BEE1020 – Basic Mathematical Economics	Juliette Stephenson Amr Algarhi
Class Exercises -Solution **Week 5**	Department of Economics University of Exeter

Exercise 1 a) Find all second derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial y^2}$ of the function

$$z = f(x, y) = 2x^2 - 2xy - 16x + 5y^2 + 2y + 34$$

b) Calculate the determinant of the Hessian matrix

$$\det H = \det \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial y \partial x} \right)^2$$

Is it positive or negative?

c) Is the function convex, concave or saddle shaped? Does the function have a peak, a trough or a maximum?

Solution 1 a) The partial derivatives are

$$\begin{aligned} \frac{\partial z}{\partial x} &= 4x - 2y - 16 \\ \frac{\partial z}{\partial y} &= -2x + 10y + 2 \end{aligned}$$

Hence the Hessian matrix with the four partial derivatives as entries is

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 10 \end{bmatrix}$$

b)

$$\det H = 4 \times 10 - (-2)^2 = 36$$

c) Because $\det H > 0$ the function is *not* saddle-shaped. Since $\frac{\partial^2 z}{\partial x^2} > 0$ it is strictly convex. The critical point $(x, y) = \left(4\frac{1}{3}, \frac{2}{3}\right)$ is hence a trough (and indeed an absolute minimum) of the function.

Exercise 2 A T-shirt shop carries two competing shirts, one endorsed by Michael Jordan and the other by Shaq O'Neal. The owner of the store can obtain both at a cost of \$2 per shirt and estimates that if Jordan shirts are sold for x dollars apiece and O'Neal shirts for y dollars apiece, consumers will buy approximately $40 - 50x + 40y$ Jordan shirts and $20 + 60x - 70y$ O'Neil shirts each day.

a) Express as functions of x and y: i) the revenue from selling Jordan shirts, ii) the revenue from selling O'Neal shirts iii) the costs for shirts and iv) the overall profit.

b) Find the critical point of the profit function.

c) How should the owner price the shirts in order to generate the largest possible profit?

d) Calculate he Hessian matrix for this problem and its determinant. Is the solution found in b) indeed an absolute maximum?

Solution 2 c) Jordan T-shirts should be sold for \$2.70 and O'Neal shirts for \$2.50 (see class exercises week 16), provided this is a maximum

d) The Hessian matrix is

$$H = \begin{bmatrix} \frac{\partial \Pi^2}{\partial x^2} & \frac{\partial \Pi^2}{\partial x \partial y} \\ \frac{\partial \Pi^2}{\partial y \partial x} & \frac{\partial \Pi^2}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -100 & 100 \\ 100 & -140 \end{bmatrix}$$

and its determinant is

$$\det H = (-100)(-140) - (100)^2 = 14000 - 10000 = 4000 > 0.$$

The function has a relative maximum at the critical point found because $\frac{\partial \Pi^2}{\partial x^2} < 0$ and det $H > 0$. Since the function is defined on the entire plane (a convex set) and since there is only one relative maximum, this is also an absolute maximum.

Exercise 3 Calculate the partial derivatives of

$$z = \frac{x}{y} = xy^{-1}$$

Suppose now that x and y are functions of t, i.e., $x = u(t)$, $x = v(t)$. Using the chain rule for two variables, what is $\frac{dz}{dt}$?

Solution 3

$$\frac{\partial z}{\partial x} = \frac{1}{y} \qquad \frac{\partial z}{\partial y} = -xy^{-2} = -\frac{x}{y^2}$$

By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{v(t)}u'(t) - \frac{u(t)}{v^2(t)}v'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{v^2(t)}$$

Exercise 4 A price-taking firm in a competitive market has the total costs $TC(Q)$. Let $\Pi(Q,P) = PQ - TC(Q)$, let $Q^*(P)$ be the optimal quantity and let $\Pi^*(P)$ be the optimal profit the firm can get when the price or output is P. Use the chain rule to calculate $\frac{d\Pi^*}{dP}$. Assuming an interior solution, simplify the expression using the first order condition for a maximum. Give an intuition for the result.

Solution 4 $\Pi^*(P) = \Pi(Q^*(P), P)$ and so, by the chain rule

$$\frac{d\Pi^*}{dP} = \frac{\partial \Pi}{\partial Q} \times \frac{dQ^*(P)}{dP} + \frac{\partial \Pi}{\partial P} \times \frac{dP}{dP} = \frac{\partial \Pi}{\partial Q} \times 0 + Q \times 1 = Q^*(P)$$

If the price of output is increased by a penny, profit goes up in pennies by the amount produced. The change in the optimal output produced can be ignored by the envelope theorem.