: Two matrices A and B *of the same kind* (i.e., with the same number of columns and rows) can be ADDED or SUBTRACTED by adding or subtracting the corresponding entries. For instance,

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 5.8 & 2 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 0 & 1 \\ 5 & 2.7 & 3 & 8 \\ 2 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 2 & 0 \\ -2 & 3.1 & -1 & 1 \\ -2 & -2 & 0 & 0 \end{bmatrix}$$

αA: Any matrix A can be MULTIPLIED BY A SCALAR (i.e., a number) α by multiplying each entry of the matrix with the scalar. For instance,

$$\begin{bmatrix} \frac{5}{7} & \frac{3}{7} & \frac{4}{7} \\ 1 & \frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{6}{7} & 1\frac{2}{7} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5 & 3 & 4 \\ 7 & 2 & 1 \\ 3 & 6 & 9 \end{bmatrix}.$$

Similarly, *vectors* of the same kind can be added or subtracted or be multiplied by a scalar.

A': For every $m \times n$-matrix A one can obtain an $n \times m$-matrix A' by turning the rows into columns and the columns into rows. For instance,

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 5.8 & 2 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5.8 & 0 \\ 2 & 2 & 0 \\ 1 & 9 & 1 \end{bmatrix}.$$

The resulting matrix is called the TRANSPOSED of A. Transposition transforms an $m \times 1$-column vector into a $1 \times m$-row vector and vice versa. Obviously, $A'' = A$.

Matrix multiplication

ad hoc? "multiplication" misleading term?
Order of factors not interchangable!
purpose compact form to handle simultaneous linear systems of equations

product of an $1 \times k$-row vector $\vec{a}'$ with a $k \times 1$-column vector $\vec{b}$:

$$\vec{a}'\vec{b} = \left[\, a_1 \;\; a_2 \;\; \cdots \;\; a_k \,\right] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} := a_1 b_1 + a_2 b_2 + \ldots + a_k b_k$$

Corresponding entries are multiplied and then one forms the sum. meaningful because both vectors have the same numbers of entries. Otherwise no product is defined. The result of the "multiplication" is a *number* or a 1×1-matrix.

For instance

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ -3 \\ 2 \end{bmatrix} = 1 \times 5 + 2 \times 4 + 3 \times (-3) + 4 \times 2 = 12$$

A linear equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = b$$

can now be written as

$$\vec{a}' \vec{x} = b$$

Provided the **matrix** A **has as many columns as the matrix** B **has rows** then one can multiply each row of A with each column of B as above.
obtain a new matrix $C = AB$ is called the PRODUCT of the two matrices and written $C = AB$.

More explicitly:
A is a $m \times k$-matrix.
B is a $k \times n$ matrix. $(m \times k) \leftrightsquigarrow (k \times n)$
A consists of m $1 \times k$ row vectors $\vec{a}_i'$. Each of the n columns of the $k \times n$-matrix
B consists of n $k \times 1$-column vectors $\vec{b}_j$ $(1 \leq j \leq n)$.

can form $m \times n$ products $\vec{a}'_i \cdot \vec{b}_j$

$$C = \begin{bmatrix} \vec{a}'_1\vec{b}_1 & \vec{a}'_1\vec{b}_2 & \cdots & \vec{a}'_1\vec{b}_j & \cdots & \vec{a}'_1\vec{b}_n \\ \vec{a}'_2\vec{b}_1 & \vec{a}'_2\vec{b}_2 & \cdots & \vec{a}'_2\vec{b}_j & \cdots & \vec{a}'_2\vec{b}_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \vec{a}'_i\vec{b}_1 & \vec{a}'_i\vec{b}_2 & \cdots & \vec{a}'_i\vec{b}_j & \cdots & \vec{a}'_i\vec{b}_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \vec{a}'_m\vec{b}_1 & \vec{a}'_m\vec{b}_2 & \cdots & \vec{a}'_m\vec{b}_j & \cdots & \vec{a}'_m\vec{b}_n \end{bmatrix}$$

Example 4

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ 6 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \cdot 1 + 0 \cdot 2 + 3 \cdot 6) & (1 \cdot 3 + 0 \cdot 5 + 3 \cdot 2) & (1 \cdot 0 + 0 \cdot 1 + 3 \cdot 2) \\ (2 \cdot 1 + 1 \cdot 2 + 5 \cdot 6) & (2 \cdot 3 + 1 \cdot 5 + 5 \cdot 2) & (2 \cdot 0 + 1 \cdot 1 + 5 \cdot 2) \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 9 & 6 \\ 34 & 21 & 11 \end{bmatrix}$$

2×2-matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. $(A+B)+C = A+(B+C)$

$$\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\right) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left(-\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -3 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$$

2. neutral element for addition: $A + \mathbf{0} = \mathbf{0} + A = A$ for any matrix A.

