BEE1024 Mathematics for Economists

Multivariate Functions

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Week 4

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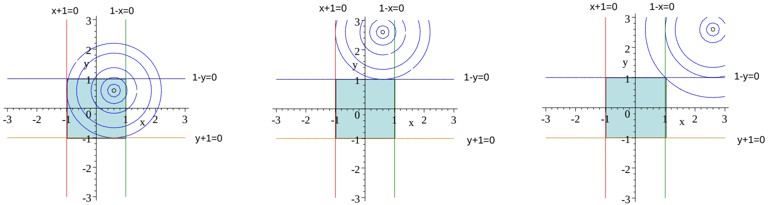
The Lagrangian Approach



- Is the optimum inside, on an edge or at a cornerpoint of the region carved out by the inequality constraints?
- I.e. which constraints hold with equality in the optimum?
- A constraint is "binding" at the optimum, if it holds there with equality.
- A constraint is "binding" if it has bite and restrains the optimum.

- A constraint which is not binding can be ignored without changing the optimum.

The Lagrangian:

$$\mathcal{L}(x, y) = f(x, y) + \lambda_1 g_1(x, y) + \lambda_2 g_2(x, y) + \cdots + \lambda_K g_K(x, y)$$

Theorem

Suppose we are given numbers $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$ *and a pair of numbers* (x^*, y^*) *such that*

1. $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$, *i.e. Lagrange mutlipliers are non-negative,*
2. (x^*, y^*) *satisfies all the constraints, i.e.,* $g_k(x^*, y^*) \geq 0$ *for all* $1 \leq k \leq K$.
3. (x^*, y^*) *is an unconstrained maximum of the Lagrangian* $\mathcal{L}(x, y)$.
4. *The complementarity conditions*

   $$\lambda_k g_k(x^*, y^*) = 0$$

   are satisfied, i.e., either the k-Lagrange multiplier is zero or the k-th constraint binds for $1 \leq k \leq K$.

Then (x^*, y^*) *is a maximum for the constrained maximization problem.*

- The Lagrangian approach does not immediately tell you which constraints are binding in the optimum, you will have to start with an informed guess using all problem-specific information.
- You write down the Lagrangian assuming that only certain constraints bind. For the others the Lagrange multipliers and the corresponding terms $\lambda_k g_k(x, y)$ are zero. (Often these are already ignored in the beginning.)
- You solve the simultaneous system of equations consisting of the FOC and the equations $g_k(x, y) = 0$ for the constraints assumed to be binding.
- You check that the solution satisfies the other constraints (which are not assumed to be binding) and that the Lagrange multiplicators are non-negative.
- You check that the solution found is an unconstrained optimum of the Lagrangian.
- If all this holds, you have found the optimum. Otherwise, try again.

Notice:

- If only one constraint binds, say g_1, the FOC become

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\lambda_1 \frac{\partial g_1}{\partial x} \\ \frac{\partial f}{\partial y} &= -\lambda_1 \frac{\partial g_1}{\partial y}\end{aligned}$$

  and must hold together with $g_1(x, y) = 0$. This simplifies to the two equations

$$\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = \frac{\partial g_1}{\partial x} \Big/ \frac{\partial g_1}{\partial y} \qquad g_1(x, y) = 0$$

- $$\frac{\partial \mathcal{L}}{\partial \lambda_k} = g_k(x, y)$$

Maximize $f(x,y)$ subject to the constraints $h_k(x,y) \leq 0$ $(k = 1, \cdots, K)$. Equivalently $g_k(x,y) \geq 0$ with $g_k(x,y) = -h_k(x,y)$.
The Lagrangian:

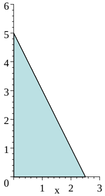
$$\begin{aligned}\mathcal{L}(x,y) &= f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y) + \cdots + \lambda_K g_K(x,y) \\ &= f(x,y) - \lambda_1 h_1(x,y) - \lambda_2 h_2(x,y) - \cdots - \lambda_K h_K(x,y)\end{aligned}$$

- Textbooks do not "converge" on one approach.
- For the algebra all this does not matter, at the worst you may get negative Lagrange multipliers.
- To avoid further confusion I rewrite minimization problems as maximization problems.

