BEE1024 Mathematics for Economists

Multivariate Functions

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Week 2

4 The Tangent Plane

- The Total Differential
- The slope of level curves
- The Chain Rule

Optimization 1

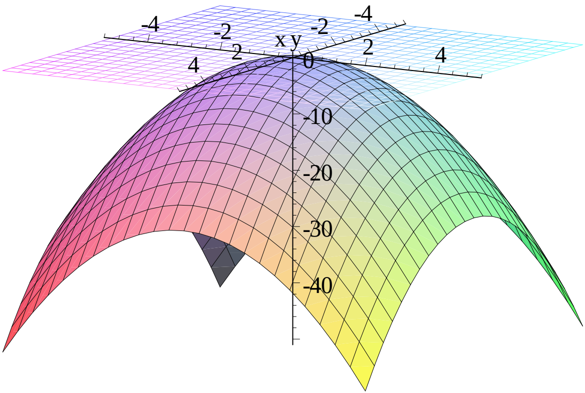
"Since the fabric of the universe is most perfect, and is the work of a most perfect creator, nothing whatsoever takes place in the universe in which some form of maximum or minimum does not appear."

Leonhard Euler, 1744

Objectives

- Subject: Optimization of multivariate functions
- Two basic types of problems:
  1. unconstrained (e.g. profit maximization)
  2. constrained (e.g. utility maximization with budget constraint)
- This week:
  1. "first order conditions" for optimum
  2. Simultaneous system of equations (→Review)
  3. the tangent plane / the marginal rate of substitution
- Next weeks:
  1. Lagrangian approach for constrained problems
  2. 2nd order conditions

Unconstrained Optimization



Unconstrained Optimization

Objective: Find (absolute) maximum of function

$$z = f(x, y)$$

i.e., find a pair (x^*, y^*) such that

$$f(x^*, y^*) \geq f(x, y)$$

holds for all pairs (x, y).

Hereby both pairs of numbers (x^*, y^*) and (x, y) must be in the domain of the function.

For an absolute minimum require $f(x^*, y^*) \leq f(x, y)$.

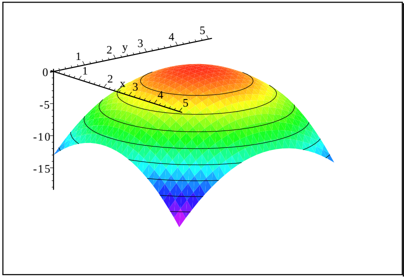
$f(x, y)$ is called the **"objective function"**.

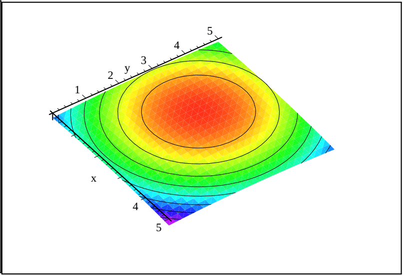
Example

The function

$$z = f(x, y) = -(x-2)^2 - (y-3)^2$$

has a maximum at $(x^*, y^*) = (2, 3)$.





First order conditions

The following must hold: freeze the variable y at the optimal value y^*, vary only x then the function in one variable

$$F(x) = f(x, y^*)$$

must have maximum at x^*: $\frac{dF}{dx}(x^*) = 0$. Thus we obtain the *first order conditions*

$$\frac{\partial z}{\partial x}_{|x=x^*,y=y^*} = 0 \qquad \frac{\partial z}{\partial y}_{|x=x^*,y=y^*} = 0$$

Example 1

$$\begin{aligned} z &= f(x,y) = -(x-2)^2 - (y-3)^2 \\ \frac{\partial z}{\partial x} &= -2(x-2)\times(+1) = 0 \qquad \frac{\partial z}{\partial y} = -2(y-3)\times(+1) = 0 \\ x^* &= 2 \qquad y^* = 3 \end{aligned}$$

The maximum (at least the only *critical or stationary point*) is at $(x^*, y^*) = (2, 3)$

Maximizing profits

Production function $Q(K, L)$.

