BEE1024 – Mathematics for Economists	Juliette Stephenson
Week 6	Department of Economics
Integration	University of Exeter

Applications: – cumulative distribution associated with a density
– solving differential equations.
– calculation of areas
– producer- and consumer surplus
1 indefinite integral
2 definite integral

The Indefinite Integral
Integration is the inverse operation of differentiation!

$$f(x) = 2x.$$

A function which has $f(x)$ as derivative is

$$F(x) = x^2.$$

We call $F(x)$ the *antiderivative* of a function $f(x)$ when the derivative of $F(x)$ is $f(x)$: $\frac{dF}{dx} = f(x)$.
$F(x) = x^2$ is *not the only* antiderivative of $f(x) = 2x$.
Also $G(x) = x^2 + 10$ and $H(x) = x^2 - 5$ are antiderivatives.
$\Leftarrow$ because additive constants vanish when differentiating.
– two antiderivatives can differ only by an additive constant.

In contrast the following question has a unique answer:
Find the antiderivative $F(x)$ of $f(x) = 2x$ with $F(5) = 2$.
Solution: must be of form $F(x) = x^2 + C$ with suitable constant C.
$F(5) = 2$ implies

$$\begin{aligned} F(5) &= 5^2 + C = 25 + C = 2 \\ C &= 2 - 25 = -23 \end{aligned}$$

Hence $F(x) = x^2 - 23$ unique solution.

important antiderivatives:[1]

$f(x)$	$F(x)$
0	C
1	$x + C$
x^{-1}	$\ln\lvert x\rvert + C$
x^{α}, $\alpha \neq -1$	$\frac{1}{\alpha+1} x^{\alpha+1} + C$
e^x	$e^x + c$

[1] For positive x one has $\ln|x| = \ln x$ and hence $\frac{d|\ln x|}{dx} = \frac{1}{x}$. For negative $x$one has $\ln|x| = \ln(-x)$ and hence

$$\frac{d\ln|x|}{dx} = \frac{1}{(-x)} \times (-1) = \frac{1}{x}$$

by the chain rule.

Notation and terminology:

If $F(x)$ is an antiderivative of $f(x)$ one writes

$$\int f(x)\,dx = F(x) + C$$

$$\text{e.g. } \int \frac{1}{2}x\,dx = x^2 + C$$

and calls $\int f(x)\,dx$ the *indefinite integral* because the result is definite only up to a constant. $f(x)$ is called the *integrand.*

Rules of Integration

These rules are just inverted rules of differentiation. We have – just restating the above findings –

$$\int 0dx = C$$

$$\int kdx = kx + C$$

$$\int x^{-1}dx = \ln|x| + C$$

$$\int x^{\alpha}dx = \frac{1}{\alpha+1}x^{\alpha+1} + C \qquad \text{for } \alpha \neq -1$$

$$\int e^x dx = e^x + C$$

Moreover,

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \text{(rule for sums)}$$

$$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx$$

(rule for multiplicative constants)

EXAMPLE:

$$\int \left(5e^x + \sqrt[3]{x^2} + \frac{5}{x}\right)dx = \int \left(5e^x + x^{\frac{2}{3}} + 5x^{-1}\right)dx =$$

$$5\int e^x dx + \int x^{\frac{2}{3}}dx + 5\int x^{-1}dx = 5e^x + C_1 + \frac{3}{5}x^{\frac{5}{3}} +$$

$$+C_2 + 5\ln|x| + C_3$$

$$= 5e^x + \frac{3}{5}x^{\frac{5}{3}} + 5\ln|x| + C$$

where $C = C_1 + C_2 + C_3$.

Integration by Parts

Integration by parts is the inverse of the product rule

$$(uv)' = u'v + uv'$$

for functions $u(x)$, $v(x)$. We can rewrite this as

$$u'v = (uv)' - uv'$$

or

$$\int u'v dx = uv - \int uv' dx$$

since uv is the antiderivative of $(uv)'$.

