BEE1024 Mathematics for Economists

Exponential and logarithmic functions, Elasticities

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Week 5

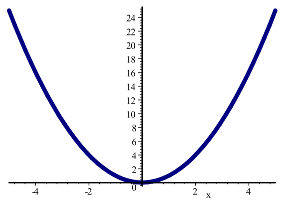
Objectives

- Exponential functions: describes growth processes with constant growth rate
  population growth, growth of GDP, inflation etc...
- logarithm: the exponent required to produce a given number
  inverse function, transforms multiplication into addition:
  $10^a \times 10^b = 10^{a+b}$
  Logarithmic differentiation
- Elasticities

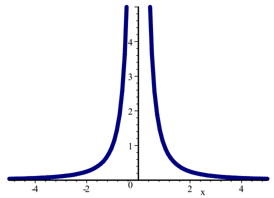
The Exponential Function

- power: x^y
- base: x
- index or exponent: y
- power function like x^2: vary x
- exponential function 2^y: vary y
- admissible values for y: positive integers, integers, rationals, real numbers
- problem: for general y the power x^y can only be defined for *positive* x

Power functions

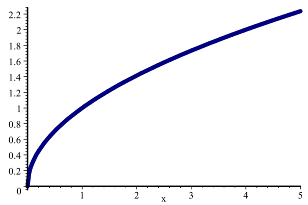


$y = 2$: $f(x) = x^2$

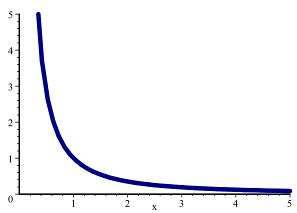


$y = -2$:
$f(x) = x^{-2} = \frac{1}{x^2}$

Power functions



$y = \frac{1}{2}$: $f(x) = x^{\frac{1}{2}} = \sqrt{x}$

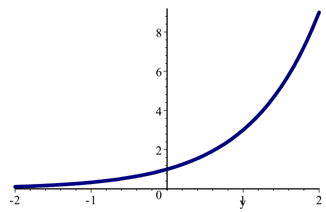


$y = -\frac{3}{2}$:
$f(x) = x^{-\frac{3}{2}} = \frac{1}{x\sqrt{x}}$

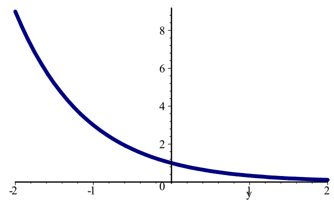
Exponential Functions

approximate irrational index y by fraction $\frac{m}{n}$:

$$x^y := \lim_{\frac{m}{n} \to y} x^{\frac{m}{n}}.$$

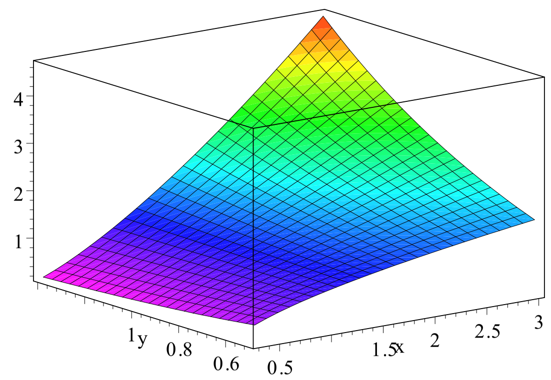


$x = 3$: $g(y) = 3^y$



$x = \frac{1}{3}$: $g(y) = \left(\frac{1}{3}\right)^y = 3^{-y}$

Exponential Functions



$z = x^y \qquad x \geq 0$

Properties of exponential functions

- An exponential function a^x :
- is strictly convex and has strictly positive values;
- is for $a > 1$ strictly increasing with $\lim_{x \to -\infty} a^x = 0$ and $\lim_{x \to \infty} a^x = +\infty$;
- is for $0 < a < 1$: decreasing with $\lim_{x \to -\infty} a^x = +\infty$ and $\lim_{x \to \infty} a^x = 0$.