r interest rate

w wage rate

P price of output

profit

$$\Pi(K, L) = PQ(K, L) - rK - wL.$$

FOC for profit maximum:

$$\frac{\partial \Pi}{\partial K} = P\frac{\partial Q}{\partial K} - r = 0 \quad (1)$$

$$\frac{\partial \Pi}{\partial L} = P\frac{\partial Q}{\partial L} - w = 0 \quad (2)$$

Intuition: Suppose $P\frac{\partial Q}{\partial K} - r > 0$. By using one unit of capital more the firm could produce $\frac{\partial Q}{\partial K}$ units of output more. The revenue would increase by $P\frac{\partial Q}{\partial K}$, the cost by r and so profit would increase. Thus we cannot have a profit optimum. If $P\frac{\partial Q}{\partial K} - r < 0$ it would symmetrically pay to reduce capital input. Hence $P\frac{\partial Q}{\partial K} - r = 0$ must hold in optimum.

Rewrite FOC as

$$\frac{\partial Q}{\partial K} = \frac{r}{P} \quad (3)$$

$$\frac{\partial Q}{\partial L} = \frac{w}{P} \quad (4)$$

Division yields:

$$MRS = -\frac{dL}{dK} = \frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L} = \frac{r}{P} \Big/ \frac{w}{P} = \frac{r}{w} \quad (5)$$

The *marginal rate of substitution* must equal the ratio of the input prices!

Why? If the firm uses one unit of capital less and $\frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L}$ units of labour more, output remains (approximately) the same. The firm would save r on capital and spend $\frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L} \times w$ on more labour while still making the same revenue. Profit would increase unless $r \leq \frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L} \times w$. As symmetric argument interchanging the role of capital and labour shows that $r \geq \frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L} \times w$ must hold if the firm optimizes profits.

Example

$$Q(K,L) = K^{\frac{1}{6}} L^{\frac{1}{2}}$$

$$\frac{\partial Q}{\partial K} = \frac{1}{6} K^{\frac{1}{6}-1} L^{\frac{1}{2}} \qquad \frac{\partial Q}{\partial L} = \frac{1}{2} K^{\frac{1}{6}} L^{\frac{1}{2}-1}$$

$$\frac{\partial Q}{\partial K} = \frac{1}{6} K^{-\frac{5}{6}} L^{\frac{1}{2}} \qquad \frac{\partial Q}{\partial L} = \frac{1}{2} K^{\frac{1}{6}} L^{-\frac{1}{2}}$$

FOC:

$$\frac{1}{6} K^{-\frac{5}{6}} L^{\frac{1}{2}} = \frac{r}{P} \qquad \frac{1}{2} K^{\frac{1}{6}} L^{-\frac{1}{2}} = \frac{w}{P}$$

$$\frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L} = \frac{1}{3} K^{-\frac{5}{6}-\frac{1}{6}} L^{\frac{1}{2}-\left(-\frac{1}{2}\right)} = \frac{r}{w}$$

$$\frac{\partial Q}{\partial K} \Big/ \frac{\partial Q}{\partial L} = \frac{1}{3} \frac{L}{K} = \frac{r}{w}$$

Example

$P = 12$, $r = 1$ and $w = 3$; FOC:

$$\frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}} = \frac{1}{12} \qquad \frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}} = \frac{3}{12}$$

$$2K^{-\frac{5}{6}}L^{\frac{1}{2}} = 1 \ (*) \qquad 2K^{\frac{1}{6}}L^{-\frac{1}{2}} = 1$$

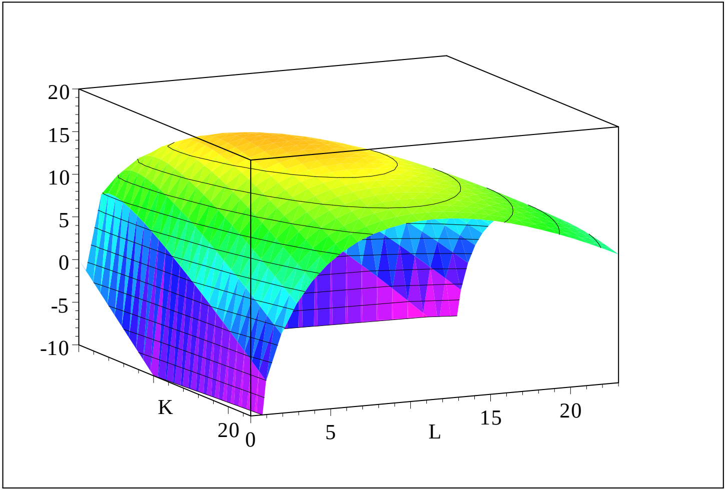
$$\frac{1}{3}\frac{L}{K} = \frac{1}{3} \Longrightarrow K = L$$

