BEE1024 Mathematics for Economists

Optimization 4

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Week 5

Absolute Maxima

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3. local (relative) versus global (absolute) maximum
   - highest point nearby or overall

Theorem

Suppose that the function $z = f(x, y)$ has a critical point at (x^*, y^*). If the determinant of the Hessian $\det H =$

$$\begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

is negative at (x^*, y^*) then (x^*, y^*) is a saddle point.
If this determinant is positive at (x^*, y^*) then (x^*, y^*) is a peak or a trough. In this case the signs of $\frac{\partial^2 z}{\partial x^2}|_{(x^*,y^*)}$ and $\frac{\partial^2 z}{\partial y^2}|_{(x^*,y^*)}$ are the same. If both signs are positive, then (x^*, y^*) is a trough. If both signs are negative, then (x^*, y^*) is a peak.

$$\det H \begin{cases} < 0: & \text{saddle point} \\ > 0: & \begin{cases} \frac{\partial^2 z}{\partial x^2} > 0: & \text{trough} \\ \frac{\partial^2 z}{\partial x^2} < 0: & \text{peak} \end{cases} \end{cases}$$

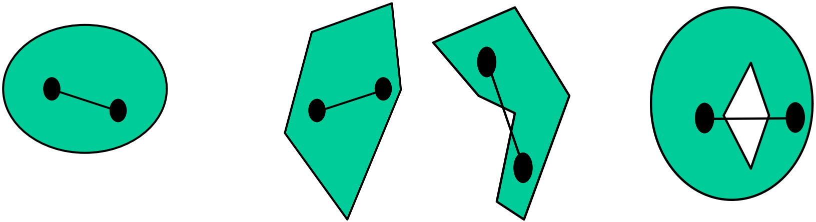
criterium for *local* optima only.

Convex Sets

intervals $\approx$ convex sets   for one variable

A region of the (x, y)-plane is *convex* if:

With every two points (x_1, y_1) and (x_2, y_2) in the region the whole line segment connecting these two points is in the region.



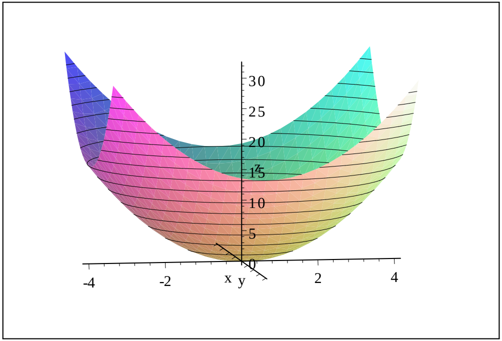
two convex sets   two non-convex sets

The (x, y)-plane and the the set of all non-negative coordinate pairs (x, y) are also convex sets.

(Notice: There are concave and convex functions and there are concave and convex lenses, but there is no such thing as a "concave set".)

Convex Functions

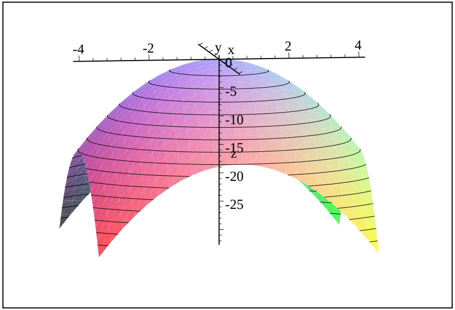
A function $f(x, y)$ is CONVEX if the area *above* the graph of the function, i.e., the set of all points (x, y, z) with (x, y) in the domain of the function and $z \geq f(x, y)$, is convex.



A convex function

Concave Functions

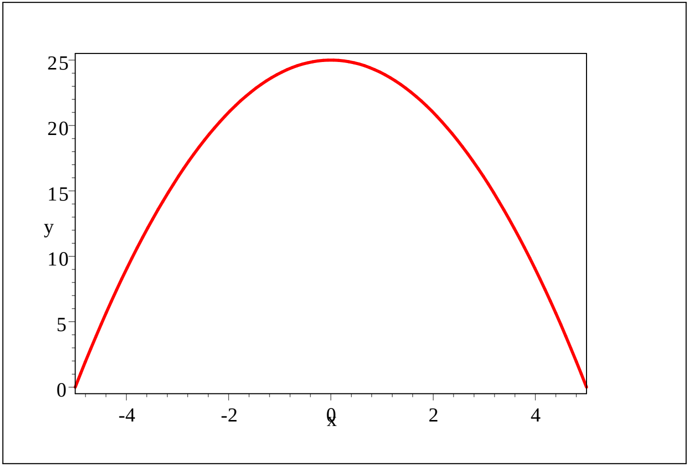
A function $f(x, y)$ is CONCAVE if the area *below* the graph of the function, i.e., the set of all points (x, y, z) with (x, y) in the domain of the function and $z \leq f(x, y)$, is convex.



A concave function

For a concave function every local maximum is a global maximum.

Con-



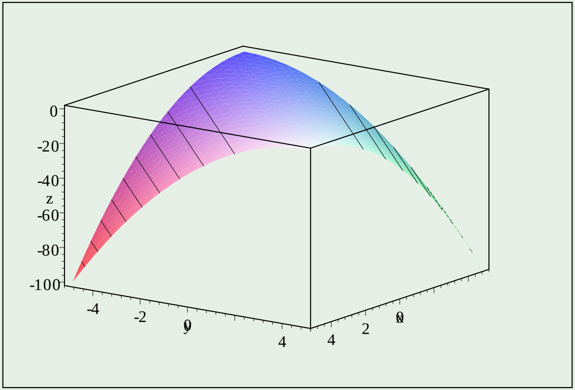
cave

Theorem

Suppose the function $f(x, y)$ is defined on a convex set. Suppose the determinant of the Hessian is always positive in the region and that $\frac{\partial^2 f}{\partial x^2}$ is always negative. Then the function is concave and every peak is a maximum.

