BEE1024 Mathematics for Economists

Multivariate Functions

Juliette Stephenson and Amr (Miro) Algarhi
Author: Dieter Balkenborg

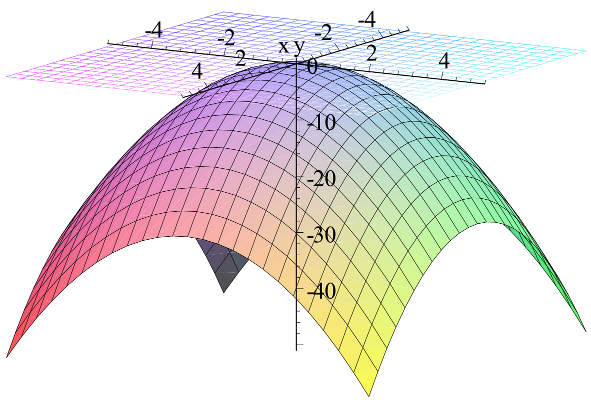
Department of Economics, University of Exeter

Week 3

Objectives

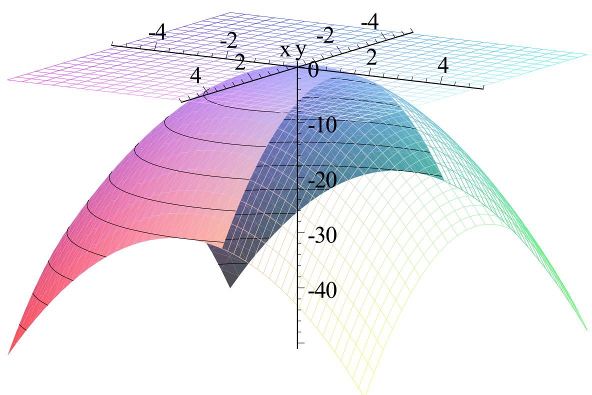
- Subject: Optimization of multivariate functions
- This lecture:
  1. Lagrangian approach for constrained optimization problems
- Next lecture:
  1. Envelope theorems
  2. 2nd order conditions

Unconstrained Optimization



Constrained Optimization:

$y \geq 1$



Constrained Optimization

Examples:

1. A consumer maximizes his utility subject to his budget constraint.
2. A producer minimizes costs subject to the constraint that a certain amount is produced.
3. Moral hazard: An insurer tries to select an insurance contract that maximizes profits subject to the constraints that it is valuable to the consumer ("Participation Constraint") and that the consumer has an incentive to be careful ("Incentive Constraint"). Basic result: Full insurance is not optimal because it would make consumer act careless.

Constrained Optimization Problem

Objective: Find the (absolute) maximum of the function

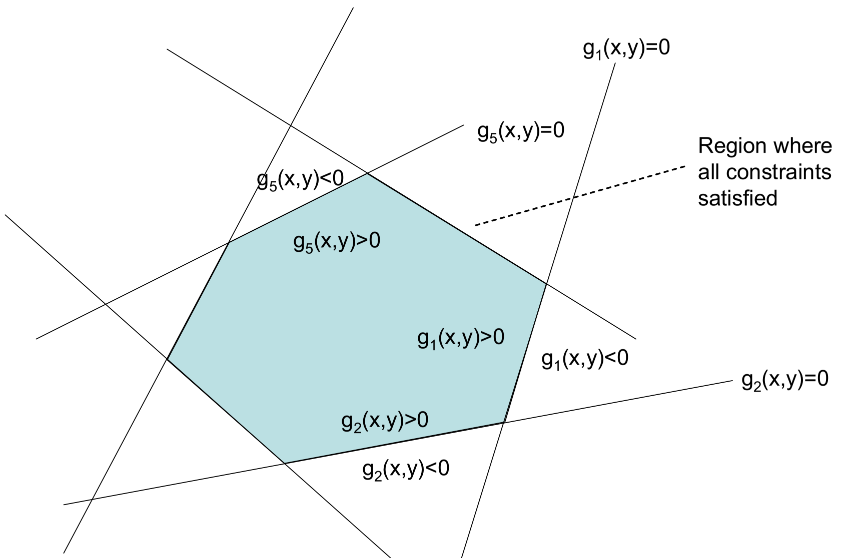
$$z = f(x, y)$$

subject to the inequality constraints

$$\begin{aligned} g_1(x,y) &\geq 0 \\ g_2(x,y) &\geq 0 \\ &\vdots \\ g_K(x,y) &\geq 0 \end{aligned}$$

Thus find pair (x^*, y^*) satisfying the constraints such that we have for all other pairs (x, y) satisfying the constraints: $f(x^*, y^*) \geq f(x, y)$.
$f(x, y)$ is called the "objective function" and I call $g_1(x, y), \ldots, g_K(x, y)$ the "constraining functions" of the problem.