Utility maximization

Maximize $u(x, y) = (x+1)(y+1)$, subject to the budget constraint $p_x x + p_y y \leq b$ and the non-negativity constraints $x, y \geq 0$.

Where can the consumer optimum be in the budget set carved out by the budget- and the non-negativity constraints?



In principle there are 7 different possibilities to consider: It could be in the interior of the triangle, on one of the three sides or it could be one of the three corner points.

Utility maximization

$u(x, y) = (x + 1)(y + 1)$ is "monotonic", more of each commodity is better.

Lemma

If the utility function is monotonic then the budget constraint must be binding in the consumer optimum, i.e. the optimum is on the budget line.

Proof.

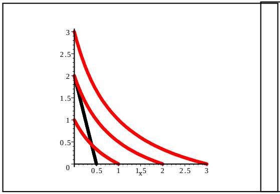
If (x, y) is a consumption bundle which costs less than the budget then one can, for instance, slightly increase the consumption of x without violating the budget constraint. The first factor in $(x + 1)(y + 1)$ increases and hence, since $y + 1 > 0$ the whole product. Utility goes up and so (x, y) cannot be the optimum. □

This argument holds generally for all *monotonic preferences*.

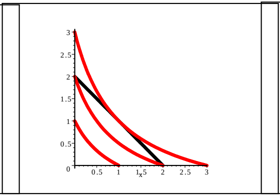
Utility maximization

- This reduces the search to three possibilities: The budget line and its two corner points. (Different: “bads”, satiated consumers.)

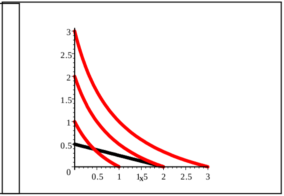
Three possibilities:



Optimum (0,2)



Optimum (1,1)



Optimum (2,0)

Utility maximization

- For a utility function like $u(x, y) = xy$ *one can say even more*, namely that consumption of both commodities must be strictly positive in optimum. (Utility is zero when $x = 0$ or $y = 0$ while a strictly positive utility can be achieved with a strictly positive budget.) Thus *only* the budget constraint can be binding.

Utility maximization - The Lagrangian

$$\begin{aligned}\mathcal{L}(x,y) &= u(x,y) + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y \\ &= (x+1)(y+1) + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y\end{aligned}$$

where $p_x, p_y, b > 0$
FOC:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x + \lambda_2 = (y+1) - \lambda_1 p_x + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y + \lambda_3 = (x+1) - \lambda_1 p_y + \lambda_3 = 0\end{aligned}$$

Case 1: Only the budget equation binds

Hence $\lambda_2 = \lambda_3 = 0$ by the complementarity condition. We get the FOC

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x = (y+1) - \lambda_1 p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y = (x+1) - \lambda_1 p_y = 0 \end{aligned}$$

and the budget constraint must hold with equality

$$p_x x + p_y y = b.$$

Notice that if x and y are postive, then the FOC imply $\lambda_1 > 0$.

Case 1: Only the budget equation binds

When only one constraint binds, it is easy to eliminate the Lagrange multiplier:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x = 0\\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y = 0\end{aligned}$$

implies

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{p_x}{p_y}$$

Together with

$$p_x x + p_y y = b.$$

we must then solve a system of two equations with two unknowns.

Case 1: Only the budget equation binds

The FOC imply

$$\frac{\partial u}{\partial x} = \lambda_1 p_x \qquad \frac{\partial u}{\partial y} = \lambda_1 p_y$$

Division of the two left hand sides and the two right hand sides yields

$$\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{p_x}{p_y}$$

Thus the marginal rate of substitution (see previous lectures) must equal the price ratio or, in other words, in the consumer optimum the indifference curve is tangential to the budget line.