Substituting $L = K$ into (*) we get

$$2K^{-\frac{5}{6}}K^{\frac{1}{2}} = 2K^{-\frac{2}{6}} = 2K^{-\frac{1}{3}} = 1 \Longrightarrow 2 = K^{\frac{1}{3}} \Longrightarrow K^* = L^* = 2^3 = 8$$

$$Q^* = K^{\frac{1}{6}}L^{\frac{1}{2}} = 8^{\frac{1}{6}+\frac{1}{2}} = 8^{\frac{4}{6}} = 8^{\frac{2}{3}} = 2^2 = 4$$

$$\Pi^* = 12Q^* - 1K^* - 3L^* = 48 - 8 - 24 = 16$$


20
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Example 3: Price Discrimination

A monopolist with total cost function $TC(Q) = Q^2$ sells his product in two different countries. When he sells Q_A units of the good in country A he will obtain the price

$$P_A = 22 - 3Q_A$$

for each unit. When he sells Q_B units of the good in country B he obtains the price

$$P_B = 34 - 4Q_B.$$

How much should the monopolist sell in the two countries in order to maximize profits?

Solution

Total revenue in country A:

$$TR_A = P_A Q_A = (22 - 3Q_A)\, Q_A$$

Total revenue in country B:

$$TR_B = P_B Q_B = (34 - 4Q_B)\, Q_B$$

Total production costs are:

$$TC = (Q_A + Q_B)^2$$

Profit:

$$\Pi\,(Q_A, Q_B) = (22 - 3Q_A)\, Q_A + (34 - 4Q_B)\, Q_B - (Q_A + Q_B)^2$$

Profit:

$$\Pi(Q_A, Q_B) = (22 - 3Q_A)\,Q_A + (34 - 4Q_B)\,Q_B - (Q_A + Q_B)^2$$

FOC:

$$\begin{aligned}\frac{\partial \Pi}{\partial Q_A} &= -3Q_A + (22 - 3Q_A) - 2(Q_A + Q_B) = 22 - 8Q_A - 2Q_B = 0\\ \frac{\partial \Pi}{\partial Q_B} &= -4Q_B + (34 - 4Q_B) - 2(Q_A + Q_B) = 34 - 2Q_A - 10Q_B\\ &= 0\end{aligned}$$

or

$$\begin{aligned}8Q_A + 2Q_B &= 22 \qquad (6)\\ 2Q_A + 10Q_B &= 34.\end{aligned}$$

linear simultaneous system

Review: Simultaneous systems of equations

Methods:

1. Using the slope-intercept form
2. The method of substitution
3. The method of elimination
4. Cramer's rule

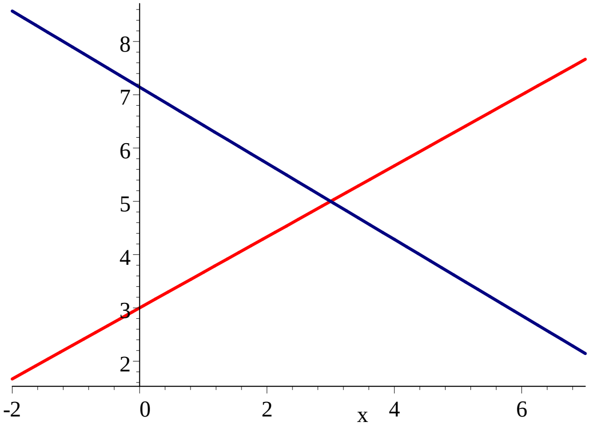
Using the slope-intercept form

Example:

$$\begin{aligned} 5x + 7y &= 50 \\ 4x - 6y &= -18 \end{aligned} \tag{7}$$

slope-intercept form:

$$\begin{aligned} 7y &= 50 - 5x \\ y &= \frac{50}{7} - \frac{5}{7}x \\ 4x - 6y &= -18 \\ 4x + 18 &= 6y \\ y &= \frac{2}{3}x + 3 \end{aligned}$$


