BEE1024

UNIVERSITY OF EXETER

SCHOOL OF BUSINESS AND ECONOMICS

May / June 2007

MATHEMATICS FOR ECONOMISTS

Duration : **TWO HOURS**

In order to pass the module, you must obtain at least 8 marks from the 25 marks allocated to **each** of the 4 sections of this paper. The sections are Section A (Multivariate Functions and Partial Differentiation), Section B (Optimization), Section C (Logarithms, Exponentials and Integrals) and Section D (Linear Algebra).

You are welcome to consult **any** material, including lecture notes, formula books, graphical calculators admitted by the school, and dictionaries of any sort. However, **full work must be shown** on your script. **Please write legibly.**

Section A (Multivariate Functions and Partial Integration)

Question A1 (6 marks): A producer has the production function

$$Q = \sqrt[5]{KL}$$

1. Calculate the marginal products of labour and capital.
2. What is the marginal rate of substitution?
3. What is the marginal rate of substitution when 2 units of capital and 3 units of labour are used?
4. Describe the isoquant through the point $(2, 3)$ as a univariate function $L = f(K)$.
5. Determine the derivative $f'(K)$. Calculate $f'(2)$.
6. How does $f'(2)$ relate to your answer in (2).

Question A2 : (7 marks) Find the partial derivatives of the following functions.

$$\begin{aligned} \text{a) } z &= 5x^5 + 5xy^4 - 16x^3y^2 + 3y + 10 \\ \text{b) } z &= \frac{1}{\sqrt{xy^2}} \\ \text{c) } z &= \sqrt[3]{(x^8y + xy^8)} \end{aligned}$$

Question A3 (6 marks): Find the first and second partial derivatives of the following functions. Verify Young's theorem.

$$\begin{aligned} \text{a) } z &= 7xy^7 + x^2y^3 \\ \text{b) } z &= \ln\left(x^2y^3\right) \end{aligned}$$

Question A4 (6 marks): Consider the demand function

$$Q^d = 1000 - 4P^2 + 6P^* + 5Y$$

which describes how the demand Q^d for a good depends on its price P, the price P^* of another good and income Y. Calculate the own-price elasticity, cross price elasticity and income elasticity when $P = 10$, $P^* = 20$ and $Y = 1000$. Is demand there elastic or inelastic? Are the good complements or substitutes? Is demand normal?

Section B (Optimization)

Question B1 : (10 marks) A monopolist produces two commodities in quantities Q_1 and Q_2. The prices for the two commodities are P_1 and P_2, respectively. Demand for each commodity is given by

$$Q_1 = 50 - 2P_1 + P_2 \qquad Q_2 = 75 + 2P_1 - 3P_2$$

Total costs for producing Q_1 (respectively Q_2) units of the first (second) commodity are $15Q_1$ and $10Q_2$, respectively.

1 Express the total revenue from sales of the two commodities in terms of the variables Q_1, Q_2, P_1, P_2. Then, using the demand functions express total revenue as a function $TR(P_1, P_2)$ of the prices only.

2 Assuming that he sells what he is producing, express total production costs as a function $TC(P_1, P_2)$ of the prices only.

3 Find a critical point of the profit function.

4 Use second partial derivatives and the Hessian matrix to show that the critical point is a relative profit maximum.

Question B2 : (10 marks) A consumer has the utility function

$$u(x, y) = 15xy + 9x + 10y$$

for apples and oranges. Apples cost £0.30 a piece and oranges £0.50. The consumer has a budget of £1.10. Use the Lagrangian approach to determine how many apples and oranges the consumer should buy, given his budget, in order to maximize his utility. In particular,

1 Write down the budget equation.

2 Write down the complete Lagrangian $\mathcal{L}(x, y)$ for the constrained optimization problem.

3 Calculate the partial derivatives $\frac{\partial \mathcal{L}}{\partial x}$ and $\frac{\partial \mathcal{L}}{\partial y}$ of the Lagrangian.

4 Assume that only the budget constraint is binding. Use the first-order conditions $\frac{\partial \mathcal{L}}{\partial x} = 0$ and $\frac{\partial \mathcal{L}}{\partial y} = 0$ to obtain an equation in x and y only.

5 Use your result from (4) together with the budget equation to find the optimal consumption bundle. What is the value of the Lagrange multiplier?

6 Due to a severe drought the price of oranges increases to £2.50. Use the Lagrangian approach to discuss how this will affect the demand of the consumer.

Question B3 : (5 marks) A price-taking firm with total cost function $TC(Q)$ has to pay an excise tax t to be subtracted from the price p paid by consumers. Assuming an interior equilibrium show that the profits of the profit maximizing firm decrease as the excise tax increases.

Section C (Logarithms, Exponentials and Integrals)

Question C1 : (5 marks) A principal, \$8000, is invested at 8% interest for 9 years. determine its future value if interest is compounded (a) annually, (b) monthly, (c) daily, (d) continuously. Give the explicit formula before you evaluate it.

Question C2 : (5 marks) Use logarithmic differentiation to find the derivative of

$$\frac{\left(x^2+8\right)^{10} x^3}{\left(x^3+x^2+1\right)^7}$$

Question C3 : (5 marks) According to a certain logistic model, the world's population (in billions) t years after 1960 would be approximately

$$P(t) = \frac{40}{1+12e^{-0.06t}}$$

1. If the model is correct, what would have been the world's population size in the year 2000? At what *percentage* rate was the population increasing at this time?
2. When will the population be growing most rapidly?
3. Sketch the graph of $P(t)$. What feature occurs on the graph at the time found in part (2)? What happens to $P(t)$ "in the long run" (as $t \to \infty$)?

Question C4 : (5 marks) Calculate the following integrals.

a) $\displaystyle\int \left(x^5 + \frac{1}{\sqrt[6]{x^5}} - \frac{5}{x} + 5e^x\right) dx$

b) $\displaystyle\int \left(x^4 + x^2\right) dx$

c) $\displaystyle\int_{-1}^{1} \left(x^4 + x^2\right) dx$

Question C5 : (5 marks) a) Use the substitution $u = 2x^3 + 3x^2$ to integrate

$$\int \left(2x^3+3x^2\right)^{10} \left(5x^2+5x\right) dx$$

b) Use integration by parts to find

$$\int \ln(x)\, x dx$$

Section D (Linear Algebra)

Question D1 : (6 marks) Consider the vectors and the matrix

$$\vec{x} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

In D1.1 and D1.2 you may write the final results only.

1. Which of the following products exist? $\vec{x}AB$, $\vec{x}BA'$, $\vec{x}'A'B$, $\vec{x}'BA$, $AB'\vec{x}$.
2. Calculate the product if it exists.

Question D2 : (6 marks)

1. Find the inverse of the matrix

$$B = \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix}$$

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2. Solve the system of equations

$$\begin{aligned} 2x + 5y &= 13 \\ 7x + 3y &= 11 \end{aligned}$$

using either the inverse or Cramer's rule.

Question D3 : (7 marks) Consider the matrix

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 4 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

1. Calculate the inverse A^{-1} of the matrix A.
2. Solve the simultaneous system of equations

$$\begin{aligned} -x + 3y &= 0 \\ 2x + 4y + 3z &= 16 \\ 2y + z &= 4 \end{aligned}$$

Question D4 : (6 marks) Which vectors of length $\sqrt{14}$ are perpendicular to the vectors

$$\vec{a} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}?$$