Case 1: Only the budget equation binds

In our particular example this yields

$$\frac{y+1}{x+1} = \frac{p_x}{p_y} \qquad y = \frac{p_x}{p_y}(x+1) - 1$$

Substitution into the budget equation yields.

$$\begin{aligned} b &= p_x x + p_y y = p_x x + p_y \left(\frac{p_x}{p_y}(x+1) - 1 \right) \\ b &= 2p_x x + p_x - p_y \end{aligned}$$

and so

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

where the last formula holds because

$$y = \frac{p_x}{p_y}(x+1) - 1 = \frac{p_x}{p_y} \frac{b - p_x + p_y}{2p_x} + \frac{2p_x}{2p_y} - \frac{2p_y}{2p_y} = \frac{b + p_x - p_y}{2p_y}$$

Case 1: Only the budget equation binds

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

can only be the solution when both numbers are non-negative. This requires

$$\begin{aligned} b - p_x + p_y &\geq 0 \qquad b + p_x - p_y \geq 0 \\ b &\geq p_x - p_y \qquad b \geq -(p_x - p_y) \\ b &\geq |p_x - p_y| \end{aligned}$$

where $||$ denotes the "absolute value"

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Intuitively, the price difference cannot be too large in comparison to the budget.

Case 1: Only the budget equation binds

To summarize, provided $b \geq |p_x - p_y|$ the Lagrangian approach yields a positive Lagrange multiplier λ_1 (see the argument further above) and the non-negative solution

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

Provided we can show that this solution is indeed an unconstrained maximum of the Lagrangian (and not a minimum etc.) it is the solution to our constrained optimization problem.

Case 2: The budget equation binds and x*=0

So one of the non-negativity constraint is binding. Thus the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial u}{\partial x} - \lambda_1 p_x + \lambda_2 = (y+1) - \lambda_1 p_x + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial u}{\partial y} - \lambda_1 p_y + \lambda_3 = (x+1) - \lambda_1 p_y + \lambda_3 = 0 \end{aligned}$$

must holds together with the budget equation

$$p_x x + p_y y = b$$

and

$$x^* = 0$$

The budget equation simplifies to $p_y y = b$ and our only solution candidate is

$$x^* = 0 \qquad y^* = b/p_y$$

Case 2: The budget equation binds and x*=0

Because $y^* > 0$ the non-negativity constraint $y = 0$ does not bind and therefore $\lambda_3 = 0$ by the complementarity conditions. The FOC simplify to

$$\begin{aligned} b/p_y + 1 - \lambda_1 p_x + \lambda_2 &= 0 \\ 1 - \lambda_1 p_y &= 0 \end{aligned}$$

So $\lambda_1 = 1/p_y > 0$ and the first FOC yields

$$\begin{aligned} b/p_y + 1 - \frac{p_x}{p_y} + \lambda_2 &= -b/p_y - 1 + \frac{p_x}{p_y} \\ \frac{b + p_y - p_x}{p_y} + \lambda_2 &= 0 \\ \lambda_2 &= \frac{p_x - p_y - b}{p_y} \end{aligned}$$

For λ_2 to be non-negative we need that the price difference $p_x - p_y$ is bigger than the budget.

Case 3: The budget equation binds and $y^*=0$

This case is handled completely symmetrically to case 2.

Summary

Apart form showing that we have indeed found unconstrained optima of the Lagrangian we get the following result

Theorem

- *When the price of x is very high, namely when $p_x \geq p_y + b$ the consumer only wants to buy y and so $x^* = 0$, $y^* = b/p_y$.*
- *When the price of y is very high, namely when $p_y \geq p_x + b$ the consumer only wants to buy y and so $x^* = b/p_x$, $y^* = 0$.*
- *In all other cases the consumer wants to buy of both commodities the amounts*

$$x^* = \frac{b - p_x + p_y}{2p_x} \qquad y^* = \frac{b + p_x - p_y}{2p_y}$$

The Envelope Theorem

- Suppose the optimization problem depends on "exogeneous parameter", i.e. variables which represent constants to be fixed outside the model or which have to be estimated in contrast to endogenous variables which are to be solved for. In the consumer problem p_x, p_y, b are exogeneous, whereas x, y are endogenous.
- How does the optimal value (e.g. utility) change if the exogeneous parameter changes?

