

BEE1020 – Basic Mathematical Economics	Juliette Stephenson Amr Algarhi
Class Exercises -Solution	Department of Economics
Week 4	University of Exeter

Exercise 1 A consumer with utility function $u(x, y) = (x + 1)(y + 1)$ has a budget $b = 100$, while the prices are $p_x = 20$, $p_y = 40$. Write down the Lagrangian for the problem and find its partial derivatives. Assume that only the budget constraint is binding. Which Lagrange multipliers must be then be zero? What system of equations must be satisfied and what is its solution? Is the solution reasonable?

Solution 1 The Lagrangian is

$$\begin{aligned}\mathcal{L}(x, y) &= u(x, y) + \lambda_1(b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y \\ &= (x + 1)(y + 1) + \lambda_1(100 - 20x - 40y) + \lambda_2 x + \lambda_3 y\end{aligned}$$

The FOC are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= (y + 1) - 20\lambda_1 + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= (x + 1) - 40\lambda_1 + \lambda_3 = 0\end{aligned}$$

If only the budget constraint binds, then $\lambda_2 = \lambda_3 = 0$ and

$$20x + 40y = 100$$

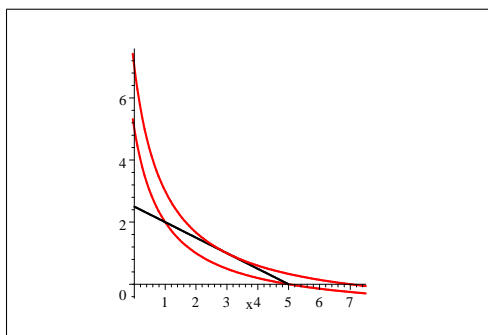
The FOC yield

$$\begin{aligned}y + 1 &= 20\lambda_1 & x + 1 &= 40\lambda_1 \\ \frac{y + 1}{x + 1} &= \frac{20}{40} = \frac{1}{2} & y &= \frac{1}{2}(x + 1) - 1 = \frac{1}{2}x - \frac{1}{2}\end{aligned}$$

Substitution into the budget equation yields

$$\begin{aligned}20x + 40y &= 20x + 40\left(\frac{1}{2}x - \frac{1}{2}\right) = 40x - 20 = 100 \\ 40x &= 120 \\ x &= 3 \\ y &= \frac{1}{2}x - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1\end{aligned}$$

The solution $x^* = 3$, $y^* = 1$ makes economic sense.



Exercise 2 A consumer with utility function $u(x, y) = (x + 1)(y + 1)$ has a budget $b = 44$, while the prices are $p_x = 100$, $p_y = 16$. Write down the Lagrangian for the problem and find its partial derivatives. Assume that only the budget constraint is binding. Which Lagrange multipliers must then be zero? Which three equations do you have to solve simultaneously? Which solution do you get? Is it reasonable?

Solution 2 The Lagrangian is

$$\begin{aligned}\mathcal{L}(x, y) &= u(x, y) + \lambda_1(b - p_x x - p_y y) + \lambda_2 x + \lambda_3 y \\ &= (x + 1)(y + 1) + \lambda_1(44 - 100x - 16y) + \lambda_2 x + \lambda_3 y\end{aligned}$$

The FOC are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= (y + 1) - 100\lambda_1 + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= (x + 1) - 16\lambda_1 + \lambda_3 = 0\end{aligned}$$

If only the budget constraint binds, then $\lambda_2 = \lambda_3 = 0$ and

$$100x + 16y = 44$$

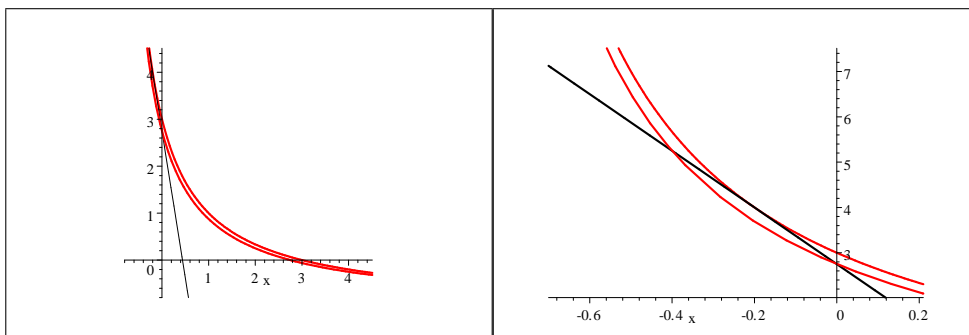
The FOC yield

$$\begin{aligned}y + 1 &= 100\lambda_1 & x + 1 &= 16\lambda_1 \\ \frac{y + 1}{x + 1} &= \frac{100}{16} = \frac{25}{4} & y &= \frac{25}{4}(x + 1) - 1 = \frac{25}{4}x + \frac{21}{4}\end{aligned}$$

Substitution into the budget equation yields

$$\begin{aligned}100x + 16y &= 100x + 16\left(\frac{25}{4}x + \frac{21}{4}\right) = 200x + 84 = 44 \\ 200x &= -40 \\ x &= -\frac{1}{5} \\ y &= \frac{25}{4}\left(-\frac{1}{5}\right) + \frac{21}{4} = -\frac{5}{4} + \frac{21}{4} = \frac{16}{4} = 4\end{aligned}$$

The solution $x^* = -1/5, y^* = 4$ does not make economic sense because $x^* < 0$.



Exercise 3 What is the consumer optimum in the last exercise?