Calculational rules for generalized powers:

$$a^{s+t} = a^s a^t \qquad a^{st} = (a^s)^t \qquad (ab)^s = a^s b^s$$

but

$$(a^s)^t \neq a^{(s^t)}$$

Compounded interests and the number e

- Put $P_0 > 0$ (the *principal*) in savings account
- fixed *nominal annual interests rate* $r > 0$
- Interests paid n times during the year
- amount P_t in your savings account after t years:
- **formula for compounded interests**

$$P_t = P_0 \left(1 + \frac{r}{n}\right)^{nt}$$

- $\frac{r}{n}$ interest paid per period
- nt total number of interest payments.

Compounded interests and the number e

The *(natural) exponential function:*
balance in account after one year if interests paid continuously:

$$\exp(r) = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n .$$

The table below shows the value of $\left(1 + \frac{r}{n}\right)^n$ for various n and r:

	$r = 5.4\%$	$r = 5.5\%$	$r = 100\%$
$n = 4$	1.055103375	1.056144809	2.44140625
$n = 12$	1.055356752	1.05640786	2.61303529
$n = 364$	1.055480375	1.056536225	2.714557303
$n = 8736$	1.055484426	1.056540432	2.718126265
$n = 524\,160$	1.055484599	1.056540612	2.718279235
$n = 31\,449\,600$	1.055484602	1.056540613	2.718281796
$n \to +\infty$	1.055484602	1.056540615	2.718281828

Compounded interests and the number e

The Euler number e is defined as

$$\boxed{e = \exp(1) = \lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n.}$$

The 'natural exponential function' is indeed the exponential function with base e:

$$\exp(r) = e^r$$

"proof" for rational r:

$$\begin{aligned}\exp(r) &= \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{m\to\infty, n=rm}\left(1+\frac{r}{n}\right)^n \\ &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{rm} = \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^r \\ &= \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^r = e^r\end{aligned}$$

Compounded interests and the number e

formula for continuously compounded interests:

$$\boxed{P_t = P_0 e^{rt}.}$$

Properties of the natural exponential function

1. $e^0 = 1$,
2. $e^1 = e$
3. $e^x > 0$ for all x
4. $$\boxed{\frac{d(e^x)}{dx} = e^x}$$
   In particular, e^x is strictly increasing and convex. *instantaneous growth rate* of a function $y = f(x)$: $\frac{dy}{dx}\Big/ y$ when x is increased by a exponential function has *constant growth rate* 1.
5. $$\boxed{e^{a+b} = e^a e^b \qquad (e^a)^b = e^{ab}.}$$
   In particular $\frac{1}{e^x} = e^{-x}$ (because $e^{-x}e^x = e^0 = 1$).

Let $f_n(x) = \left(1+\frac{x}{n}\right)^n$.
Intuition for property 3: $\left(1+\frac{x}{n}\right)$ is positive when $x > 0$ or when n large compared to $|x|$. Then $f_n(x) > 0$ and so hence $e^x > 0$.
Intuition for property 4:

$$\frac{df_n}{dx} = n\left(1+\frac{x}{n}\right)^{n-1}\frac{1}{n} = \left(1+\frac{x}{n}\right)^{n-1} \approx \left(1+\frac{x}{n}\right)^n = f_n(x)$$

for n very large compared to $|x|$ since $1+\frac{x}{n}$ is then very close to 1.

Theorem

There is one and only one function $y = f(x)$ which satisfies the "initial condition" $f(0) = 1$ and the "differential equation"

$$\frac{dy}{dx} = y$$

and this is the exponential function $f(x) = \exp x = e^x$.