Example

Every point (x, x) is a maximum of the function $z = -(y - x)^2$



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Example

Profit function

$$\Pi(K, L) = 12K^{\frac{1}{6}}L^{\frac{1}{2}} - K - 3L$$

has critical point at $K^* = L^* = 8$. We have

$$\frac{\partial \Pi}{\partial K} = 2K^{-\frac{5}{6}}L^{\frac{1}{2}} - 1 \qquad \frac{\partial \Pi}{\partial L} = 6K^{\frac{1}{6}}L^{-\frac{1}{2}} - 3$$

and hence

$$H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial K^2} & \frac{\partial^2 \Pi}{\partial L \partial K} \\ \frac{\partial^2 \Pi}{\partial K \partial L} & \frac{\partial^2 \Pi}{\partial L^2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{3}K^{-\frac{11}{6}}L^{\frac{1}{2}} & K^{-\frac{5}{6}}L^{-\frac{1}{2}} \\ K^{-\frac{5}{6}}L^{-\frac{1}{2}} & -3K^{\frac{1}{6}}L^{-\frac{3}{2}} \end{bmatrix}$$

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$$= 5K^{-\frac{10}{6}}L^{-1} - K^{-\frac{10}{6}}L^{-1} = 4K^{-\frac{5}{3}}L^{-1} > 0$$

and $\frac{\partial^2 \Pi}{\partial K^2} = -\frac{5}{3}K^{-\frac{11}{6}}L^{\frac{1}{2}} < 0$ for all $K, L > 0$. Therefore the function is concave and $K^* = L^* = 8$ is a global maximum.

The Envelope Theorem

- Suppose the optimization problem depends on "exogeneous parameters", i.e. variables which represent constants to be fixed outside the model or which have to be estimated in contrast to endogenous variables which are to be solved for. In the consumer problem p_x, p_y, b are exogeneous, whereas x, y are endogenous.

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- How does the optimal value (e.g. utility) change if the exogeneous parameter changes?

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- $f^*(t) = \mathcal{L}^*(t) = f(x^*(t), y^*(t), t) + \lambda_1 g_1(x^*(t), y^*(t), t) + \lambda_2 g_2(x^*(t), y^*(t), t) + \cdots + \lambda_K g_K(x^*(t), y^*(t), t)$

The Envelope Theorem

Theorem

$$\frac{df^*}{dt}_{|t} = \frac{\partial \mathcal{L}}{\partial t}_{|t,x^*(t),y^*(t),\lambda_1^*(t),\cdots,\lambda_K^*(t)}$$

Proof.

(Basic idea) By the chain rule

$$\begin{aligned}\frac{df^*}{dt} &= \frac{\partial \mathcal{L}^*}{\partial t} \\ &= \frac{\partial \mathcal{L}}{\partial x}\frac{dx^*}{dt} + \frac{\partial \mathcal{L}}{\partial y}\frac{dy^*}{dt} + \frac{\partial \mathcal{L}}{\partial \lambda_1}\frac{d\lambda_1^*}{dt} + \cdots + \frac{\partial \mathcal{L}}{\partial \lambda_K}\frac{d\lambda_K^*}{dt} + \frac{\partial \mathcal{L}}{\partial t}\frac{dt}{dt} \\ &= \frac{\partial \mathcal{L}}{\partial t}\frac{dt}{dt} = \frac{\partial \mathcal{L}}{\partial t}\end{aligned}$$

using the first order conditions and the complementarity conditions which must hold in optimum. □

Application to consumer optimum

Let $u^*(b)$ be the maximal utility the consumer can get given budget and prices. We keep the prices fixed and vary the budget. Lagrangian:

$$\begin{aligned}\mathcal{L}(x, y, b) &= u(x, y) + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y \\ \frac{\partial \mathcal{L}}{\partial b} &= \lambda_1\end{aligned}$$

The envelope theorem tells us that the Lagrange multiplier for the budget constraint is equal to the (approximate) increase in utility when the consumer's budget is increased by a money unit. Alternative interpretation: λ_1 describes what the marginal utility for saving a pound for future consumption must be, such that the consumer neither wants to make debt nor save.

Cost minimization

Let $TC(Q_0)$ be the total costs of the producer to produce Q_0 units of output. So $TC(Q_0) = rK^* + wL^*$ where (K^*, L^*) minimize costs subject the constraint $Q(K, L) \geq Q_0$. By the envelope theorem, $-MC(Q_0) = -\frac{dTC}{dQ_0} = \frac{\partial \mathcal{L}}{\partial Q_0}$
Lagrangian:

$$\begin{aligned} \mathcal{L}(K, L, Q_0) &= -rK - wL + \lambda_1 (Q(K, L) - Q_0) \\ &= [\lambda_1 Q(K, L) - rK - wL] - \lambda_1 Q_0 \\ \frac{\partial \mathcal{L}}{\partial Q_0} &= -\lambda_1 \end{aligned}$$

We see that the Lagrange multiplier is equal to the marginal cost, $MC(Q_0) = \lambda_1$.
Notice that the term in square brackets above describes profit at the output price λ_1. So, maximizing the Lagrangian is the same as maximizing profits at the price λ_1.