BEE1020 – Basic Mathematical Economics	Juliette Stephenson Amr Algarhi
Homework -Solution **Week 6**	Department of Economics University of Exeter

Problem 1 *Bank A offers the interest rate of 7% compounded annually. Bank B offers the annual interest rate 6.8% compounded continuously. Where should you open the account?*

In order for money to double in 5 years what would the nominal annual interest rate have to be if interests were paid a) continuously b) monthly or c) yearly?

Solution 1 If I leave £1 in bank A for a year, I will get £1.07 at the end, so the effective interest rate is 7%. If I leave £1 in bank B for a year, I will get $£e^{0.068} = £1.0704$ in the end, so the effective interest rate is 7.04%. Hence I should open my account at bank B, at least if I want to save my money for a year or longer.

Suppose I save £1 for five years. a) With continuous interest payments and an interest rate r I will get e^{5r} after 5 years. In order for money to double, this amount must be equal to 2. Thus

$$e^{5r} = 2 \qquad 5r = \ln\left(e^{5r}\right) = \ln 2 \qquad r = \frac{\ln 2}{5} \approx 0.13863$$

and the interest rate must be 13.863%.

b) With monthly interest payments we must have

$$\left(1 + \frac{r}{12}\right)^{60} = 2 \qquad 1 + \frac{r}{12} = \sqrt[60]{2} \qquad r = 12\left(\sqrt[60]{2} - 1\right) \approx 0.13943$$

and the interest rate must be 13.943%.

c) With yearly interest payments we must have

$$(1+r)^5 = 2 \qquad 1 + r = \sqrt[5]{2} \qquad r = \sqrt[5]{2} - 1 \approx 0.14870$$

and the interest rate must be 14.870%. (One percent more than with continuous interest payments.

Problem 2 *Differentiate the functions*

$$a)\ y = 3e^{1-4x} \qquad b)\ y = e^{\frac{1}{x^2}} \qquad c)\ y = \frac{\ln x}{x} \qquad d)\ y = 2^x \ln x$$

Solution 2

$$\begin{aligned}
\text{a) } \frac{dy}{dx} &= 3e^{1-4x}\left(\frac{d(1-4x)}{dx}\right) = -12e^{1-4x} \\
\text{b) } \frac{dy}{dx} &= e^{x^{-2}}\left(\frac{d(x^{-2})}{dx}\right) = -\frac{2}{x^3}e^{\frac{1}{x^2}} \\
\text{c) } \frac{dy}{dx} &= \frac{(\ln x)' x - (\ln x) x'}{x^2} = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1-\ln x}{x^2} \\
\text{d) } \frac{dy}{dx} &= (2^x)' \ln x + 2^x (\ln x)' = \left(e^{x\ln 2}\right)' \ln x + e^{x\ln 2}(\ln x)' \\
&= (\ln 2) e^{x\ln 2} \ln x + e^{x\ln 2}\left(\frac{1}{x}\right) = \left(\ln 2 \ln x + \frac{1}{x}\right) 2^x \ln x
\end{aligned}$$

Problem 3 *Use logarithmic differentiation to find the derivative of*

$$a)\ y = \sqrt[4]{\frac{2x+1}{1-3x}} = \left(\frac{2x+1}{1-3x}\right)^{\frac{1}{4}} \qquad b)\ y = x^{1-x^2} = e^{\left(1-x^2\right)\ln x}$$

Solution 3

$$\begin{aligned}
\text{a) } \ln y &= \frac{1}{4}\ln\frac{2x+1}{1-3x} = \frac{1}{4}\ln(2x+1) - \frac{1}{4}\ln(1-3x) \\
\frac{d\ln y}{dx} &= \frac{1}{4}\frac{1}{2x+1}\frac{d(2x+1)}{dx} - \frac{1}{4}\frac{1}{1-3x}\frac{d(1-3x)}{dx} = \frac{1}{2(2x+1)} + \frac{3}{4(1-3x)} \\
\frac{dy}{dx} &= \left[\frac{1}{2(2x+1)} + \frac{3}{4(1-3x)}\right] y(x) = \left[\frac{1}{2(2x+1)} + \frac{3}{4(1-3x)}\right]\left(\frac{2x+1}{1-3x}\right)^{\frac{1}{4}}
\end{aligned}$$

$$\begin{aligned}
\text{b) } \ln y &= \ln\left(e^{\left(1-x^2\right)\ln x}\right) = \left(1-x^2\right)\ln x \\
\frac{d\ln y}{dx} &= \left(1-x^2\right)' \ln x + \left(1-x^2\right)(\ln x)' = -2x\ln x + \frac{1-x^2}{x} \\
\frac{dy}{dx} &= \left(-2x\ln x + \frac{1-x^2}{x}\right) y(x) = \left(-2x\ln x + \frac{1-x^2}{x}\right) x^{1-x^2}
\end{aligned}$$

Problem 4 *A consumer has the utility function*

$$u(x,y) = \ln\left(x^{\frac{1}{6}} y^{\frac{2}{3}}\right)$$

a) Simplify the utility function and find its two partial derivatives.

b) If a unit of commodity x costs £1 and each unit of commodity y costs £4, how much would the consumer buy in optimum of both commodities if he had a budget of £100 available?

Solution 4 a)

$$u(x,y) = \ln\left(x^{\frac{1}{6}} y^{\frac{2}{3}}\right) = \ln x^{\frac{1}{6}} + \ln y^{\frac{2}{3}} = \frac{1}{6}\ln x + \frac{2}{3}\ln y$$

b) The consumer must maximize utility $u(x, y)$ subject to the constraint $x + 4y \leq 100$. The Lagrangian for this problem is

$$\mathcal{L}(x, y) = \frac{1}{6} \ln x + \frac{2}{3} \ln y - \lambda (x + 4y - 100)$$

The condition for a critical point of the Lagrangian are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{6}\frac{1}{x} - \lambda = 0 \qquad \frac{1}{6x} = \lambda \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{2}{3}\frac{1}{y} - 4\lambda = 0 \qquad \frac{2}{3y} = 4\lambda \end{aligned}$$

Division of the two equations yields

$$\frac{\frac{1}{6x}}{\frac{2}{3y}} = \frac{1}{6x}\frac{3y}{2} = \frac{3y}{12x} = \frac{\lambda}{4\lambda} = \frac{1}{4}$$

Thus $\frac{3y}{12x} = \frac{1}{4}$ or $12y = 12x$ or $y = x$. Substituting this into the budget equation yields $x + 4y = 5x = 100$ or $x = y = 20$. Thus the consumer wants to buy 20 units of each commodity.