BEE1024 Mathematics for Economists

Optimization 4

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Week 5

Absolute Maxima

1. unconstrained versus constrained maximization problem
   - the problem has constraints or not
2. interior versus boundary maximum
   - in the interior or on the boundary of the admissible region
3. local (relative) versus global (absolute) maximum
   - highest point nearby or overall

Theorem

Suppose that the function $z = f(x, y)$ has a critical point at (x^*, y^*). If the determinant of the Hessian $\det H =$

$$\begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

is negative at (x^*, y^*) then (x^*, y^*) is a saddle point.
If this determinant is positive at (x^*, y^*) then (x^*, y^*) is a peak or a trough. In this case the signs of $\frac{\partial^2 z}{\partial x^2}_{|(x^*,y^*)}$ and $\frac{\partial^2 z}{\partial y^2}_{|(x^*,y^*)}$ are the same. If both signs are positive, then (x^*, y^*) is a trough. If both signs are negative, then (x^*, y^*) is a peak.

$$\det H \begin{cases} < 0: & \text{saddle point} \\ > 0: & \begin{cases} \frac{\partial^2 z}{\partial x^2} > 0: & \text{trough} \\ \frac{\partial^2 z}{\partial x^2} < 0: & \text{peak} \end{cases} \end{cases}$$

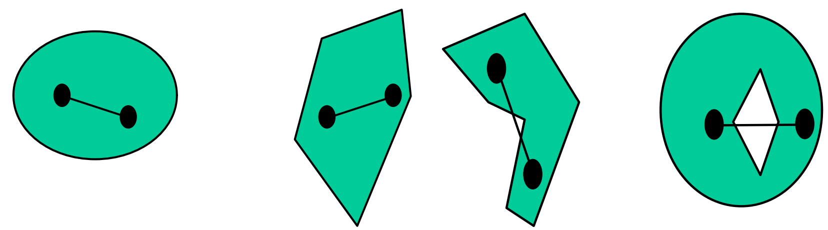
criterium for *local* optima only.

Convex Sets

intervals ≈ convex sets for one variable

A region of the (x, y)-plane is *convex* if:

With every two points (x_1, y_1) and (x_2, y_2) in the region the whole line segment connecting these two points is in the region.



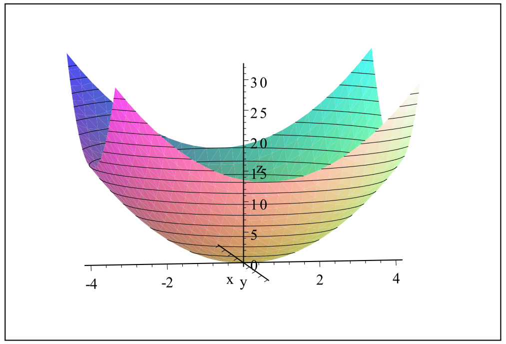
two convex sets two non-convex sets

The (x, y)-plane and the the set of all non-negative coordinate pairs (x, y) are also convex sets.

(Notice: There are concave and convex functions and there are concave and convex lenses, but there is no such thing as a "concave set".)

Convex Functions

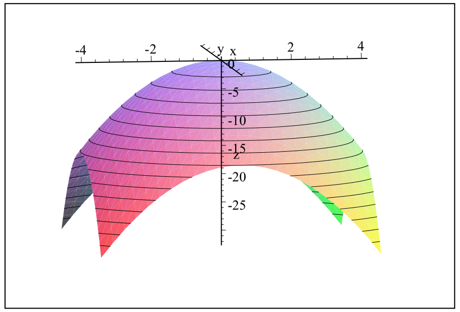
A function $f(x, y)$ is CONVEX if the area *above* the graph of the function, i.e., the set of all points (x, y, z) with (x, y) in the domain of the function and $z \geq f(x, y)$, is convex.