Properties of the natural exponential function

- Exponential versus polynomial growth: For any polynomial $P(x)$

$$\lim_{x \to +\infty} \frac{e^x}{P(x)} = +\infty$$

Intuition: Suppose $P(x) = a_m x^m + \ldots$ has degree m. Approximate e^x by $\left(1 + \frac{x}{n}\right)^n$ with n larger than m. Then

$$\lim_{x \to +\infty} \frac{e^x}{P(x)} \approx \lim_{x \to +\infty} \frac{\left(1 + \frac{x}{n}\right)^n}{P(x)} = \lim_{x \to +\infty} \frac{\left(\frac{1}{n}\right)^n x^n + \ldots}{a_m x^m + \ldots} =$$

$$\lim_{x \to +\infty} Cx^{n-m} = +\infty$$

A quicker way to calculate e^x: is to use the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

Logarithmic functions

Let $a > 0$. The logarithmic function $\log_a x$ to the base a is defined as the inverse of the exponential function a^y

$$y = \log_a x \Leftrightarrow a^y = x$$

For instance, $1000 = 10^3$, so $\log_{10} 1000 = 3$; $\frac{1}{8} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

Differentiating the natural logarithm

$x = e^{\ln x}$

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x} = x^{-1}$$

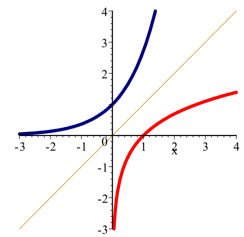
because by the chain rule

$$1 = \frac{dx}{dx} = e^{\ln x} \times (\ln x)' = x \times (\ln x)'$$

The natural logarithm

natural logarithm function

$$\boxed{y = \ln(x) \Leftrightarrow x = e^y}$$



Properties of the natural logarithm

1. $\ln(y)$ is only defined for strictly positive $y > 0$.
2. $\frac{d\ln(y)}{dy} = \frac{1}{y}$. In particular, $\ln(y)$ is strictly increasing and concave.
3. $\ln(1) = 0$, $\ln(e) = 1$.
4. $\lim_{y \to 0} \ln(y) = -\infty$, $\lim_{y \to +\infty} \ln(y) = +\infty$.
5. $\ln(ab) = \ln(a) + \ln(b)$, $\ln\left(a^b\right) = b\ln(a)$. In particular $\ln\left(\frac{1}{a}\right) = -\ln(a)$.
6. LOGARITHMIC DIFFERENTIATION: Combined with the chain rule one obtains the following useful formula where $y = g(x)$ is any differentiable function:

$$\boxed{\frac{d\ln(g(x))}{dx} = \frac{g'(x)}{g(x)}} \quad (1)$$

Theorem

The natural logarithm $y(x) = \ln x$ is the unique solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x}$$

which satisfies the initial condition $y(1) = 0$.

Differentiating general exponential and logarithmic functions

$\ln(x^y) = y \ln(x)$
general powers can be rewritten as

$$\boxed{x^y = e^{y \ln(x)} = e^{\text{index} \times \ln(\text{base})}.}$$

Partial differentiation yields

$$\begin{aligned}\frac{\partial x^y}{\partial y} &= e^{y \ln(x)} \ln(x) = x^y \ln(x) \\ \frac{\partial x^y}{\partial x} &= e^{y \ln(x)} \left(y \frac{1}{x}\right) = yx^y \frac{1}{x} = yx^{y-1}.\end{aligned}$$

Derivative of an exponential function $y = a^x$ is

$$\boxed{\frac{da^x}{dx} = \ln(a)\, a^x.}$$

The instantaneous growth rate of $y = a^x$ is $\ln(a)$
The derivative of a power function $y = x^b$ is

$$\boxed{\frac{dx^b}{dx} = bx^{b-1}}$$

even if b is irrational.