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. Additive inverse: unique matrix B with $A + B = B + A = \mathbf{0}$.

$$B = -A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

4. addition "commutes": $A + B = B + A$.

5. product AB always defined and again 2×2-matrix.

6. $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$

7. Multiplication by zero gives 0: $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$.

8. $A(BC) = (AB)C$

$$\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\right) \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = ?$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \left(\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) = ?$$

9. 1 is a neutral element for multiplication with numbers: $1 \times a = a \times 1 = a$. IDENTITY MATRIX

$$\mathrm{Id}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(\mathrm{Id}_2)\, A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$\begin{bmatrix} (1 \times a) + (0 \times c) & (1 \times b) + (0 \times d) \\ (0 \times a) + (1 \times c) & (0 \times c) + (1 \times d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

So $(\mathrm{Id}_2)\, A = A$ and similarly $A\, (\mathrm{Id}_2) = A$.

10. numbers: equation $ab = ba = 1$, solved by $b = \frac{1}{a}$ if $a \neq 0$. Is there B such that $AB = BA = \mathrm{Id}_2$? Not always:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} a+2b & 3a+6b \\ c+2d & 3c+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

not possible: $3a+6b=0$ implies $a+2b=0$, but cannot have $a+2b=1$.

When $\det A = ad - bc \neq 0$, **inverse exists.** It is denoted by A^{-1} and is calculated as follows for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Set

$$A^{-1} = \frac{1}{\det A} \operatorname{ad}(A)$$

where the ADJOINT MATRIX is

$$\operatorname{ad}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Then

$$\left(A^{-1}\right) A = A\left(A^{-1}\right) = \mathrm{Id}_2$$

$$(\mathrm{ad}\,(A))\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + da \end{bmatrix} =$$

$$\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad - bc)\,\mathrm{Id}_2 = (\det A)\,\mathrm{Id}_2$$

and similarly $A\,(\mathrm{ad}\,(A)) = (ad - bc)\,\mathrm{Id}_2$.

11. **Not true:** $AB = BA$

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}, \; BA = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

For numbers:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

for matrices only:

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2.$$

Application

We want to solve the linear system of equations

$$\begin{aligned} 5x + 3y &= 2 \\ 2x + 7y &= 4 \end{aligned}$$

short-hand:

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 7 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\det A = 35 - 6 = 29 \neq 0$$

$$
\begin{aligned}
A^{-1}\left(A\vec{x}\right) &= A^{-1}\vec{b} \\
\left(A^{-1}A\right)\vec{x} &= A^{-1}\vec{b} \\
\mathrm{Id}_2\,\vec{x} &= A^{-1}\vec{b} \\
\vec{x} &= A^{-1}\vec{b}
\end{aligned}
$$

Hence

$$\vec{x} = \frac{1}{\det A}\left(\mathrm{ad}\,A\right)\vec{b}.$$

In our case

$$
\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{35-6}\begin{bmatrix} 7 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} \\
&= \frac{1}{29}\begin{bmatrix} 14-12 \\ -4+20 \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} \frac{2}{29} \\ \frac{16}{29} \end{bmatrix}.
\end{aligned}
$$

econometrics: T observation $(y_t, x_{t1}, x_{t2}, \cdots, x_{tn})$
estimate a linear relation

$$y = \beta_1 x_2 + \beta_2 x_2 + \ldots + \beta_n x_n.$$

$T \times 1$-column vector $\vec{y}$,
$n \times 1$-column vector $\vec{\beta}$
$T \times n$-matrix X.
ordinary least squares estimator

$$\vec{\beta} = \left(X'X\right)^{-1} X'\vec{y}.$$