The Envelope Theorem

- exogeneous parameter: t
- endogeneous variables: x, y
- Maximize $f(x, y, t)$ subject to the constraints $g_k(x, y, t) \geq 0$ $(k = 1, \cdots, K)$.
- $\mathcal{L}(x, y, t) = f(x, y, t) + \lambda_1 g_1(x, y, t) + \lambda_2 g_2(x, y, t) + \cdots + \lambda_K g_K(x, y, t)$
- Optimum $x^*(t), y^*(t)$ with corresponding Lagrange multipliers $\lambda_k^*(t)$ $(k = 1, \cdots, K)$
- Optimal value: $f^*(t) = f(x^*(t), y^*(t), t)$ (is a univariate function)
- $f^*(t) = \mathcal{L}^*(t) = f(x^*(t), y^*(t), t) + \lambda_1 g_1(x^*(t), y^*(t), t) + \lambda_2 g_2(x^*(t), y^*(t), t) + \cdots + \lambda_K g_K(x^*(t), y^*(t), t)$

The Envelope Theorem

Theorem

$$\frac{df^*}{dt}_{|t} = \frac{\partial \mathcal{L}}{\partial t}_{|t,x^*(t),y^*(t),\lambda_1^*(t),\cdots m\lambda_K^*(t)}$$

Proof.

(Basic idea) By the chain rule

$$\begin{aligned}\frac{df^*}{dt} &= \frac{\partial \mathcal{L}^*}{\partial t} \\ &= \frac{\partial \mathcal{L}}{\partial x}\frac{dx^*}{dt} + \frac{\partial \mathcal{L}}{\partial y}\frac{dy^*}{dt} + \frac{\partial \mathcal{L}}{\partial \lambda_1}\frac{d\lambda_1^*}{dt} + \cdots + \frac{\partial \mathcal{L}}{\partial \lambda_K}\frac{d\lambda_K^*}{dt} + \frac{\partial \mathcal{L}}{\partial t}\frac{dt}{dt} \\ &= \frac{\partial \mathcal{L}}{\partial t}\frac{dt}{dt} = \frac{\partial \mathcal{L}}{\partial t}\end{aligned}$$

using the first order conditions and the complementarity conditions which must hold in optimum. □

Application to consumer optimum

Let $u^* (b)$ be the maximal utility the consumer can get given budget and prices. We keep the prices fixed and vary the budget. Lagrangian:

$$\mathcal{L}(x, y, b) = u(x, y) + \lambda_1 (b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y$$
$$\frac{\partial \mathcal{L}}{\partial b} = \lambda_1$$

The envelope theorem tells us that the Lagrange multiplier for the budget constraint is equal to the (approximate) increase in utility when the consumer's budget is increased by a money unit. Alternative interpretation: λ_1 describes what the marginal utility for saving a pound for future consumption must be such that the consumer neither wants to make debt nor save.

Cost minimization

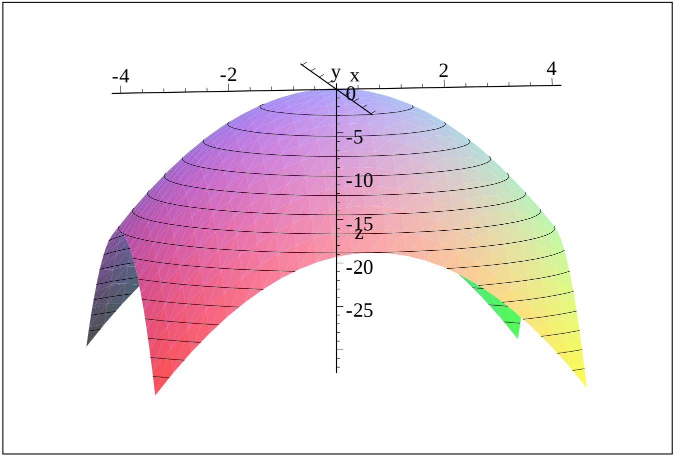
Let $TC(Q_0)$ be the total costs of the producer to produce Q_0 units of output. So $TC(Q_0) = rK^* + wL^*$ where (K^*, L^*) minimize costs subject the constraint $Q(K, L) \geq Q_0$. By the envelope theorem, $-MC(Q_0) = -\frac{dTC}{dQ_0} = \frac{\partial \mathcal{L}}{\partial Q_0}$
Lagrangian:

$$\begin{aligned}
\mathcal{L}(K, L, Q_0) &= -rK - wL + \lambda_1 (Q(K, L) - Q_0) \\
&= [\lambda_1 Q(K, L) - rK - wL] - \lambda_1 Q_0 \\
\frac{\partial \mathcal{L}}{\partial Q_0} &= -\lambda_1
\end{aligned}$$

We see that the lagrange multiplier is equal to the marginal cost, $MC(Q_0) = \lambda_1$.
Notice that the term in square brackets above describes profit at the output price λ_1. So, maximizing the Lagrangian is the same as maximizing profits at the price λ_1.

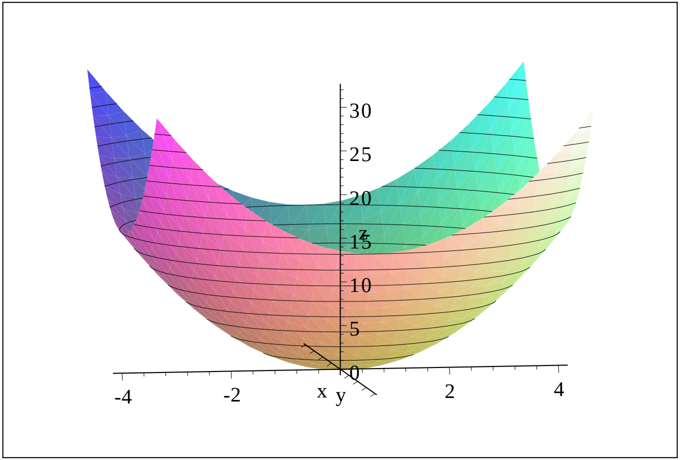
Second order conditions: A peak

critical point of function $z = f(x, y) = -x^2 - y^2$: $(0, 0)$



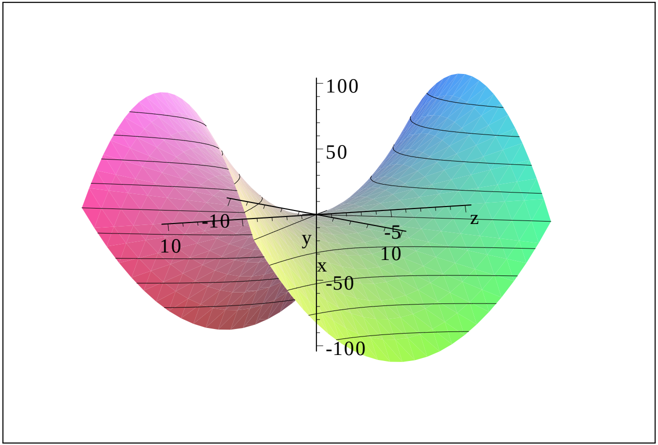
Second order conditions: A trough

critical point of function $z = f(x, y) = +x^2 + y^2$: $(0, 0)$



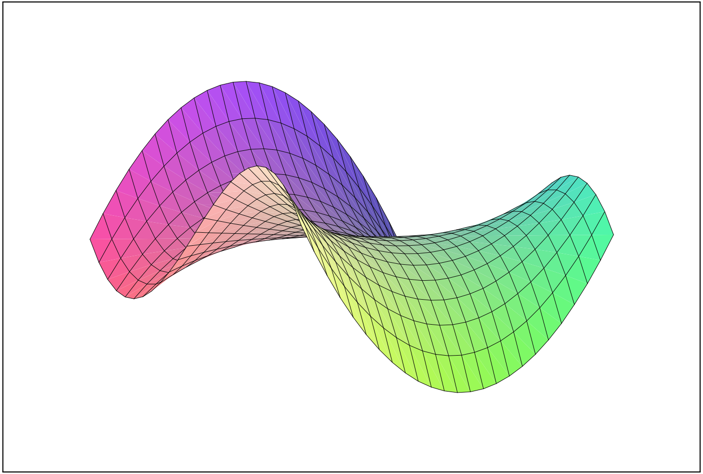
Second order conditions: A saddle point

critical point of function $z = f(x, y) = +x^2 - y^2$: $(0, 0)$



Second order conditions: A monkey saddle

critical point of function $z = f(x, y) = yx^2 - y^3$: $(0, 0)$



This and more complex possibilities will be ignored.

