BEE1024 Mathematics for Economists

Multivariate Functions

Juliette Stephenson and Amr (Miro) Algarhi
Author: Dieter Balkenborg

Department of Economics, University of Exeter

Week 1

Objectives for the week

- *Functions in two independent variables.*

The lecture should enable you for instance to calculate the marginal product of labour.

Objectives for the week

- *Functions in two independent variables.*
- *Level curves* $\longleftrightarrow$ indifference curves or isoquants

The lecture should enable you for instance to calculate the marginal product of labour.

Objectives for the week

- *Functions in two independent variables.*
- *Level curves* $\longleftrightarrow$ indifference curves or isoquants
- *Partial differentiation* $\longleftrightarrow$ partial analysis in economics

The lecture should enable you for instance to calculate the marginal product of labour.

Functions in two variables

A *function*

$$z = f(x, y)$$

or simply

$$z(x, y)$$

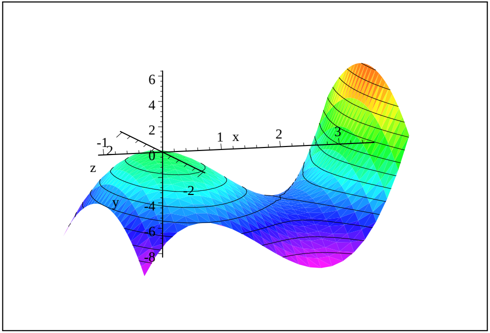
in two independent variables with one dependent variable assigns to each pair (x, y) of (decimal) numbers from a certain domain D in the two-dimensional plane a number $z = f(x, y)$.
x and y are hereby the *independent variables*
z is the *dependent variable*.

Outline

Example: The cubic polynomial

The *graph* of f is the surface in 3-dimensional space consisting of all points $(x, y, f(x, y))$ with (x, y) in D.

$$z = f(x, y) = x^3 - 3x^2 - y^2$$



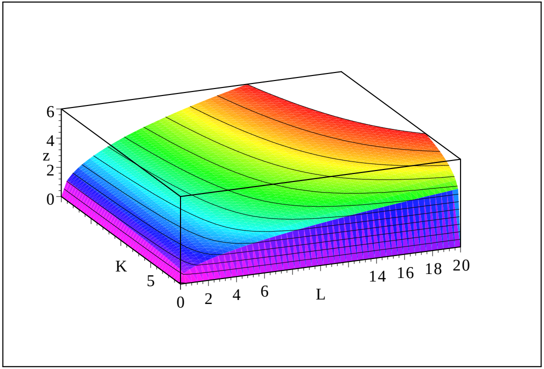
Exercise: Evaluate $z = f(2, 1)$, $z = f(3, 0)$, $z = f(4, -4)$, $z = f(4, 4)$

Outline

Example: production function

$$Q = \sqrt[6]{K}\sqrt{L} = K^{\frac{1}{6}} L^{\frac{1}{2}}$$

capital $K \geq 0$, labour $L \geq 0$, output $Q \geq 0$



Outline

Example: profit function

Assume that the firm is a price taker in the product market and in both factor markets.

- P is the price of output

Example: profit function

Assume that the firm is a price taker in the product market and in both factor markets.

- P is the price of output
- r the interest rate (= the price of capital)

Example: profit function

Assume that the firm is a price taker in the product market and in both factor markets.

- P is the price of output
- r the interest rate (= the price of capital)
- w the wage rate (= the price of labour)

Example: profit function

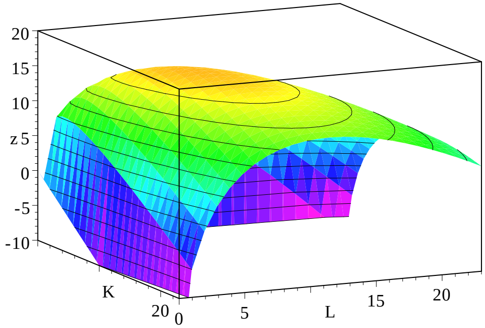
Assume that the firm is a price taker in the product market and in both factor markets.