8
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2
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6
x

At intersection point two linear functions must have same y-value. Hence

$$\begin{aligned}
\frac{50}{7} - \frac{5}{7}x &= y = \frac{2}{3}x + 3 \\
\frac{50}{7} - 3 &= \frac{2}{3}x + \frac{5}{7}x \qquad \mid \times 3 \times 7 \\
150 - 63 &= 14x + 15x \\
87 &= 29x \\
x &= \frac{87}{29} = 3
\end{aligned}$$

Found value of x. Calculate the value of y:

$$y = \frac{2}{3} \times 3 + 3 = 5$$

$$\begin{aligned} 5x + 7y &= 50 \\ 4x - 6y &= -18 \end{aligned}$$

solution: $x^* = 3$, $y^* = 5$
It is strongly recommended to check, in particular in exams:

$$\begin{aligned} 5 \times 3 + 7 \times 5 &= 15 + 35 = 50 \\ 4 \times 3 - 6 \times 5 &= 12 - 30 = -18 \end{aligned}$$

The method of substitution

Solve one of the equations for one of the variables:

$$\begin{aligned} 4x - 6y &= -18 \\ 4x + 18 &= 6y \\ y &= \frac{4}{6}x + 3 = \frac{2}{3}x + 3 \end{aligned} \tag{8}$$

Replace y in the other equation, obtain equation in one variable. Do not forget to put brackets around the expression which replaces y!

$$
\begin{aligned}
5x + 7y &= 50 \\
5x + 7\left(\frac{2}{3}x + 3\right) &= 50 \\
5x + \frac{14}{3}x + 21 &= 50 \\
\frac{15}{3}x + \frac{14}{3}x + 21 &= 50 \\
\frac{29}{3}x &= 50 - 21 = 29 \\
x &= \frac{3}{29} \times 29 = 3
\end{aligned}
$$

Use (8) to find y:

$$
y = \frac{2}{3}x + 3 = \frac{2}{3} \times 3 + 3 = 5
$$

The method of elimination

Multiply first equation by coefficient of x in second equation.
Multiply second equation by the coefficient of x in first equation.

$$\begin{aligned} 5x + 7y &= 50 \quad &| \times 4 \\ 4x - 6y &= -18 \quad &| \times 5 \\ 20x + 28y &= 200 \\ 20x - 30y &= -90 \end{aligned}$$

Subtract second equation from first equation

$$
\begin{aligned}
&20x + 28y = 200 \\
&20x - 30y = -90 \qquad |- \\
&0 + 28y - (-30y) = 200 - (-90) \\
&58y = 290 \\
&y = \tfrac{290}{58} = 5
\end{aligned}
$$

Use one of the original equations to find x

$$5x + 7y = 50 \qquad 5x + 35 = 50 \qquad 5x = 15 \qquad x = 3$$

Cramer's Rule

$\rightarrow$ linear algebra. Write system of equations as

$$\begin{bmatrix} 5 & 7 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ -18 \end{bmatrix}$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 50 \\ -18 \end{bmatrix}$ are "column vectors"

$$\begin{bmatrix} 5 & 7 \\ 4 & -6 \end{bmatrix}$$

is the 2×2–"matrix of coefficients".

The determinant

With each 2×2-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

associate a new *number* called the *determinant*

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

vertical lines, not square brackets!!!