A convex function

Concave Functions

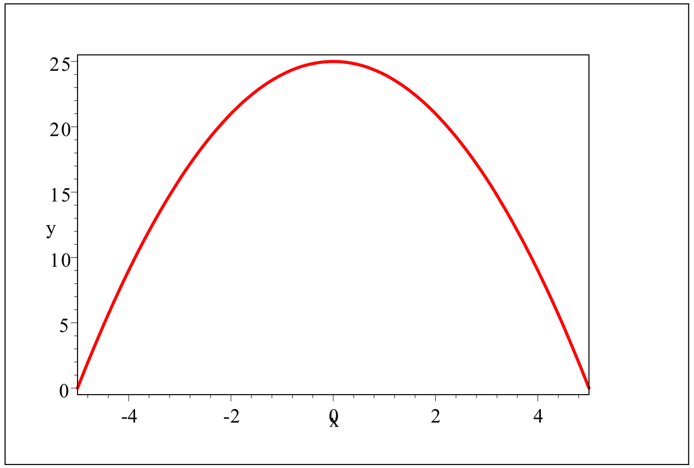
A function $f(x, y)$ is CONCAVE if the area *below* the graph of the function, i.e., the set of all points (x, y, z) with (x, y) in the domain of the function and $z \leq f(x, y)$, is convex.



A concave function

For a concave function every local maximum is a global maximum.

Con-



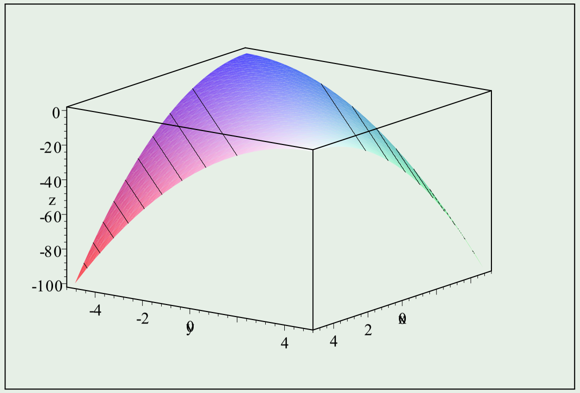
cave

Theorem

Suppose the function $f(x, y)$ is defined on a convex set. Suppose the determinant of the Hessian is always positive in the region and that $\frac{\partial^2 f}{\partial x^2}$ is always negative. Then the function is concave and every peak is a maximum.

Example

Every point (x, x) is a maximum of the function $z = -(y - x)^2$



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Example

Profit function

$$\Pi(K,L) = 12K^{\frac{1}{6}}L^{\frac{1}{2}} - K - 3L$$

has critical point at $K^* = L^* = 8$. We have

$$\frac{\partial \Pi}{\partial K} = 2K^{-\frac{5}{6}}L^{\frac{1}{2}} - 1 \qquad \frac{\partial \Pi}{\partial L} = 6K^{\frac{1}{6}}L^{-\frac{1}{2}} - 3$$

and hence

$$H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial K^2} & \frac{\partial^2 \Pi}{\partial L \partial K} \\ \frac{\partial^2 \Pi}{\partial K \partial L} & \frac{\partial^2 \Pi}{\partial L^2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{3}K^{-\frac{11}{6}}L^{\frac{1}{2}} & K^{-\frac{5}{6}}L^{-\frac{1}{2}} \\ K^{-\frac{5}{6}}L^{-\frac{1}{2}} & -3K^{\frac{1}{6}}L^{-\frac{3}{2}} \end{bmatrix}$$

Example

$$\begin{aligned} \det H &= \begin{vmatrix} -\frac{5}{3}K^{-\frac{11}{6}}L^{\frac{1}{2}} & K^{-\frac{5}{6}}L^{-\frac{1}{2}} \\ K^{-\frac{5}{6}}L^{-\frac{1}{2}} & -3K^{\frac{1}{6}}L^{-\frac{3}{2}} \end{vmatrix} \\ &= 5K^{-\frac{10}{6}}L^{-1} - K^{-\frac{10}{6}}L^{-1} = 4K^{-\frac{5}{3}}L^{-1} > 0 \end{aligned}$$

and $\frac{\partial^2 \Pi}{\partial K^2} = -\frac{5}{3}K^{-\frac{11}{6}}L^{\frac{1}{2}} < 0$ for all $K, L > 0$. Therefore the function is concave and $K^* = L^* = 8$ is a global maximum.