The Hessian matrix

$$\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

Theorem

Suppose that the function $z = f(x, y)$ *has a critical point at* (x^*, y^*). *If the determinant of the Hessian* $\det H =$

$$\begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

is negative at (x^*, y^*) *then* (x^*, y^*) *is a saddle point.*
If this determinant is positive then at (x^*, y^*) *then* (x^*, y^*) *is a peak or a trough. In this case the signs of* $\frac{\partial^2 z}{\partial x^2}_{|(x^*,y^*)}$ *and* $\frac{\partial^2 z}{\partial y^2}_{|(x^*,y^*)}$ *are the same. If both signs are positive, then* (x^*, y^*) *is a trough. If both signs are negative, then* (x^*, y^*) *is a peak.*

- Nothing can be said if the determinant is zero.
- Notice that $\det H > 0$ implies $\operatorname{sign} \frac{\partial^2 z}{\partial x^2} = \operatorname{sign} \frac{\partial^2 z}{\partial y^2}$

$$\det H \begin{cases} < 0: & \text{saddle point} \\ > 0: & \begin{cases} \frac{\partial^2 z}{\partial x^2} > 0: & \text{trough} \\ \frac{\partial^2 z}{\partial x^2} < 0: & \text{peak} \end{cases} \end{cases}$$

Example

$z = x^2 - y^2$. The partial derivatives are $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = -2y$. Clearly, $(0,0)$ is the only critical point. The determinant of the Hessian is

$$\det H = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = (2)(-2) - 0 \times 0 = -4 < 0$$

Hence the function has a saddle point at $(0,0)$.

Example

$z = -x^2 - y^2$. The partial derivatives are $\frac{\partial z}{\partial x} = -2x$ and $\frac{\partial z}{\partial y} = -2y$. Clearly, $(0,0)$ is the only critical point. The determinant of the Hessian is

$$\det H = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$$

and $\frac{\partial^2 z}{\partial x^2} < 0$. Hence the function has a peak at $(0,0)$.

Example

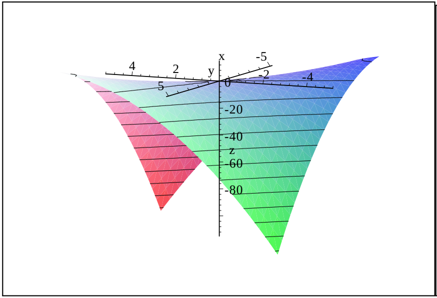
$$\begin{aligned} z &= -x^2 + \frac{5}{2}xy - y^2 \\ \frac{\partial z}{\partial x} &= -2x + \frac{5}{2}y \\ \frac{\partial z}{\partial y} &= \frac{5}{2}x - 2y \end{aligned}$$

$(0,0)$ critical point.

Determinant of the Hessian is

$$\det H = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \begin{vmatrix} -2 & \frac{5}{2} \\ \frac{5}{2} & -2 \end{vmatrix} = (-2)(-2) - \left(\frac{5}{2}\right)^2$$
$$= -\frac{9}{4} < 0$$

Hence $(0,0)$ saddle point although *both* $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ negative.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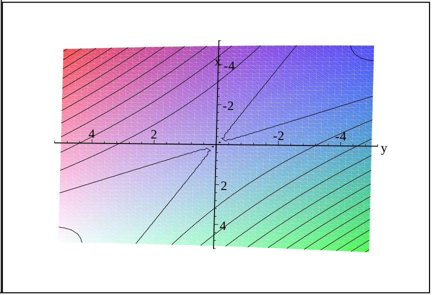
-60
-80


x
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