BEE1020 – Basic Mathematical Economics	Dieter Balkenborg
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Optimization	University of Exeter

"Since the fabric of the universe is most perfect, and is the work of a most perfect creator, nothing whatsoever takes place in the universe in which some form of maximum or minimum does not appear."

Leonhard Euler, 1744

1 Objective

Two types of optimization problems frequently occur in economics: unconstrained optimization problems like profit maximization problems and constrained optimization problems like the utility maximization problem of consumer constrained by a budget. In this handout we discuss the main step to solve an unconstrained optimization problem - namely to find the two "first order conditions" and how to solve them. The solutions for this system of two equations in two unknowns can be local maxima, minima or saddle points. How to distinguish between these types ("second order conditions") will be discussed in the next handout.

In order to solve the first order conditions one has to solve a simultaneous system of equations. We will review methods to solve linear simultaneous systems.

Thereafter we develop the Lagrangian approach for constrained optimization problems.

2 Optimization problems

2.1 Unconstrained optimization

Suppose we want to find the absolute maximum of a function

$$z = f(x, y)$$

in two variables, i.e., we want to find the pair of numbers (x^*, y^*) such that

$$f(x^*, y^*) \geq f(x, y)$$

holds for all (x, y). If (x^*, y^*) is such a maximum the following must hold: If we freeze the variable y at the optimal value y^* and vary only the variable x then the function

$$f(x, y^*)$$

– which is now just a function in just one variable – has a maximum at x^*. Typically the maximum of this function in one variable will be a critical point. Hence we expect the partial derivative $\frac{\partial z}{\partial x}$ to be zero at the optimum. Similarly we expect the partial derivative $\frac{\partial z}{\partial y}$ to be zero at the optimum. We are thus led to the *first order conditions*

$$\begin{aligned} \frac{\partial z}{\partial x}_{|x=x^*,y=y^*} &= 0 \\ \frac{\partial z}{\partial y}_{|x=x^*,y=y^*} &= 0 \end{aligned}$$

which must typically be satisfied. We speak of first order conditions because only the first derivatives are involved. The conditions do not tell us whether we have a maximum, a minimum or a saddle point. To find out the latter we will have to look at the second derivatives and the so-called "second order conditions", as will be discussed in a later lecture.

Example 1 Consider a price-taking firm with the production function $Q(K, L)$. Let r be the interest rate (the price of capital K), let w be the wage rate (the price of labour L) and let P be the price of the output the firm is producing. When the firm uses K units of capital and L units labour she can produce $Q(K, L)$ units of the output and hence make a revenue of

$$P \times Q(K, L)$$

by selling each unit of output at the market price P. Her production costs will then be her total expenditure on the inputs capital and labour

$$rK + wL.$$

Her profit will be the difference

$$\Pi(K, L) = PQ(K, L) - rK - rL.$$

The firm tries to find the input combination (K^*, L^*) which maximizes her profit. To find it we want to solve the first order conditions

$$\frac{\partial \Pi}{\partial K} = P\frac{\partial Q}{\partial K} - r = 0 \quad (1)$$

$$\frac{\partial \Pi}{\partial L} = P\frac{\partial Q}{\partial L} - w = 0 \quad (2)$$

These conditions are intuitive: Suppose for instance $P\frac{\partial Q}{\partial K} - r > 0$. $\frac{\partial Q}{\partial K}$ is the marginal product of capital, i.e., it tells us how much more can be produced by using one more unit of capital. $P\frac{\partial Q}{\partial K}$ is the marginal increase in revenue when one more unit of capital is used and the additional output is sold on the market. r is the marginal cost of using one more unit of capital. $P\frac{\partial Q}{\partial K} - r > 0$ means that the marginal profit from using one more unit of capital is positive, i.e., it increases profits to produce more by using more capital and therefore we cannot be in a profit optimum. Correspondingly, $P\frac{\partial Q}{\partial K} - r < 0$ would mean that profit can be increased by producing less and using less capital. So (1) should hold in the optimum. Similarly (2) should hold.

