BEE1024

UNIVERSITY OF EXETER

SCHOOL OF BUSINESS AND ECONOMICS

April 2008

Mock Exam

MATHEMATICS FOR ECONOMISTS

Duration : **TWO HOURS**

This mock exam is in the same format as the actual exam, with the following instructions and same 4 sections.

This will form the basis for revision of the course and you are required to bring solutions to the week 9 tutorials (April 30th and May 1st)

In order to pass the module, you must obtain in the exam at least 10 marks from the 25 marks allocated to each of the 4 sections of this paper. The sections are Section A (Multivariate Functions and Partial Differentiation), Section B (Optimization), Section C (Logarithms, Exponentials and Integrals) and Section D (Linear Algebra).

You are welcome to consult **any** material, including lecture notes, formula books, graphical calculators admitted by the school, and dictionaries of any sort. However, **full work must be shown** on your script. **Please write legibly.**

Section A (Multivariate Functions and Partial Differentiation)

Question A1 (6 marks): A producer has the production function

$$Q = \sqrt[7]{KL^2}$$

1. Calculate the marginal products of labour and capital.
2. What is the marginal rate of substitution?
3. What is the marginal rate of substitution when 4 units of capital and 3 units of labour are used?
4. Describe the isoquant through the point $(4,3)$ as a univariate function $L = f(K)$.
5. Determine the derivative $f'(K)$. Calculate $f'(4)$.

Question A2 : (7 marks) Find all partial derivatives of the following functions.

A2.i. $z = 5x^3y^2 - 3x^2y^2 + 10x - 3y^3 + 5$

A2.ii. $z = \frac{x+1}{x-1} - y(\ln(y) - 1)$

A2.iii. $z = x\sqrt{1-y^2}$

Question A3 (6 marks): Calculate the Hessian matrix of the function

$$f(x,y) = \ln(2^x + y).$$

Is the function convex or concave near $x = y = 0$?

Question A4 (6 marks): Demand Q^d for apples depends as follows on the price of apples p, oranges p^* and income b of consumers:

$$Q^d = 500 - 2p^2 + 3p^* + 5b^3$$

Suppose $p = 4$, $p^* = 20$ and $b = 2$.

A4.i. Determine demand at the given prices and income.

A4.ii. Determine own-price elasticity, cross price elasticity and income elasticities.

A4.iii. Which of the following terms are appropriate to describe the situation: complement / substitute, necessity/ luxury, elastic / inelastic demand, inferior / normal?

Section B (Optimization)

Question B1 : (10 marks) A monopolist sells the same commodity in two countries, A and B. When she sells the commodity at the price p in country A the quantity demanded in country A will be

$$x = 40 - p$$

When she sells the commodity at the price q in country B the quantity demanded in country B will be

$$y = 60 - 2q$$

When she produces on total $x + y$ units for the two countries her total production costs will be

$$\frac{1}{2}(x+y)^2$$

B1.i. Express the total revenue from sales in the two countries in terms of the variables x, y, p, q. Then, using the demand functions express total revenue as a function $TR(p,q)$ of the prices only.

B1.ii. Assuming that she sells what she is producing, express total production costs as a function $TC(p,q)$ of the prices only.

B1.iii. Find a critical point of the profit function.

B1.iv. Use second partial derivatives and the Hessian matrix to show that the critical point is a relative profit maximum.

B1.v. How much is demanded in each country at the profit maximum? Calculate the demand elasticity $\frac{\partial y}{\partial q}\frac{q}{y}$ in the second country at the price set in the profit optimum. Is demand there elastic or inelastic or none of both?

Question B2 : (10 marks) A consumer has the following utility when he consumes x units of apples and y units of oranges:

$$u(x,y) = -x^2 + 4x - y^2 + 16y$$

Suppose the consumer has a budget of £3.20 to be spend on oranges and apples. Each apple and each orange costs £0.40. Use the method of Lagrange to find the optimal consumption bundle:

B2.i. Write down the budget constraint and the Lagrangian.

B2.ii. Write down the first-order conditions for a critical point for the Lagrangian. Assume that the non-negativity conditions are not binding. Find a condition for a critical point that does not involve the Lagrange multiplier.

B2.iii. Use the latter condition and the budget constraint to find a candidate for the optimum. (One can show that it is indeed the optimum.)

Question B3 : (5 marks) A firm has the revenue function has the revenue function $TR(Q)$ where Q is the quantity of output produced. The total costs are given by $TC(Q,c) = cQ$ where $c > 0$ is the constant marginal cost. Assuming that the profit optimum is uniquely characterized by the first order condition, show that the optimal profit

$$\Pi^*(c) = \max_Q [TR(Q) - TC(Q,c)]$$

is decreasing in c.

Section C (Logarithms, Exponentials and Integrals)

Question C1 : (5 marks) How quickly will money triple if it is invested at a nominal annual interest rate of 3 percent when interest is compounded (a) continuously, (b) quarterly, (c) annually?

Question C2 : (5 marks) Use logarithmic differentiation to find the derivative of the function

$$y = (2x+2)^{-5}(x+2)^{\frac{7}{2}}$$

Question C3 : (5 marks) An epidemic spreads throughout a community so that t weeks after its outbreak, the number of residents who have been infected is given by a function of the form

$$f(t) = \frac{A}{1 + Ce^{-kt}}$$

where A is the total number of susceptible residents. Show that the epidemic is spreading most rapidly when half of the residents have been infected.

Question C4 : (5 marks) Evaluate the integrals

C4.i. $\displaystyle\int \left(\frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x}\right) dx$

C4.ii. $\displaystyle\int (x^3 - 4)\, dx$

C4.iii. $\displaystyle\int_{-1}^{3} (x^3 - 4)\, dx$

Question C5 : (5 marks)

C5.i. Use the substitution $u = 1 - 2x$ to integrate $\int 6e^{1-2x} dx$.

C5.ii. Use integration by parts to evaluate $\int (1 \times \ln x)\, dx$.Calculate the following integrals.

$$\text{a)} \int \frac{e^{(x+2)^2}}{4} (x+2)\, dx$$

$$\text{b)} \int \ln (x+2)\, dx$$

Section D (Linear Algebra)

Question D1 : (6 marks) Consider the the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

D1.i. Which of the following products exist? ABC, $AB'C$, $AB'C'$, $B'CA$

D1.ii. Evaluate the products in a) when they exist.

Question D2 : (6 marks) Use matrix algebra to find the solution of the simultaneous system of equations

$$\begin{aligned} 3x + 5y &= 7 \\ 7x - 13y &= 2 \end{aligned}$$

In doing so, find the inverse of the coefficient matrix. Express your solution using fractions, not decimal numbers.

Question D3 : (7 marks) Calculate the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

and use this information to solve the linear simultaneous system of equations

$$\begin{aligned} y + 2z &= 4 \\ x + z &= 4 \\ 2x + y &= 8 \end{aligned}$$

Question D4 : (12 marks) Consider the matrix

$$P = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

D4.i. Calculate P^2. What is the inverse of P? What is the determinant?

D4.ii. Draw the six vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ \vec{c} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$P\vec{a}$, $P\vec{b}$ and $P\vec{c}$ on graph paper. What is the geometric interpretation of the mapping from any vector $\vec{x}$ to the vector $P\vec{x}$?

D4.iii. Show that for any two vectors

$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \ \vec{y} = \begin{bmatrix} c \\ d \end{bmatrix}$$

that $(P\vec{x})' (P\vec{y}) = \vec{x}'\vec{y}$.