Remark: Formula not symmetric with respect to the factors

Example

$$\int x^2 e^x dx$$

Set $v = x^2$ and $u' = e^x$. Then $v' = 2x$ and $u = e^x$. Therefore

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx.$$

To solve the last integral set $v = x$ and $u' = e^x$. Then $v' = 1$ and $u = e^x$. Therefore

$$\int xe^x dx = xe^x - \int 1 \times e^x dx = xe^x - e^x + C$$

and overall

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x - C$$

Example

$$\int x^2 e^x dx$$

Try $u' = x^2$, $v = e^x$. Then $u = \frac{1}{3}x^3$ and $v' = e^x$ and therefore by integration by parts

$$\int x^2 e^x dx = \frac{1}{3}x^3 e^x - \frac{1}{3}\int x^3 e^x dx$$

This does not simplify matters.

Integration by Substitution

$\int e^{5x+2} dx = ?$ or $\int \frac{1}{3-2x} dx = ?$.

Differentiating $y = ae^{5x+2}$ gives $y' = 5ae^{5x+2}$ according to the chain rule: Set $u = 5x + 2$. Then $\frac{du}{dx} = 5$, $y = ae^u$, $\frac{dy}{dx} = ae^u$ and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = ae^u \times 5 = 5ae^{5x+2}.$$

Hence

$$\int e^{5x+2} dx = \frac{1}{5}e^{5x+2} + C$$

Similarly $\ln|3-2x|$ has the derivative $\frac{1}{3-2x} \times (-2)$ and hence

$$\int \frac{1}{3-2x} dx = -\frac{1}{2}\ln|3-2x| + C.$$

More generally, integration by substitution

$$\int g\left(u\left(x\right)\right)\frac{du}{dx}dx = \int g\left(u\right)du$$

is the inversion of the chain rule. This can be seen as follows: Suppose $y = G\left(u\right)$ is an antiderivative of $g\left(u\right)$, i.e., $\frac{dy}{du} = g\left(u\right)$. Let $u = u\left(x\right)$. By the chain rule the function $y = G\left(u\left(x\right)\right)$ has the derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = g\left(u\left(x\right)\right)\frac{du}{dx}$$

so $G\left(u\left(x\right)\right)$ is an antiderivative of $g\left(u\left(x\right)\right)\frac{du}{dx}$.

Example

$$\int e^{5x+2}dx$$

Set $u = 5x+2$, so $\frac{du}{dx} = 5$ or $du = 5dx$ or $dx = \frac{1}{5}du$.

Substituting $5x+2$ by u and dx by $\frac{1}{5}du$ we obtain

$$\int e^{5x+2}dx = \int e^{u} \times dx = \int e^{u} \times \frac{1}{5}du = \frac{1}{5}\int e^{u}du = \frac{1}{5}e^{u} + C$$

$$= \frac{1}{5}e^{5x+2} + C$$

Example

$$\int xe^{x^2}dx$$

Set $u = x^2$. Then $du = 2xdx$ or $dx = \frac{1}{2x}du$. Therefore

$$\int xe^{x^2}dx = \int xe^{u}dx = ?$$

Example

$$\int xe^{x^2}dx$$

Try $u = e^{x^2}$. Then $du = 2xe^{x^2}dx = 2xudx$. Hence $dx = \frac{1}{2xu}du$

$$\int xe^{x^2}dx = \int xu\frac{1}{2xu}du = \frac{1}{2}\int du = \frac{1}{2}u + C = \frac{1}{2}e^{x^2} + C$$

Example

$$\int x^2 e^x dx$$

Try $u = e^x$. Then $du = e^x dx = u dx$, $x = \ln u$ and hence

$$\int x^2 e^x dx = \int x^2 u dx = \int x^2 u \left(\frac{1}{u}\right) du = \int x^2 du = \int (\ln u)^2 du$$

which does not really help.

Method: Substitute some sub-expression of x in the integral by u.

Then $du = u'(x)\, dx$ yields $dx = \frac{1}{u'(x)} du$.