It is useful to rewrite the two first order conditions as

$$\frac{\partial Q}{\partial K} = \frac{r}{P} \tag{3}$$

$$\frac{\partial Q}{\partial L} = \frac{w}{P} \tag{4}$$

Thus, for both inputs it must be the case that the marginal product is equal to the price ratio of input- to output price. When we divide here the two left-hand sides and equate them with the quotient of the right hand sides we obtain as a consequence

$$\frac{\partial Q}{\partial K}\Big/\frac{\partial Q}{\partial L} = \frac{r}{P}\Big/\frac{w}{P} = \frac{r}{w} \tag{5}$$

so the marginal rate of substitution must be equal to the price ratio of the input price (which is the relative price of capital in terms of labour).

Example 2 Let us be more specific and assume that the production function is

$$Q(K,L) = K^{\frac{1}{6}}L^{\frac{1}{2}}$$

and that the output- and input prices are $P = 12$, $r = 1$ and $w = 3$. Then

$$\frac{\partial Q}{\partial K} = \frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}} \qquad \frac{\partial Q}{\partial L} = \frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}}$$

$$\frac{\partial Q}{\partial K}\Big/\frac{\partial Q}{\partial L} = \frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}}\Big/\frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}} = \frac{1}{3}K^{-\frac{5}{6}}L^{\frac{1}{2}}K^{-\frac{1}{6}}L^{\frac{1}{2}} = \frac{1}{3}\frac{L}{K}$$

The first order conditions (3) and (4) become

$$\frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}} = \frac{1}{12} \tag{6}$$

$$\frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}} = \frac{3}{12}. \tag{7}$$

This is a simultaneous system of two equations in two unknowns. The implied condition (5) becomes

$$\frac{1}{3}\frac{L}{K} = \frac{1}{3}$$

which simplifies to

$$L = K$$

Example 3 A monopolist with total cost function $TC(Q) = Q^2$ sells his product in two different countries. When he sells Q_A units of the good in country A he will obtain the price

$$P_A = 22 - 3Q_A$$

for each unit. When he sells Q_B units of the good in country B he obtains the price

$$P_B = 34 - 4Q_B.$$

How much should the monopolist sell in the two countries in order to maximize profits?

To solve this problem we calculate first the profit function as the sum of the revenues in each country minus the production costs. Total revenue in country A is

$$TR_A = P_A Q_A = (22 - 3Q_A)\, Q_A$$

Total revenue in country B is

$$TR_B = P_B Q_B = (34 - 4Q_B)\, Q_B$$

Total production costs are

$$TC = (Q_A + Q_B)^2$$

Therefore the profit is

$$\Pi\,(Q_A, Q_B) = (22 - 3Q_A)\, Q_A + (34 - 4Q_B)\, Q_B - (Q_A + Q_B)^2\,,$$

a quadratic function in $Q_A$and Q_B. In order to find the profit maximum we must find the critical points, i.e., we must solve the first order conditions

$$\begin{aligned}\frac{\partial \Pi}{\partial Q_A} &= -3Q_A + (22 - 3Q_A) - 2\,(Q_A + Q_B) = 22 - 8Q_A - 2Q_B = 0 \\ \frac{\partial \Pi}{\partial Q_B} &= -4Q_B + (34 - 4Q_B) - 2\,(Q_A + Q_B) = 34 - 2Q_A - 10Q_B = 0\end{aligned}$$

or

$$\begin{aligned}8Q_A + 2Q_B &= 22 \\ 2Q_A + 10Q_B &= 34.\end{aligned} \tag{8}$$

Thus we have to solve a *linear* simultaneous system of two equations in two unknowns.

3 Simultaneous systems of equations

3.1 Linear systems

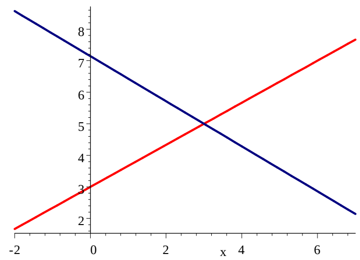
We illustrate four methods to solve a linear system of equations using the example

$$\begin{aligned}5x + 7y &= 50 \\ 4x - 6y &= -18\end{aligned} \tag{9}$$

3.1.1 Using the slope-intercept form

We rewrite both linear equations in slope-intercept form

$$\begin{aligned}7y &= 50 - 5x \\ y &= \frac{50}{7} - \frac{5}{7}x \\ 4x - 6y &= -18 \\ 4x + 18 &= 6y \\ y &= \frac{2}{3}x + 3\end{aligned}$$