- P is the price of output
- r the interest rate (= the price of capital)
- w the wage rate (= the price of labour)
- total profit of this firm:

$$\begin{aligned} \Pi(K, L) &= TR - TC \\ &= PQ - rK - wL \\ &= PK^{\frac{1}{6}}L^{\frac{1}{2}} - rK - wL \end{aligned}$$

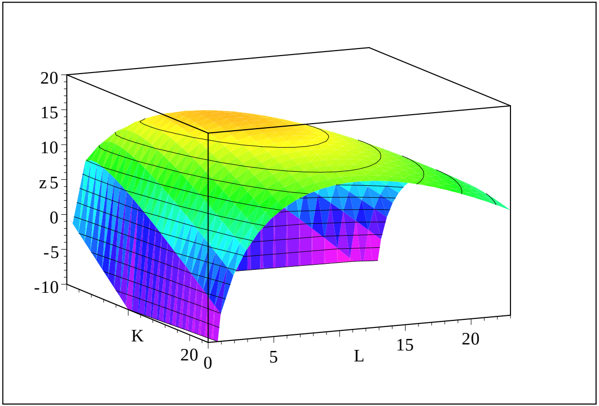
- $P = 12$, $r = 1$, $w = 3$:

$$\begin{aligned} \Pi(K, L) &= PK^{\frac{1}{6}}L^{\frac{1}{2}} - rK - wL \\ &= 12K^{\frac{1}{6}}L^{\frac{1}{2}} - K - 3L \end{aligned}$$



- $P = 12$, $r = 1$, $w = 3$:

$$\Pi(K, L) = PK^{\frac{1}{6}}L^{\frac{1}{2}} - rK - wL = 12K^{\frac{1}{6}}L^{\frac{1}{2}} - K - 3L$$



- Profits is maximized at $K = L = 8$.

Outline

Level Curves

- The *level curve* of the function $z = f(x, y)$ for the level c is the solution set to the equation

$$f(x, y) = c$$

where c is a given constant.

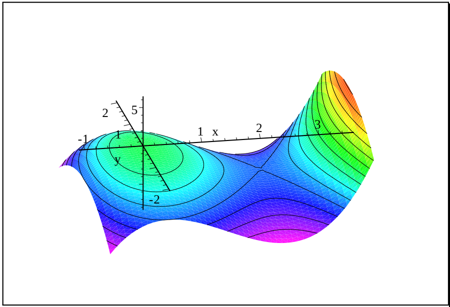
Level Curves

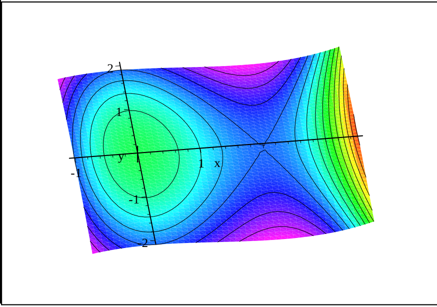
- The *level curve* of the function $z = f(x, y)$ for the level c is the solution set to the equation

$$f(x, y) = c$$

where c is a given constant.

- Geometrically, a level curve is obtained by intersecting the graph of f with a horizontal plane $z = c$ and then projecting into the (x, y)-plane. This is illustrated on the next page for the cubic polynomial discussed above:

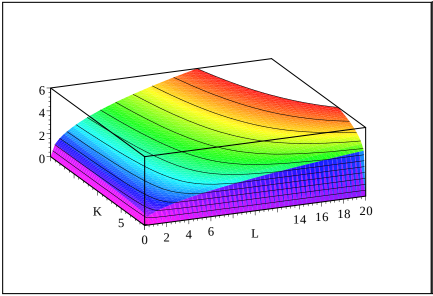


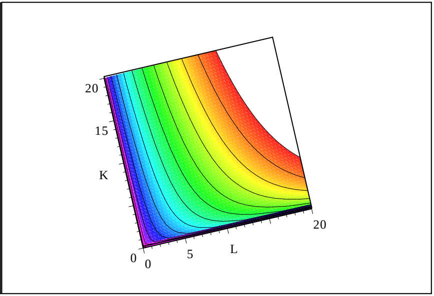


compare: topographic map

Isoquants

In the case of a production function the level curves are called *isoquants*. An isoquant shows for a given output level capital-labour combinations which yield the same output.

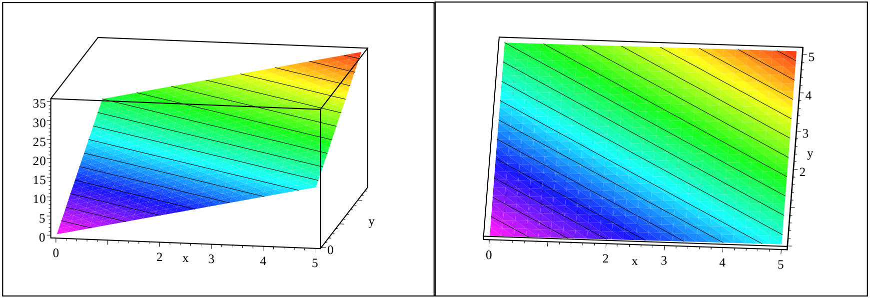




Finally, the linear function

$$z = 3x + 4y$$

has the graph and the level curves:



The level curves of a linear function form a family of parallel lines:

$$c = 3x + 4y \qquad 4y = c - 3x \qquad y = \frac{c}{4} - \frac{3}{4}x$$

slope $-\frac{3}{4}$, variable intercept $\frac{c}{4}$.