Replace dx by $\frac{1}{u'(x)} du$ in the integral.

Hopefully $u'(x)$ is a factor in the integral and hence cancels against $\frac{1}{u'(x)}$

so that a simple expression in u only remains.

Otherwise try a different method.

The Definite Integral

Notation: a) If $F(x)$ is a function and a and b are numbers we write

$$F(x)\,|_a^b \qquad \text{for} \qquad F(b) - F(a)$$

Typically we will have $a < b$. Example:

$$\left(2x^3 - \frac{1}{2}x^2\right)\Big|_{-2}^{3} = \left(2 \times 3^3 - \frac{1}{2} \times 3^2\right) - \left(2 \times (-2)^3 - \frac{1}{2} \times (-2)^2\right)$$
$$= 54 - 4\frac{1}{2} + 16 + 2 = 67.5$$

b) If $F(x)$ is an antiderivative of $f(x)$ we write

$$\int_a^b f(x)\, dx = F(x)\,|_a^b$$

and call $\int_a^b f(x)\, dx$ the *definite integral* of $f(x)$ from a to b.

b *upper limit*

a *lower limit* of the integral.

$f(x)$ the *integrand.*

Notice that if $G(x) = F(x) + c$ is another antiderivative of $f(x)$ then

$$G(x)\,|_a^b = G(b) - G(a) = (F(b) - c) - (F(a) - c)$$
$$= F(b) - F(a) = F(x)\,|_a^b$$

Example:

$$\int_{-2}^{3}\left(6x^2 - x\right)dx = \left(2x^3 - \frac{1}{2}x^2\right)\Big|_{-2}^{3} = 67.5$$

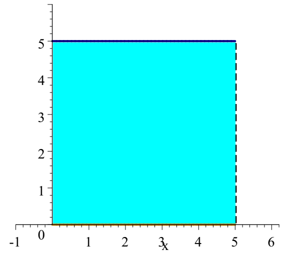
→indefinite integral gives a *function*
→definite integral gives a *number*.

Theorem: Suppose the function $f(x)$ is continuous and positive-valued in the interval $a \leq x \leq b$. Then $\int_a^b f(x)\,dx$ is the area of the region below the graph of $f(x)$ and above the horizontal axis from a to b.

Example

$$\int_0^5 5dx = 5x|_0^5 = 25 - 0$$

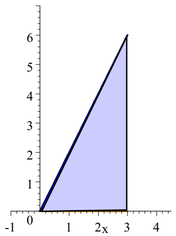
Indeed, the region below the graph of the constant function $f(x) = 5$ from 0 to 5 gives us a square of length 5.



Example

$$\int_0^3 2xdx = x^2|_0^3 = 9$$

Indeed, the region below graph of the constant function $f(x) = 2x$ from 0 to 3 gives us a triangle with width 3 and height 6, which has area $\frac{3\times6}{2} = 9$.



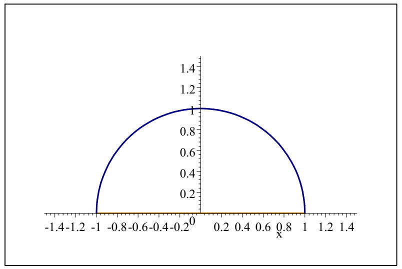
Example

$$\int\sqrt{1-x^2}dx = \frac{1}{2}x\sqrt{(1-x^2)} + \frac{1}{2}\arcsin x + C$$

is a difficult to solve integral. However, it is the case that

$$\int_{-1}^{1}\sqrt{1-x^2}dx = \frac{1}{2}\pi$$

since the graph of the function describes a semi-circle of radius 1:

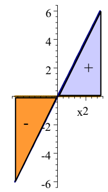


THE FUNDAMENTAL THEOREM OF CALCULUS:

If the function $f(x)$ is continuous on the interval $a \leq x \leq b$ then the integral $\int_a^b f(x)\,dx$ is the area of the region above the horizontal axis and below the graph of the function $f(x)$ minus the area of the region below the horizontal axis and above the graph of the function.