For instance,

$$\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix} = 5 \times (-6) - 4 \times 7 = -30 - 28 = -58$$

Cramer's rule

The linear system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

has the solution

$$x^* = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc} \qquad y^* = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$

$$\begin{bmatrix} 5 & 7 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ -18 \end{bmatrix}$$

$$x^* = \frac{\begin{vmatrix} 50 & 7 \\ -18 & -6 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix}} = \frac{50 \times (-6) - (-18) \times 7}{5 \times (-6) - 4 \times 7} = \frac{-300 + 126}{-30 - 28} = \frac{-1}{-5}$$

$$y^* = \frac{\begin{vmatrix} 5 & 50 \\ 4 & -18 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix}} = \frac{5 \times (-18) - 4 \times 50}{5 \times (-6) - 4 \times 7} = \frac{-90 - 200}{-58} = \frac{-290}{-58} = 5$$

$$\begin{bmatrix} 5 & 7 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ -18 \end{bmatrix}$$

$$x^* = \frac{\begin{vmatrix} 50 & 7 \\ -18 & -6 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix}} = \frac{-300+126}{-30-28} = \frac{-174}{-58} = 3$$

$$y^* = \frac{\begin{vmatrix} 5 & 50 \\ 4 & -18 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix}} = \frac{-90-200}{-58} = \frac{-290}{-58} = 5$$

Exercise

$$\begin{aligned} 8Q_A + 2Q_B &= 22 \\ 2Q_A + 10Q_B &= 34. \end{aligned}$$

Existence and uniqueness

linear simultaneous system of equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

Rewrite as

$$\begin{aligned} y &= \frac{e}{b} - \frac{a}{b}x \\ y &= \frac{f}{d} - \frac{c}{d}x \end{aligned}$$

Slopes are identical when $\frac{a}{b} = \frac{c}{d}$, i.e., when the determinant $ad - cb$ is zero. If in addition intercepts are equal, both equations describe the same line.

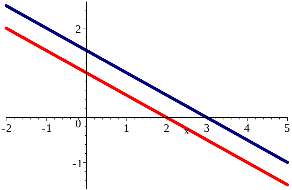
Example

$$x + 2y = 3 \qquad 2x + 4y = 4$$

has no solution: The two lines

$$y = \frac{3}{2} - \frac{1}{2}x \qquad y = 1 - \frac{1}{2}x$$

are parallel



There is no common solution. Trying to find one yields a contradiction

$$\begin{aligned}\frac{3}{2}-\frac{1}{2}x &= y = 1-\frac{1}{2}x \qquad \Big| +\frac{1}{2}x \\ \frac{3}{2} &= 1\end{aligned}$$

Summary: The determinant is non-zero if and only if the system has a unique solution. If the determinant is zero, there are either zero or infinitely many solutions.

One equation non-linear, one linear

Consider, for instance,

$$y^2 + x - 1 = 0 \qquad y + \frac{1}{2}x = 1$$

In our example it is convenient to solve second equation for x:

$$\begin{aligned} \frac{1}{2}x &= 1 - y \\ x &= 2 - 2y \\ y^2 + (2-2y) - 1 &= y^2 - 2y + 1 = (y-1)^2 = 0 \end{aligned}$$

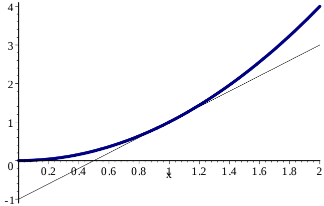
So the unique solution is $y^* = 1$ and $x^* = 2 - 2 \times 1 = 0$.

Two non-linear equations

No general method.

The Tangent Plane

For a function $y = f(x)$ in one variable the derivative $\frac{df}{dx}_{|x=x_0}$ at x_0 is the slope of the tangent to the graph of f through the point $(x_0, f(x_0))$.



The tangent of $f(x) = x^2$ at $x = 1$.