The constraints carve out a region of the plane.



Example 1: Consumer Optimization

A consumer wants to maximize his utility

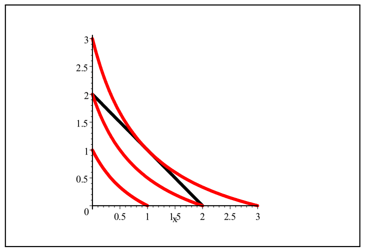
$$u(x, y) = (x + 1)(y + 1)$$

subject to his budget constraint

$$b - p_x x - p_y y \geq 0$$

and the non-negativity constraints

$$x \geq 0 \qquad y \geq 0$$



Example 2: Cost Minimization

A producer with production function $Q(K,L) = K^{\frac{1}{6}} L^{\frac{1}{2}}$ in a perfectly competitive market wants to *minimize* costs subject to producing at least Q_0 units.
Maximize

$$-(rK + wL)$$

subject to

$$\begin{aligned} Q(K,L) - Q_0 &\geq 0 \\ K &\geq 0 \\ L &\geq 0 \end{aligned}$$

Example 3: Shortest Route

A swimmer who is currently at the coordinates (a,b) wants to swim along the shortest route to the square island with corner points $(-1,1)$, $(1,-1)$, $(-1,1)$, $(1,1)$.
Instead of minimizing the distance we can maximize the negative of the square of the distance

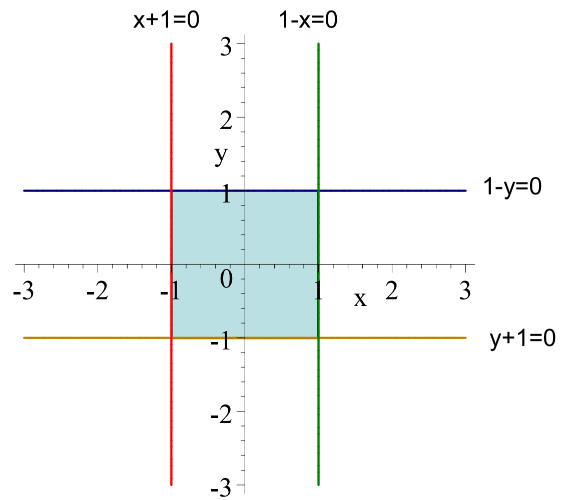
$$-(x-a)^2 - (y-b)^2$$

subject to the constraints

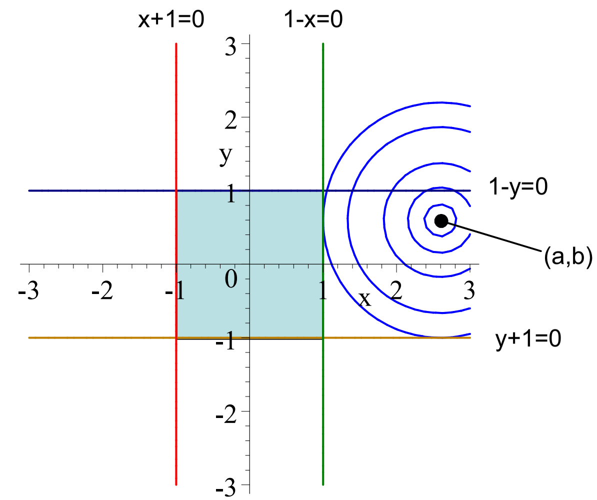
$$\begin{aligned} x+1 &\geq 0 \\ 1-x &\geq 0 \\ y+1 &\geq 0 \\ 1-y &\geq 0 \end{aligned}$$

(x,y) is a point in the square carved out by the solutions to the four inequalities.

The Island

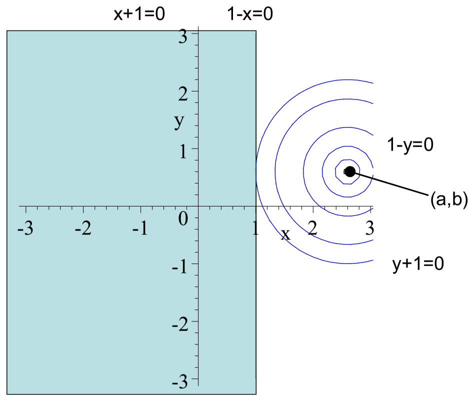


The Swimmer



Binding constraints

A constraint is *binding at the optimum* if it holds with equality in the optimum. In the above picture only one of the four the constraints is binding. All non-binding constraints can be ignored. If they are left out the optimum does not change.