$$\boxed{x = \log_a(y) \Leftrightarrow y = a^x.}$$

We have

$$y = a^x \Leftrightarrow y = e^{x \ln(a)} \Leftrightarrow \ln(y) = x \ln(a) \Leftrightarrow x = \frac{\ln(y)}{\ln(a)}$$

so

$$\boxed{\log_a y = \frac{\ln(y)}{\ln(a)}}$$

$\log_a y$ has hence the derivative

$$\boxed{\frac{d(\log_a(y))}{dy} = \frac{1}{\ln(a) y}}$$

Compounded Interests

Example: Suppose Bank A offers the annual nominal interest rate $r_A = 5.5\%$ and pays interests monthly. Bank B offers the annual nominal interest rate $r_B = 5.4\%$ and pays interests daily. Which bank offers the better deal?
Solution:

$$r_{\text{eff},A} = \left(1 + \frac{r_A}{12}\right)^{12} - 1 = \left(1 + \frac{0.055}{12}\right)^{12} - 1 = 5.64\%$$

$$r_{\text{eff},B} = \left(1 + \frac{r_B}{364}\right)^{364} - 1 = \left(1 + \frac{0.054}{364}\right)^{364} - 1 = 5.55\%$$

so Bank A offers better deal.

Exponential decay

most radioactive substances decay exponentially
sample of initial size Q_0 weights $Q(t) = Q_0 e^{-kt}$ at time t.
k measures rate of decay, "*half-life*" of the radioactive substance:
Example: Show that a radioactive substance that decays according to the formula $Q(t) = Q_0 e^{-kt}$ has a half-life of $\bar{t} = \frac{\ln 2}{k}$.
Solution: find value $\bar{t}$ for which $Q(\bar{t}) = \frac{1}{2} Q_0$, that is

$$\frac{1}{2} Q_0 = Q_0 e^{-k\bar{t}}.$$

Divide by Q_0 and take natural logarithm:

$$\ln \frac{1}{2} = -k\bar{t}.$$

Thus the half-life is

$$\bar{t} = \frac{-\ln \frac{1}{2}}{k} = \frac{\ln 2}{k}$$

Differentiation

Example: Find the derivative of $g(x) = x^x$.
Solution: Using logarithmic differentiation we obtain

$$\begin{aligned} \frac{g'(x)}{g(x)} &= \frac{d(\ln x^x)}{dx} = \frac{d\left(\ln e^{x\ln(x)}\right)}{dx} = \frac{d(x\ln(x))}{dx} \\ &= 1 \times \ln(x) + x \times \frac{1}{x} = \ln(x) + 1 \\ g'(x) &= (\ln(x) + 1)\, x^x. \end{aligned}$$

The logistic curve

The graph of the function of the form

$$Q(t) = \frac{B}{1 + Ae^{-Bkt}}$$

where A, B, k are positive constants, is called a *logistic curve*. describes growth processes when environmental factors impose a "braking" effect on the rate of growth.
Example: Show that the growth rate of the logistic curve $Q(t) = \frac{1}{1+e^{-t}}$ is $1 - Q(t)$.

Solution: We have

$$Q'(t) = \frac{-e^{-t}(-1)}{(1+e^{-t})^2} = \frac{e^{-t}}{(1+e^{-t})^2}$$

$$\frac{Q'(t)}{Q(t)} = \frac{e^{-t}}{1+e^{-t}}$$

$$1 - Q(t) = \frac{1+e^{-t}-1}{1+e^{-t}} = \frac{e^{-t}}{1+e^{-t}}$$

Theorem

The logistic curve $y(x) = \frac{1}{1+e^{-x}}$ is the unique solution to the differential equation

$$\frac{dy}{dx} = y(1-y)$$

which satisfies the initial condition $y(0) = \frac{1}{2}$.