Solution 3 The consumer does not want to consume commodity x because it is too expensive, so $x^* = 0$ and from the budget constraint $y^* = 44/16 = 11/4 = 3.75$. This fits with the Lagrangian approach: The budget equation holds per construction. The Lagrange multiplier λ_3 for the constraint $y \geq 0$ can be set to zero. The FOC

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= (y + 1) - 100\lambda_1 + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= (x + 1) - 16\lambda_1 = 0\end{aligned}$$

give for $x^* = 0, y^* = 11/4$: $\lambda_1 = 1/16 \geq 0$ when we use the second equation and

$$\lambda_2 = -\left(\frac{11}{4} + 1\right) + \frac{100}{16} = \frac{5}{2} \geq 0$$

when we use the first. Thus, provided $x^* = 0, y^* = 11/4$ is indeed the maximum of the Lagrangian for the values of the Lagrangre multipliers just calculated, we have solved the constrained optimization problem.

Exercise 4 The highway department is planning to build a picnic area for motorists along a major highway. It is to be rectangular with an area of 5,000 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job?

- Identify this problem as a constraint optimization problem. What objective function $f(x, y)$ is to be maximized / minimized subject to what constraint $g(x, y) \geq 0$?
- Write down the Lagrangian $\mathcal{L}(x, y)$ for this problem.
- Find the solution to the three equations a) $\frac{\partial \mathcal{L}}{\partial x} = 0$ b) $\frac{\partial \mathcal{L}}{\partial y} = 0$ c) $g(x, y) \geq 0$.

Solution 4 a) Let x denote the length of the side of the rectangle adjacent to the highway and let y denote the length of the other side. Then we have to minimize the amount of fencing $x + 2y$ subject of the constraint that the area xy is at least 5000 ($xy \geq 5000$). Equivalently, we have to maximize $-x - 2y$ subject to the constraint $xy - 5000 \geq 0$.

b) The Lagrangian is

$$\mathcal{L}(x, y) = -x - 2y + \lambda(xy - 5000)$$

c)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= -1 + \lambda y = 0 & \text{or} & & 1 &= \lambda y \\ \frac{\partial \mathcal{L}}{\partial y} &= -2 + \lambda x = 0 & \text{or} & & 2 &= \lambda x \\ 5000 &= xy\end{aligned}$$

Division of the two equations on the right yields $\frac{1}{2} = \frac{\lambda y}{\lambda x} = \frac{y}{x}$ or $x = 2y$. Hence $5000 = xy = 2y^2$, $y^2 = 2500$, $y = 50$ (since the solution -50 does not make sense). We obtain $x^* = 100$, $y^* = 50$ and $\lambda = 1/50 \geq 0$.

Exercise 5 A consumer has the following utility when he consumes x units of apples and y units of oranges:

$$u(x, y) = -x^2 + 4x - y^2 + 16y$$

Suppose the consumer has a budget of £3.20 to be spend on oranges and apples. Each apple and each orange costs £0.40. Use the method of Lagrange to find the optimal consumption bundle:

- Write down the budget constraint and the Lagrangian. Assume that only the budget constraint is binding.
- Write down the first-order conditions for a critical point for the Lagrangian. Find a condition for a critical point that does not involve the Lagrange multiplier.
- Use the latter condition and the budget constraint to find a candidate for the optimum. (One can show that it is indeed the optimum.)

Solution 5 a) With the given prices and budget the budget constraint is

$$0.4x + 0.4y = 3.2.$$

The Lagrangian for the constrained utility maximization problem is hence

$$\begin{aligned}\mathcal{L}(x, y) &= u(x, y) + \lambda [3.2 - 0.4x - 0.4y] + \lambda_2 x + \lambda_3 y \\ &= -x^2 + 4x - y^2 + 16y + \lambda [3.2 - 0.4x - 0.4y]\end{aligned}$$

since $\lambda_2 = \lambda_3 = 0$ by the complementarity conditions b) In the optimum the two partial derivatives of the Lagrangian must be zero:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= -2x + 4 - \lambda [0.4] = 0 & \text{or} & & -2x + 4 &= 0.4\lambda \\ \frac{\partial \mathcal{L}}{\partial y} &= -2y + 16 - \lambda [0.4] = 0 & \text{or} & & -2y + 16 &= 0.4\lambda\end{aligned}$$

Division of the two equations on the right yields

$$\begin{aligned}\frac{-2x + 4}{-2y + 16} &= \frac{0.4\lambda}{0.4\lambda} = 1 & \text{or} & & -2x + 4 &= -2y + 16 & \text{or} \\ -2x - 12 &= -2y & \text{or} & & y &= x + 6\end{aligned}$$

Thus $y = x + 6$ must hold at the constrained utility maximum.

c) In addition, the constraint

$$\begin{aligned} 0.4x + 0.4y &= 3.2 & \text{or} & & x + y &= \frac{3.2}{0.4} = 8 & \text{or} \\ y &= 8 - x \end{aligned}$$

must hold at the optimum. Overall $y = x + 6$ and $y = 8 - x$ must hold, so

$$x + 6 = 8 - x \quad \text{or} \quad 2x = 2 \quad \text{or} \quad x = 1$$

Moreover, $x = 1$ implies $y = 8 - x = 7$. *Thus the Lagrangian approach suggests that it is optimal for the consumer to buy one apple and seven oranges.* (Notice that we obtain $0.4\lambda = 16 - 2y = 16 - 14 = 2 > 0$ for the Lagrange multiplier.)