BEE1020 – Basic Mathematical Economics	Juliette Stephenson Amr Algarhi
Homework -Solution **Week 5**	Department of Economics University of Exeter

Exercise 1 Find the critical point of the functions

$$\begin{aligned} \text{a) } z &= f(x,y) = xy - 2x + 3y - 6 \\ \text{b) } z &= g(x,y) = 2x^2 + 2xy - 6x + 5y^2 - 6y + 5 \\ \text{c) } z &= h(x,y) = -2x^2 - 2xy + 6x - 5y^2 - 5 \end{aligned}$$

Determine whether they are troughs, peaks or saddlepoints.

Solution 1 a)

$$\begin{aligned} \frac{\partial z}{\partial x} &= y - 2 = 0 \Leftrightarrow y = 2 \\ \frac{\partial z}{\partial y} &= x + 3 = 0 \Leftrightarrow x = -3 \end{aligned}$$

The critical point is $(x,y) = (3,2)$.

$$\det H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0$$

The critical point is hence a saddle point.

b)

$$\begin{aligned} \frac{\partial z}{\partial x} &= 4x + 2y - 6 = 0 \\ \frac{\partial z}{\partial y} &= 2x + 10y - 6 = 0 | \times 2 \end{aligned}$$

$$\begin{aligned} 4x + 2y &= 6 | - \\ 4x + 20y &= 12 | + \\ -18y &= -6 \end{aligned}$$

Hence $y = \frac{1}{3}$, $4x = 6 - \frac{2}{3} = \frac{16}{3}$, $x = \frac{4}{3}$. The critical point is $(x,y) = \left(\frac{4}{3}, \frac{1}{3}\right)$. Since

$$\det H = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} = 40 - 4 = 36 > 0$$

the critical point is *not* a saddle point. Since $\frac{\partial^2 z}{\partial x^2} = 4 > 0$ it is a trough. (In fact, the function is strictly convex and the critical point a global minimum.)

c)

$$\begin{aligned} \frac{\partial z}{\partial x} &= -4x - 2y + 6 = 0 \\ \frac{\partial z}{\partial y} &= -2x - 10y = 0 | \times (-2) \end{aligned}$$

$$\begin{aligned} -4x-2y &= -6| + \\ 4x+20y &= 0| + \\ 18y &= -6 \end{aligned}$$

Hence $y=-\frac{1}{3}$, $4x=6+\frac{2}{3}=\frac{20}{3}$, $x=\frac{5}{3}$. The critical point is again $(x,y)=\left(\frac{5}{3},-\frac{1}{3}\right)$. Since

$$\det H = \begin{bmatrix} -4 & -2 \\ -2 & -10 \end{bmatrix} = 40-4=36>0$$

the critical point is *not* a saddle point. Since $\frac{\partial^2 z}{\partial x^2}=-4<0$ it is a peak. (In fact, the function is strictly concave and the critical point a global maximum.)

Exercise 2 A dairy produces whole milk and skim milk in quantities x and y gallons, respectively. Suppose that the price of whole milk is $p(x)=20-5x$ and that the price of skim milk is $q(y)=4-2y$ and assume that $C(x,y)=2xy+4$ is the total (!) joint-cost function of the commodities. What should x and y be to maximize profit?

Solution 2 The revenue from selling whole milk is

$$xp(x)=x(20-5x),$$

the revenue from selling skim milk is

$$yq(y)=y(4-2y)$$

and the total costs are $2xy+4$. The profit is therefore

$$\Pi(x,y)=x(20-5x)+y(4-2y)-2xy-4$$

The conditions for a critical point are

$$\begin{aligned} \frac{\partial \Pi}{\partial x} &= 20-5x-5x-2y=20-10x-2y=0 \\ \frac{\partial \Pi}{\partial y} &= 4-2y-2y-2x=4-2x-4y=0 \end{aligned}$$

Dividing the second equations by 2 we obtain the simultaneous system of equations

$$\begin{aligned} 10x+2y &= 20 \\ x+2y &= 2 \end{aligned}$$

Subtraction yields $9x=18$ or $x=2$ and so we obtain from the second equation $y=1-\frac{1}{2}x=1-1=0$. Thus the profit function has a critical point at $x=2$and at $y=0$. The Hessian matrix is

$$H=\begin{bmatrix} \frac{\partial \Pi^2}{\partial x^2} & \frac{\partial \Pi^2}{\partial x \partial y} \\ \frac{\partial \Pi^2}{\partial y \partial x} & \frac{\partial \Pi^2}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -10 & -2 \\ -2 & -4 \end{bmatrix}$$

and its determinant is

$$\det H=(-10)(-4)-(-2)^2=36>0.$$

The function has a relative maximum at the critical point found because $\frac{\partial \Pi^2}{\partial x^2}<0$ and det $H>0$. Since the function is defined on the entire plane (a convex set) and since there is only one relative maximum, this is also an absolute maximum.