BEE1024 – Mathematics for Economists	Juliette Stephenson Amr Algarhi
Class Solutions **Week 7**	Department of Economics University of Exeter

Exercise 1 Find the indefinite integral. Check your answers by differentiating.

$$\text{a)} \int \frac{1}{x^2} dx$$
$$\text{b)} \int \left(3\sqrt{y} + \frac{2}{y^3} + \frac{1}{y}\right) dy$$
$$\text{c)} \int \sqrt{t}\left(t^2 - 1\right) dt$$

Solution 1

$$\begin{aligned}
\text{a)} \int x^{-2}dx &= \frac{1}{-2+1}x^{-2+1} + C = -x^{-1} + C \\
\frac{d}{dx}\left(-x^{-1} + C\right) &= x^{-2} \\
\text{b)} \int \left(3\sqrt{y} + \frac{2}{y^3} + \frac{1}{y}\right) dy &= 3\int y^{\frac{1}{2}}dy + 2\int y^{-3}dy + \int y^{-1}dy = 2y^{\frac{3}{2}} - y^{-2} + \ln|y| + C \\
\frac{d}{dy}\left(2y^{\frac{3}{2}} - y^{-2} + \ln|y| + C\right) &= 3y^{\frac{1}{2}} + 2y^{-3} + \frac{1}{y} \\
\text{c)} \int \sqrt{t}\left(t^2 - 1\right) dt &= \int t^{\frac{1}{2}}\left(t^2 - 1\right) dt = \int \left(t^{\frac{5}{2}} - t^{\frac{1}{2}}\right) dt = \frac{2}{7}t^{\frac{7}{2}} - \frac{2}{3}t^{\frac{3}{2}} + C \\
\frac{d}{dt}\left(\frac{2}{7}t^{\frac{7}{2}} - \frac{2}{3}t^{\frac{3}{2}} + C\right) &= t^{\frac{5}{2}} - t^{\frac{1}{2}}
\end{aligned}$$

Exercise 2 Calculate

$$\int_2^3 \frac{1}{x^2} dx$$

Solution 2

$$\int_2^3 x^{-2}dx = \left[\frac{1}{-2+1}x^{-2+1}\right]_2^3 = \left[-\frac{1}{x}\right]_2^3 = \left[-\frac{1}{3}\right] - \left[-\frac{1}{2}\right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Exercise 3 Suppose the supply function in a market is $Q^s = P^2$ and the current market price is $P^* = 4$. What is the producer surplus?

Solution 3 The producer surplus is

$$PS = \int_0^4 Q^s(P)\, dp = \int_0^4 P^2 dP = \frac{1}{3}P^3\Big|_0^4 = \frac{64}{3} = 21\frac{1}{3}$$

Alternatively, one can work with the inverse demand function $P^s = \sqrt{Q}$ which describes at what price suppliers will supply the quantity Q. Producer surplus is then, since $Q^s(4) = 16$,

$$PS = 4 \times 16 - \int_0^{16} Q^{\frac{1}{2}} dQ = 64 - \left[\frac{2}{3}Q^{\frac{3}{2}}\right]_0^{16} = 64 - \frac{2}{3}\left(\sqrt{16}\right)^3 = 64 - \frac{2}{3} \times 64 = \frac{64}{3} = 21\frac{1}{3}$$

Exercise 4 Calculate the area above the horizontal axis and below the graph of the function

$$y = f(x) = 16 - x^4$$

Solution 4 The graph of the function is above the horizontal axis in the interval $-2 \leq x \leq 2$. This can be seen from the factorization

$$f(x) = \left(4 - x^2\right)\left(4 + x^2\right) = (2 - x)(2 + x)\left(4 + x^2\right)$$

and the sign diagram

	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$2 < x$
$2 - x$	$+$	$+$	$+$	0	$-$
$2 + x$	$-$	0	$+$	$+$	$+$
$4 + x^2$	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$-$

The area is hence given by the definite integral

$$\begin{aligned} \int_{-2}^{2} \left(16 - x^4\right) dx &= \left(16x - \frac{1}{5}x^5\right)\Big|_{-2}^{2} \\ &= \left(16 \times 2 - \frac{1}{5}2^5\right) - \left(16 \times (-2) - \frac{1}{5}(-2)^5\right) \\ &= 16 \times 4 - \frac{1}{5}2^6 = 4^3 - \frac{1}{5}4^3 = \frac{4}{5}4^3 = \frac{4^4}{5} = \frac{256}{5} = 51.2 \end{aligned}$$

Exercise 5 Demand is given by

$$Q^d(P) = 10 - \sqrt{P}$$

a) Find the interval of prices for which demand is positive.

b) Express total revenue $TR = PQ$ as a function of the price. When is total revenue maximized?

c) For which price is the own-price elasticity $ped(P) = \frac{dQ^d}{dP} \times \frac{P}{Q^d}$ equal to -1?

Solution 5 a) Demand is zero for $10 = \sqrt{P}$ or $P = 100$. For lower prices it is positive.

b)

$$\begin{aligned} TR &= PQ^d = P\left(10 - \sqrt{P}\right) = 10P - P^{\frac{3}{2}} \\ \frac{dTR}{dP} &= 10 - \frac{3}{2}P^{\frac{1}{2}} = 0 \qquad \sqrt{P} = \frac{20}{3} \qquad P = \frac{400}{9} = 44\frac{4}{9} \\ \frac{d^2TR}{d^2P} &= -\frac{3}{4}P^{-\frac{1}{2}} < 0 \end{aligned}$$

Thus total revenue has a maximum at $P = 44\frac{4}{9}$.

c)

$$\begin{aligned}
Q^d &= 10 - \sqrt{P} \\
\frac{dQ^d}{dP} &= -\frac{1}{2\sqrt{P}} \\
ped\,(P) &= \frac{dQ^d}{dP}\frac{P}{Q^d} = -\frac{1}{2\sqrt{P}}\frac{P}{10-\sqrt{P}} = -\frac{1}{2}\frac{\sqrt{P}}{10-\sqrt{P}} \\
-\frac{1}{2}\frac{\sqrt{P}}{10-\sqrt{P}} &= -1 \qquad \sqrt{P} = 2\left(10-\sqrt{P}\right) = 20 - 2\sqrt{P} \\
3\sqrt{P} &= 20 \qquad \sqrt{P} = \frac{20}{3} \qquad P = 44\frac{4}{9}
\end{aligned}$$

Thus demand elasticity is -1 for $P = 44\frac{4}{9}$.