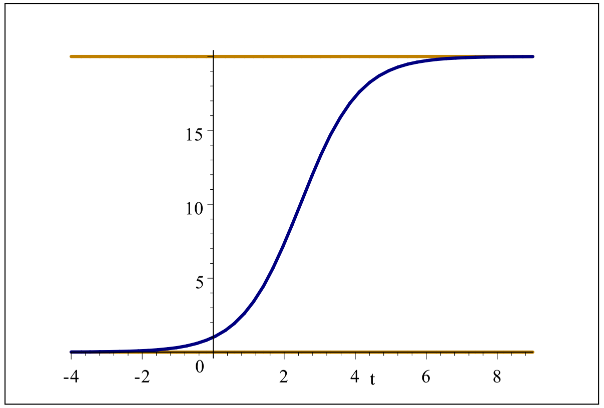
Example

Public health records indicate that t weeks after the outbreak of a certain form of influenza, approximately $Q(t) = \frac{20}{1+19e^{-1.2t}}$ thousand people had caught the disease.
a) How many people had the disease when it broke out? How many had it two weeks later?
b) At what time does the spread of the infection begin to decline?
c) If the trend continues, approximately how many people will eventually contract the disease?

Solution: a) Since $Q(0) = \frac{20}{1+19} = 1$ it follows that 1000 people initially had the disease. When $t = 2$

$$Q(2) = \frac{20}{1+19e^{-2.4}} \approx 7.343$$

so about 7.343 thousand people had contracted the disease by the second week.
b) inflection point at $\bar{t} \approx 2.5$. For $t < \bar{t}$ convex and so the number of newly infected increasing. For $t > \bar{t}$ concave and so the number of newly infected is decreasing.



c) $\lim_{t\to+\infty} Q(t) = 20$, so roughly 20000 people catch the disease on total.

Elasticities
Own-price elasticity

demand function

$$Q^d = 1000 - P^3$$
$$\frac{dQ^d}{dP} = -3P^2$$

current price is £5, price raised by a pound:
quantity demand decreases approximately by
$\frac{dQ^d}{dP}|_{P=5} = 3 \times 5^2 = 75$ tons.
one-percent increase in the price: increase by 5p$= \frac{1}{20} \times \$1$
reduce quantity demanded by approximately $3.75 = \frac{75}{20}$ tons.
demand at £5 is $1000 - 5^3 = 875$
percentage decrease in quantity demanded is $\frac{3.75}{875} \approx 0.0043 = 0.43\%$.
Thus 1% increase in price reduces quantity demanded by 0.43%.

More generally: $\frac{dQ^d}{dP}$ approximate change in demand when the price increases by a pound.
initial price is P,
increase by one pound is an increase by $\frac{100}{P}$ percent.
an increase of the price by 1% changes the quantity demanded by approximately by $\frac{P}{100} \times \frac{dQ^d}{dP}$ tons.
percentage change in quantity demanded is approximately:

$$\text{ped}(P) = \frac{100}{Q^d} \times \frac{P}{100} \times \frac{dQ^d}{dP} = \frac{dQ^d}{dP} \times \frac{P}{Q^d}$$

This is the *own-price elasticity of demand.*

rewrite this formula as

$$\text{ped}(P) = \frac{dQ^d}{Q^d} \div \frac{dP}{P}$$

where $100\frac{dP}{P}$ is the percentage increase in price and $100\frac{dQ^d}{Q^d}$ is (approximately) the induced percentage change in quantity.
In our example

$$\text{ped}(P) = \left(-3P^2\right) \times \frac{P}{Q^d} = -3\frac{P^3}{1000 - P^3}$$

When is demand inelastic?

$$3\frac{P^3}{1000-P^3} < 1$$

or

$$3P^3 < 1000 - P^3 \qquad 4P^3 < 1000 \qquad P^3 < 250 \qquad P < \sqrt[3]{250} \approx 6.3$$

Exactly when $P = \sqrt[3]{250}$ there is unit elasticity and above demand is elastic.