The Lagrangian Approach

The Lagrangian approach transfers a constrained optimization problem into

1. an unconstrained optimization problem and
2. a pricing problem.

The new function to be optimized is called the *Lagrangian.* For each constraint a shadow price is introduced, called a *Lagrange multiplier.*
In the new unconstrained optimization problem a constraint can be violated, but only at a cost.
The pricing problem is to find shadow prices for the constraints such that the solutions to the new and the original optimization problem are identical.

The Lagrangian Approach

The Lagrangian:

$$\mathcal{L}(x,y) = f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y) + \cdots + \lambda_K g_K(x,y)$$

Theorem

Suppose we are given numbers $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$ and a pair of numbers (x^*, y^*) such that

1. $\lambda_1, \lambda_2, \ldots, \lambda_K \geq 0$, *i.e. Lagrange mutlipliers are non-negative,*
2. (x^*, y^*) *satisfies all the constraints, i.e.,* $g_k(x^*, y^*) \geq 0$ *for all* $1 \leq k \leq K$.
3. (x^*, y^*) *is an unconstrained maximum of the Lagrangian* $\mathcal{L}(x,y)$.
4. *The complementarity conditions*

$$\lambda_k g_k(x^*, y^*) = 0$$

are satisfied, i.e., either the k-Lagrange multiplier is zero or the k-th constraint binds for $1 \leq k \leq K$.

Then (x^*, y^*) *is a maximum for the constrained maximization problem.*

The Lagrangian:

$$\mathcal{L}(x,y) = f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y) + \cdots + \lambda_K g_K(x,y)$$

Proof.

Because of the complementarity conditions we have $\mathcal{L}(x^*,y^*) = f(x^*,y^*)$. We have $\mathcal{L}(x,y) \le \mathcal{L}(x^*,y^*)$ for any (x,y). If (x,y) satisfies all constraints then $\lambda_k g_k(x,y) \ge 0$ for each constraint since the λ_k are non-negative. Hence $f(x,y) \le \mathcal{L}(x,y)$ and therefore $f(x,y) \le f(x^*,y^*)$ for any point (x,y) satisfying the constraints. □

The Method

1. Make an informed guess about which constraints are binding at the optimum. (Suppose there are k^* such constraints.)
2. Set the Lagrange multipliers for all other constraints zero, i.e. ignore these constraints.
3. Solve the two first -order conditions $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\frac{\partial \mathcal{L}}{\partial y} = 0$ together with the conditions that the k^* constraints are binding. (Notice that we have $2 + k^*$ constraints and equations, namely x, y and k^* Lagrange multipliers.)
4. Check whether the solution is indeed an unconstrained optimum of the Lagrangian. (May be difficult.)
5. Check that the Lagrange multipliers are all non-negative and that the solution (x^*,y^*) satisfies all constraints.
6. If 4. and 5. are violated, start again at 1. with a new guess.

The Swimmer's Problem

The Lagrangian is

$$\begin{aligned}\mathcal{L}(x,y) &= -(x-a)^2 - (y-b)^2 \\ &\quad + \lambda_1(x+1) + \lambda_2(1-x) + \lambda_3(y+1) + \lambda_4(1-y)\end{aligned}$$

FOC:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= -2(x-a) + \lambda_1 - \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2(y-b) + \lambda_3 - \lambda_4 = 0\end{aligned}$$

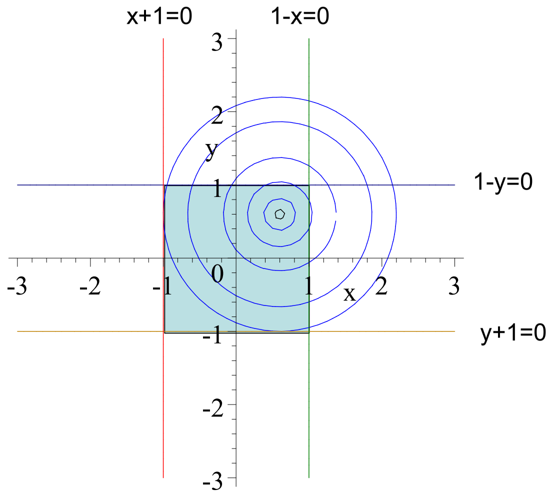
Case 1:

Suppose $-1 < x^* < 1$, $-1 < y^* < 1$.
None of the four constraints is "binding"
In this case the optimum has to be a stationary point of the objective function. This gives the conditions

$$\begin{aligned}\frac{\partial f}{\partial x} &= -2(x-a) = 0 \\ \frac{\partial f}{\partial y} &= -2(y-b) = 0\end{aligned}$$

UNIQUE SOLUTION: $(x^*,y^*) = (a,b)$. Must have: $-1 \le a, b \le 1$ for this to be the optimum.