Proposition: This tangent is the graph of the linear function

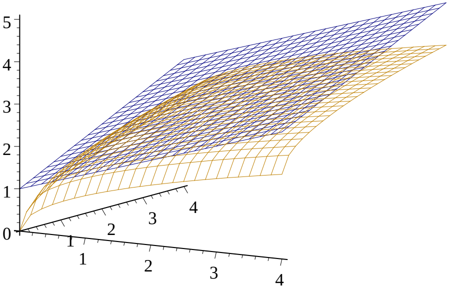
$$y = l(x) = f(x_0) + \frac{df}{dx}_{|x=x_0}(x - x_0).$$

Proof: This is so because $l(x_0) = f(x_0)$, so the graph of l goes through the point $(x_0, f(x_0))$. Moreover,

$$y = l(x) = \left(f(x_0) - \frac{df}{dx_0}_{|x=x_0} x_0 \right) + \frac{df}{dx}_{|x=x_0} x,$$

so $l(x)$ is indeed a linear function in x with intercept $\left(f(x_0) - \frac{df}{dx_0}_{|x=x_0} x_0 \right)$ and the "right" slope $\frac{df}{dx}_{|x=x_0}$.

The graph of a function in one variable is a curve and its tangent is a line. By analogy, the graph of a function $z = f(x, y)$ in two variables is a surface and its tangent at a point (x_0, y_0) is a *plane*.



Non-vertical planes are the graphs of linear functions $z = ax + by + c$. The tangent plane of the function $z = f(x, y)$ in (x_0, y_0) is, correspondingly, the graph of the linear function

$$z = l(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}_{|x=x_0, y=y_0}(x - x_0) + \frac{\partial f}{\partial y}_{|x=x_0, y=y_0}(y - y_0) \tag{9}$$

since the graph of this linear function contains the point $(x_0, y_0, f(x_0, y_0))$ and has the right slopes in the x- and y-directions.

Exercise: This is the example shown in the above graph. The tangent plane of $z = f(x, y) = \sqrt{x} + \sqrt{y}$ at the point $(1, 1)$ is obtained as follows: We have

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x}} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{y}}$$

and hence

$$\frac{\partial z}{\partial x}_{|x=1,y=1} = \frac{1}{2} \qquad \text{and} \qquad \frac{\partial z}{\partial y}_{|x=1,y=1} = \frac{1}{2}.$$

Since $f(1, 1) = 2$ the tangent plane is the graph of the linear function

$$z = l(x, y) = 2 + \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) = 1 + \frac{1}{2}x + \frac{1}{2}y$$

The Total Differential

Using the *abbreviations* $dx = x - x_0$, $dy = y - y_0$ and $dz = z - z_0$ the formula for the tangent can be neatly rewritten as

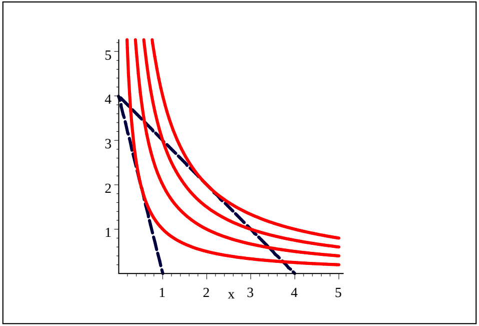
$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{10}$$

an expression which is called the **total differential**.

The slope of level curves

The marginal rate of substitution

$$xy = c$$



Suppose (x_0, y_0) is a point on the level curve $f(x, y) = c$. Then the slope of tangent to the level curve in this point in the x-y-plane is

$$-MRS = \frac{dy}{dx}_{|x=x_0,y=y_0} = -\left.\frac{\partial z}{\partial x}_{|x=x_0,y=y_0}\right/ \frac{\partial z}{\partial y}_{|x=x_0,y=y_0} \qquad (11)$$

provided $\frac{\partial z}{\partial y}_{|x=x_0,y=y_0} \neq 0$. This is known as the *rule for implicit differentiation*: If the function $y = y(x)$ is implicitly defined by the equation $f(x, y(x)) = c$, then its derivative $\frac{dy}{dx}$ is given by the above formula.

Remark: A quick way to memorize the formula (11) is to use the total differential as follows: In order to stay on the level curve one must have $dz = z - z_0 = 0$, so

$$0 = dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

hence

$$dy = -\left(\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y}\right)dx$$

or

$$\frac{dy}{dx} = -\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y}.$$

Example: A linear function

$$z = ax + by + c$$

has the partial derivatives $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$. According to the above formula its level curves have the slope

$$-\frac{\partial z}{\partial x} \Big/ \frac{\partial z}{\partial y} = -\frac{a}{b}.$$

Indeed, its level curves are the solutions to the equations

$$l = ax + by + c$$

with some constant l. Solving the equation for y we obtain

$$y = -\frac{a}{b}x + \frac{l - c}{b}$$

which is a linear function with slope $-\frac{a}{b}$.