Constant Elasticity

A function has constant elasticity ε if and only if it is of the form

$$Q = \alpha P^{\varepsilon}$$

We have

$$\begin{aligned} \frac{dQ}{dP} &= \alpha\varepsilon P^{\varepsilon-1} \\ \frac{dQ}{dP}\frac{P}{Q} &= \alpha\varepsilon P^{\varepsilon-1}\frac{P}{\alpha P^{\varepsilon}} = \varepsilon \end{aligned}$$

Total Revenue

With this demand, total revenue of the market is

$$TR = PQ^d = P\left(1000 - P^3\right)$$

Total revenue is maximized when

$$\frac{dTR}{dP} = 1000 - 4P^3 = 0$$

or $P = \sqrt[3]{250}$, i.e., exactly when there is unit elasticity.
Since $\frac{d^2TR}{dP^2} = -12P^2 < 0$, total revenue decreases to the left and increases to the right of this price.

Inverse Demand

The consumers will demand a quantity Q when the price P is such that

$$\begin{aligned} Q &= Q^d(P) = 1000 - P^3 \\ P^3 &= 1000 - Q \\ P^d &= \sqrt[3]{1000 - Q} \end{aligned}$$

inverse demand function

Marginal Revenue

use inverse demand function to express total revenue as a *function of quantity:*

$$TR = PQ = \sqrt[3]{1000 - Q} \times Q = (1000 - Q)^{\frac{1}{3}} \times Q$$

quantity demanded is decreasing in price.
total revenue is increasing in price when it decreasing in quantity and vice versa.

Marginal revenue is the change in revenue if a small unit more of the commodity is sold on the market.

$$MR = \frac{dTR}{dQ} = -\frac{1}{3}(1000 - Q)^{-\frac{2}{3}} \times Q + (1000 - Q)^{\frac{1}{3}}$$

Marginal revenue is zero when

$$\begin{aligned} (1000 - Q)^{-\frac{2}{3}} \times Q &= 3(1000 - Q)^{\frac{1}{3}} \\ Q &= 3(1000 - Q) = 3000 - 3Q \\ 4Q &= 3000 \qquad Q = 750 \end{aligned}$$

at $P = \sqrt[3]{250}$. For lower quantities it is positive and for higher ones negative.

Theorem

Marginal revenue and own-price elasticity are related by

$$MR = P\left(1 + \frac{1}{\mathrm{ped}(P)}\right)$$

This is so because

$$\frac{dTR}{dP} = \frac{d\left(PQ^d\right)}{dP} = Q^d + P\frac{dQ^d}{dP}$$

whereas by the chain rule

$$\frac{dTR}{dP} = \frac{dTR}{dQ}\frac{dQ^d}{dP}$$

and so

$$MR = \frac{dTR}{dQ} = \left(Q^d + P\frac{dQ^d}{dP}\right) \Big/ \frac{dQ^d}{dP} = \frac{Q^d}{\frac{dQ^d}{dP}} + P = \frac{P}{\frac{dQ^d}{dP}\frac{P}{Q^d}} + P$$

Elasticities and logarithms

data $(x, y) = (\ln P, \ln Q)$. Then

$$\frac{dy}{dx} = \text{ped}(P)$$

$\frac{dy}{dQ} = \frac{1}{Q}$, $P = e^x$ $\frac{dP}{dx} = e^x = P$.The chain rule applied twice yields

$$\frac{dy}{dx} = \frac{dy}{dQ}\frac{dQ}{dP}\frac{dP}{dx} = \frac{1}{Q}\frac{dQ}{dP}P = \text{ped}(P)$$

Other Elasticities

demand for a commodity function of own price, income and other prices.

$$Q^d = 100 - p + 2p^* - 3y$$

cross price elasticity:

$$\frac{\partial Q^d}{\partial p^*}\frac{p^*}{Q^d} = 2\frac{p^*}{Q^d}$$

income elasticity

$$\frac{\partial Q^d}{\partial y}\frac{y}{Q^d} = -3\frac{y}{Q^d}$$

Other Elasticities

demand for a commodity function of own price, income and other prices.

$$Q^d = 100 - p + 2p^* - 3y$$

p is the price of the commodity,
p^* the price of another commodity
y is income.

own price elasticity:

$$\frac{\partial Q^d}{\partial p}\frac{p}{Q^d} = -\frac{p}{Q^d}$$