Case 1



Case 2

Optimum is on the upper side of the square, but not a cornerpoint, i.e. the ONLY binding constraint is $y \leq 1$. All Lagrange multipliers except λ_4 must be zero. So the Lagrangian is

$$\mathcal{L}(x, y) = f(x, y) + \lambda_4 (1 - y)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x} = -2(x - a) = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2(y - b) - \lambda_4 = 0$$

Add that the constraint $y = 1$ is binding, i.e.

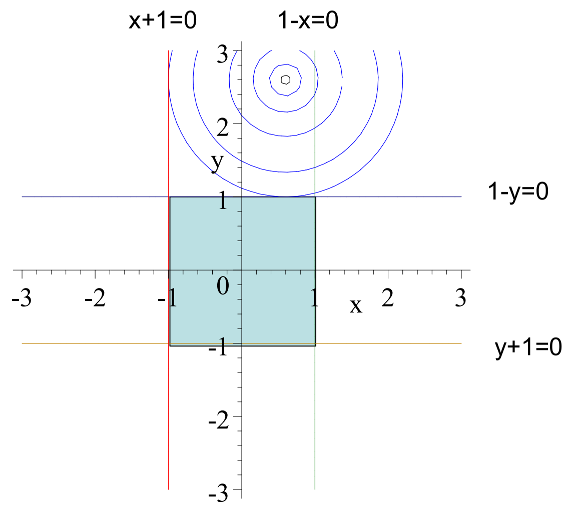
$$y = 1.$$

unique solution $x^* = a, y^* = 1, \lambda_4 = -2(1 - b)$.

Case 2

For our candidate $(x^*, y^*) = (a, 1)$ to satisfy all constraints we must have $-1 \leq a \leq 1$. In addition, $\lambda_4 = 2(b - 1)$ must be nonnegative. True only if $b - 1 \geq 0$ or $b \geq 1$.
Cases 3,4,5: Optimum is on a different side.
Solved symmetrically

Case 2



Case 6

Suppose optimum is cornerpoint $(1,1)$. Then constraints $x \leq 1$ and $y \leq 1$ are binding while $x \geq -1$ and $y \geq -1$ are not. Complementarity conditions imply $\lambda_1 = \lambda_3 = 0$. Lagrangian is

$$\mathcal{L}(x,y) = f(x,y) - \lambda_2 (x-1) - \lambda_4 (y-1)$$

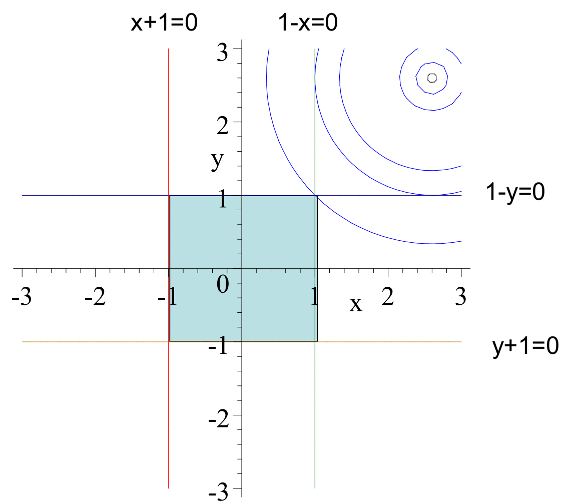
FOC:

$$\frac{\partial \mathcal{L}}{\partial x} = -2(x-a) - \lambda_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2(y-b) - \lambda_4 = 0$$

cornerpoint:

$$x = 1$$
$$y = 1.$$

Case 6



Case 6

unique solution: $(x^*, y^*) = (1,1)$, $\lambda_2 = -2(1-a)$, $\lambda_4 = -2(1-b)$. The Lagrange multipliers are non-negative if

$$a \geq 1 \quad \text{and} \quad b \geq 1.$$

Cases 7, 8, 9: Optimum is another corner point. The following table describes for each pair (a,b) what the optimum (x^*, y^*) is

	$b \leq -1$	$-1 < b < 1$	$1 \leq b$
$a \leq -1$	$(-1,-1)$	$(-1,b)$	$(-1,1)$
$-1 < a < 1$	$(a,-1)$	(a,b)	$(a,1)$
$1 \leq a$	$(1,-1)$	$(1,b)$	$(1,1)$