Example

$$\int_{-3}^{0} 2x dx = x^2|_{-3}^{0} = 0 - 9 = -9$$
$$\int_{0}^{3} 2x dx = x^2|_{0}^{3} = 9$$
$$\int_{-3}^{3} 2x dx = x^2|_{3}^{3} = 9 - 9 = 0$$



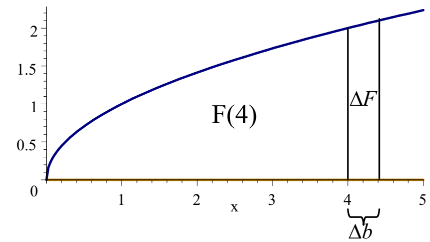
Intuition behind the first theorem:

Let $y = F(b)$ giving area below $f(x)$
for *fixed* left interval end a
and *variable* right interval end b
increase right interval end b by a small amount Δb
area increases approximately by $\Delta b \times f(b)$.
Hence rate of increase in area is

$$\frac{\Delta F}{\Delta b} = \frac{F(b + \Delta b) - F(b)}{\Delta b} \approx f(b)$$

The smaller Δb, the better the approximation.

Hence $\frac{dF}{db} = f(b)$, i.e., $F(b)$ is antiderivative of $f(b)$.



Remark: The notation $\int_a^b f(x)\,dx$ indicates calculation of area:
slice area into vertical slices of height $f(x)$
and infinitely small width dx
and then sum $(\int)$ over areas

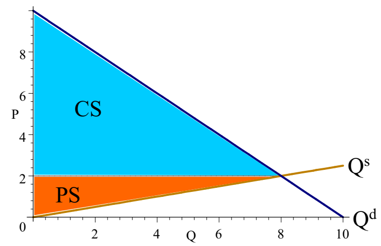
Consumer- and producer surplus

Problem: The demand in a perfectly competitive market is given by

$$Q^d = 10 - P$$

and the supply by

$$Q^s = 4P.$$



Find the market equilibrium and calculate consumer- and producer surplus.

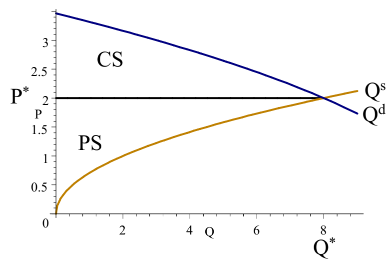
Solution: equilibrium price: $4P = 10 - P$ or $5P = 10$ or $P^* = 2$.
equilibrium quantity: $Q^* = 4P^* = 8$.
Consumer surplus: area of the upper triangle $\frac{8 \times 8}{2} = 32$.
Producer surplus: area of the lower triangle $\frac{2 \times 8}{2} = 8$.

Problem: The demand in a perfectly competitive market is given by

$$Q^d = 12 - P^2$$

and the supply by

$$Q^s = 2P^2.$$



Find the market equilibrium and calculate consumer- and producer surplus.

Solution: equilibrium price: $12 - P^2 = 2P^2$ or $3P^2 = 12$ or $P = \pm 2$, i.e., $P^* = 2$.

Equilibrium quantity: $Q^* = 2 \times 2^2 = 8$.

independent variable on the vertical axis!!!

producer surplus: area to left of supply curve below the equilibrium price

$$PS = \int_0^{P^*} Q^s(P)\,dP = \int_0^2 2P^2 dP = \frac{2}{3}P^3\Big|_0^2 = \frac{16}{3} = 5\frac{1}{3}.$$

Consumer surplus: the area to the left of the demand curve above the equilibrium price.

demand curve intersects the vertical axis at $\bar{P} = \sqrt{12} = 3.46$

$$CS = \int_2^{\sqrt{12}} \left(12 - P^2\right) dP = \left(12P - \frac{1}{3}P^3\right)\Big|_2^{\sqrt{12}} = 6.3795.$$