Exercise: Describe the isoquant of the production function

$$Q = KL$$

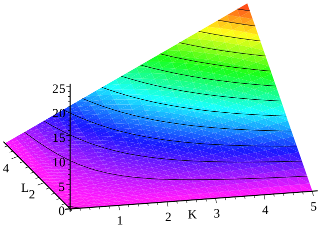
for the quantity $Q = 4$.

Exercise: Describe the isoquant of the production function

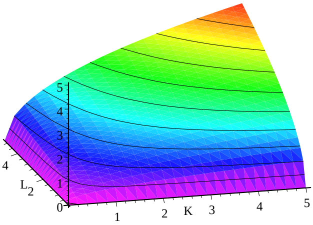
$$Q = \sqrt{KL}$$

for the quantity $Q = 2$.

Remark: The exercises illustrate the following general principle: If $h(z)$ is an increasing (or decreasing) function in one variable, then the composite function $h(f(x, y))$ has the same level curves as the given function $f(x, y)$. However, they correspond to different levels.



$Q = KL$



$Q = \sqrt{KL}$

Objectives for the week

- *Functions in two independent variables.*

The lecture should enable you for instance to calculate the marginal product of labour.

Objectives for the week

- *Functions in two independent variables*.
- *Level curves* $\longleftrightarrow$ indifference curves or isoquants

The lecture should enable you for instance to calculate the marginal product of labour.

Objectives for the week

- *Functions in two independent variables.*
- *Level curves* $\longleftrightarrow$ indifference curves or isoquants
- *Partial differentiation* $\longleftrightarrow$ partial analysis in economics

The lecture should enable you for instance to calculate the marginal product of labour.

Outline

Partial Derivatives: A Basic Example

Exercise: What is the derivative of

$$z(x) = a^3 x^2$$

with respect to x when a is a given constant?

Exercise: What is the derivative of

$$z(y) = y^3 b^2$$

with respect to y when b is a given constant?

Outline

Partial derivatives: Notation

Consider function $z = f(x, y)$. Fix $y = y_0$, vary only x:
$z = F(x) = f(x, y_0)$.

- **The derivative of this function $F(x)$ at $x = x_0$ is called the partial derivative of f with respect to x and denoted by**

$$\frac{\partial z}{\partial x}_{|x=x_0, y=y_0} = \frac{dF}{dx}_{|x=x_0} = \frac{dF}{dx}(x_0)$$

Partial derivatives: Notation

Consider function $z = f(x, y)$. Fix $y = y_0$, vary only x:
$z = F(x) = f(x, y_0)$.

- **The derivative of this function $F(x)$ at $x = x_0$ is called the partial derivative of f with respect to x and denoted by**

$$\frac{\partial z}{\partial x}_{|x=x_0, y=y_0} = \frac{dF}{dx}_{|x=x_0} = \frac{dF}{dx}(x_0)$$

- **Notation: "d"="dee", "δ"="delta", "∂"="del"**

Partial derivatives: Notation

Consider function $z = f(x, y)$. Fix $y = y_0$, vary only x:
$z = F(x) = f(x, y_0)$.

- **The derivative of this function $F(x)$ at $x = x_0$ is called the partial derivative of f with respect to x and denoted by**

$$\frac{\partial z}{\partial x}_{|x=x_0, y=y_0} = \frac{dF}{dx}_{|x=x_0} = \frac{dF}{dx}(x_0)$$

- **Notation: "d"="dee", "δ"="delta", "∂"="del"**
- It suffices to *think* of y and all expressions containing only y as exogenously fixed constants. We can then use the familiar rules for differentiating functions in one variable in order to obtain $\frac{\partial z}{\partial x}$.

Partial derivatives: Notation

Consider function $z = f(x, y)$. Fix $y = y_0$, vary only x:
$z = F(x) = f(x, y_0)$.