Example: An isoquant of the production function $Q = K^{\frac{1}{6}} L^{\frac{1}{2}}$ has according to the above formula the slope

$$\begin{aligned} \frac{dK}{dL} &= -\frac{\partial Q}{\partial L} \Big/ \frac{\partial Q}{\partial K} = -\left(\frac{1}{2} K^{\frac{1}{6}} L^{-\frac{1}{2}}\right) \Big/ \left(\frac{1}{6} K^{-\frac{5}{6}} L^{\frac{1}{2}}\right) \\ &= -\frac{6}{2} K^{\frac{1}{6}} K^{\frac{5}{6}} L^{-\frac{1}{2}} L^{-\frac{1}{2}} = -3\frac{K}{L}. \end{aligned}$$

in the point (K, L).
Indeed, an isoquant is the set of solutions to the equation $\bar{q} = K^{\frac{1}{6}} L^{\frac{1}{2}}$ for fixed $\bar{q}$. Solving for K we see that the isoquant is the graph of the function

$$K = \left(\bar{q} L^{-\frac{1}{2}}\right)^6 = \bar{q}^6 L^{-3}.$$

Differentiation yields

$$\frac{dK}{dL} = -3\bar{q}^{6} L^{-4}$$

and since (K, L) is on the isoquant

$$\frac{dK}{dL} = -3\left(K^{\frac{1}{6}} L^{\frac{1}{2}}\right)^{6} L^{-4} = -3\frac{K}{L}.$$

Economically, $-\frac{dK}{dL}$ is the **marginal rate of substitution** of labour for capital: If labour is reduced by one unit, how much more capital is (approximately) needed to produce the same output? Just reducing labour by one unit reduces output by the marginal product of labour $\frac{\partial q}{\partial L}$. Each additional unit of capital produces $\frac{\partial q}{\partial K}$ – the marginal product of capital – more units. Therefore, if now capital input is increased by $\frac{\partial q}{\partial L} \Big/ \frac{\partial q}{\partial K}$ units, output changes overall by

$$-\frac{\partial q}{\partial L} + \left(\frac{\partial q}{\partial L} \Big/ \frac{\partial q}{\partial K} \right) \frac{\partial q}{\partial K} = 0.$$

Remark: The *implicit function theorem* states that if $\frac{\partial z}{\partial y}_{|x=x_0,y=y_0} \neq 0$ for some point (x_0, y_0) on a level curve $f(x, y) = c$, then one can find a function $y = g(x)$ defined near $x = x_0$ such that the level curve is locally the graph of this function, i.e. one has $f(x, g(x)) = c$ for x near x_0.

Exercise: For the production function

$$Q = \sqrt{KL}$$

calculate the marginal rate of substitution for $K = 3$ and $L = 4$.

The Chain rule

Suppose that $z = f(x, y)$ is a function in two variables x, y. Suppose that x depends on the variable t via $x = g(t)$ and that y depends on the variable t via $y = h(t)$. Then we can define the *composite* function in one variable $z = F(t) = f(x(t), y(t))$ which has t as the independent and z as the dependent variable. The derivative of this function, if it exists, is calculated as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Notice that this is just the total differential divided by dt.

Example: Suppose $z = xy$, $x = t^3 + 2t$, $y = t^2 - t$. Then, according to the chain rule,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = y \times \left(3t^2 + 2\right) + x \times (2t - 1) \\ &= \left(t^2 - t\right)\left(3t^2 + 2\right) + (2t - 1)\left(t^3 + 2t\right)\end{aligned}$$

Since $z = xy = \left(t^3 + 2t\right)\left(t^2 - t\right)$ we get the same result using the product rule:

$$\frac{dz}{dt} = \left(3t^2 + 2\right)\left(t^2 - t\right) + \left(t^3 + 2t\right)(2t - 1).$$