- **The derivative of this function $F(x)$ at $x = x_0$ is called the partial derivative of f with respect to x and denoted by**

$$\frac{\partial z}{\partial x}_{|x=x_0, y=y_0} = \frac{dF}{dx}_{|x=x_0} = \frac{dF}{dx}(x_0)$$

- **Notation: "d" = "dee", "δ" = "delta", "∂" = "del"**
- It suffices to *think* of y and all expressions containing only y as exogenously fixed constants. We can then use the familiar rules for differentiating functions in one variable in order to obtain $\frac{\partial z}{\partial x}$.
- Other common notations for partial derivatives are $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ or f_x, f_y or f'_x, f'_y.

Partial derivatives: The example continued

Example: Let

$$z(x,y) = y^3x^2$$

Then

$$\frac{\partial z}{\partial x} = 2y^3x \qquad \frac{\partial z}{\partial y} = 3y^2x^2$$

Outline

Partial derivatives: A second example

Example: Let

$$z = x^3 + x^2y^2 + y^4.$$

Setting e.g. $y = 1$ we obtain $z = x^3 + x^2 + 1$ and hence

$$\begin{aligned} \frac{\partial z}{\partial x}_{|y=1} &= 3x^2 + 2x \\ \frac{\partial z}{\partial x}_{|x=1,y=1} &= 5 \end{aligned}$$

Partial derivatives: A second example

For fixed, but arbitrary, y we obtain

$$\frac{\partial z}{\partial x} = 3x^2 + 2xy^2$$

as follows: We can differentiate the sum $x^3 + x^2y^2 + y^4$ with respect to x term-by-term. Differentiating x^3 yields $3x^2$, differentiating x^2y^2 yields $2xy^2$ because we think now of y^2 as a constant and $\frac{d\left(ax^2\right)}{dx} = 2ax$ holds for any constant a. Finally, the derivative of any constant term is zero, so the derivative of y^4 with respect to x is zero.

Similarly considering x as fixed and y variable we obtain

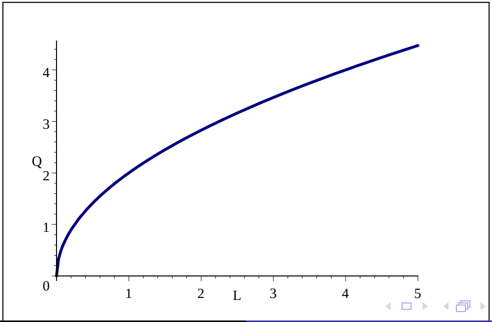
$$\frac{\partial z}{\partial y} = 2x^2y + 4y^3$$

Outline

The Marginal Products of Labour and Capital

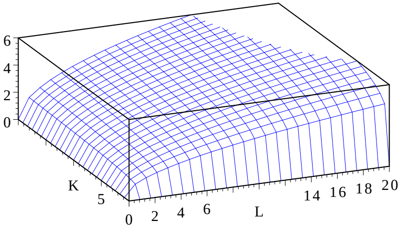
Example: The partial derivatives $\frac{\partial q}{\partial K}$ and $\frac{\partial q}{\partial L}$ of a production function $q = f(K, L)$ are called the **marginal product of capital** and (respectively) **labour.** They describe approximately by how much output increases if the input of capital (respectively labour) is increased by a small unit.

Fix $K = 64$, then $q = K^{\frac{1}{6}} L^{\frac{1}{2}} = 2L^{\frac{1}{2}}$ which has the graph



The Marginal Products

This graph is obtained from the graph of the function in two variables by intersecting the latter with a vertical plane parallel to L-q-axes.



The partial derivatives $\frac{\partial q}{\partial K}$ and $\frac{\partial q}{\partial L}$ describe geometrically the slope of the function in the K- and, respectively, the L- direction.

Diminishing productivity of labour:

The more labour is used, the less is the increase in output when one more unit of labour is employed. Algebraically:

$$\frac{\partial q}{\partial L} = \frac{1}{2} K^{\frac{1}{6}} L^{-\frac{1}{2}} = \frac{1}{2} \frac{\sqrt[6]{K}}{\sqrt[2]{L}} > 0,$$

$$\frac{\partial^2 q}{\partial L^2} = \frac{\partial}{\partial L}\left(\frac{\partial q}{\partial L}\right) = -\frac{1}{4} K^{\frac{1}{6}} L^{-\frac{3}{2}} = -\frac{1}{4} \frac{\sqrt[6]{K}}{\sqrt[2]{L^3}} < 0,$$

Exercise: Find the partial derivatives of

$$z = \left(x^2 + 2x\right)\left(y^3 - y^2\right) + 10x + 3y$$