The solutions to each equation form a line, which we have now described as the graph of linear functions. At the intersection point of the two lines the two linear functions must have the same y-value. Hence

$$\begin{aligned}
\frac{50}{7} - \frac{5}{7}x &= y = \frac{2}{3}x + 3 \\
\frac{50}{7} - 3 &= \frac{2}{3}x + \frac{5}{7}x \qquad | \times 3 \times 7 \\
150 - 63 &= 14x + 15x \\
87 &= 29x \\
x &= \frac{87}{29} = 3
\end{aligned}$$

We have found the value of x in a solution. To calculate the value of y we use one of the linear functions in slope-intercept form

$$y = \frac{2}{3} \times 3 + 3 = 5$$

The solution to the system of equations is $x^* = 3$, $y^* = 5$

It is strongly recommended to check the result for the original system of equations.

$$\begin{aligned}
5 \times 3 + 7 \times 5 &= 15 + 35 = 50 \\
4 \times 3 - 6 \times 5 &= 12 - 30 = -18
\end{aligned}$$

3.1.2 The method of substitution

First we solve one of the equations in (9), say the second, for one of the variables, say y:

$$\begin{aligned}
4x - 6y &= -18 \\
4x + 18 &= 6y \\
y &= \frac{4}{6}x + 3 = \frac{2}{3}x + 3 \qquad (10)
\end{aligned}$$

Then we replace y in the other equation by the result. (Do not forget to place brackets around the expression.) We obtain an equation in only one variable, x, which we solve for x.

$$
\begin{aligned}
5x + 7y &= 50 \\
5x + 7\left(\frac{2}{3}x + 3\right) &= 50 \\
5x + \frac{14}{3}x + 21 &= 50 \\
\frac{15}{3}x + \frac{14}{3}x + 21 &= 50 \\
\frac{29}{3}x &= 50 - 21 = 29 \\
x &= \frac{3}{29} \times 29 = 3
\end{aligned}
$$

We have found x and can now use (10) to find y:

$$y = \frac{2}{3}x + 3 = \frac{2}{3} \times 3 + 3 = 5$$

Hence the solution is $x = 3$, $y = 5$.

3.1.3 The method of elimination

To eliminate x we proceed in two stages: First we multiply the first equation by the coefficient of x in the second equation and we multiply the second equation by the coefficient of x in the first equation

$$
\begin{aligned}
5x + 7y &= 50 \qquad | \times 4 \\
4x - 6y &= -18 \qquad | \times 5 \\
20x + 28y &= 200 \\
20x - 30y &= -90
\end{aligned}
$$

Then we subtract the second equation from the first equation

$$
\begin{array}{l}
20x + 28y = 200 \\
20x - 30y = -90 \qquad |- \\
0 + 28y - (-30y) = 200 - (-90) \\
58y = 290 \\
y = \frac{290}{58} = 5
\end{array}
$$

Having found y we use one of the original equations to find x

$$
\begin{aligned}
5x + 7y &= 50 \\
5x + 35 &= 50 \\
5x &= 15 \\
x &= 3
\end{aligned}
$$

Instead of first eliminating x we could have first eliminated y:

$$\begin{array}{rcll} 5x + 7y &=& 50 & \mid \times 6 \\ 4x - 6y &=& -18 & \mid \times 7 \\ 30x + 42y &=& 300 & \\ 28x - 42y &=& -126 & \mid + \\ 58x &=& 174 & \\ x &=& 3 & \end{array}$$

3.1.4 Cramer's rule

This is a little preview on linear algebra. In linear algebra the system of equations is written as

$$\begin{bmatrix} 5 & 7 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50 \\ -18 \end{bmatrix}$$

where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} 50 \\ -18 \end{bmatrix}$ are so-called columns vectors, simply two numbers written below each other and surrounded by square brackets. $\begin{bmatrix} 5 & 7 \\ 4 & -6 \end{bmatrix}$ is the 2×2-matrix of coefficients. A 2×2-matrix is a system of four numbers arranged in a square and surrounded by square brackets.

With each 2×2-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we associate a new *number* called the *determinant*

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Notice that the determinant is indicated by vertical lines in contrast to square brackets for a matrix.

For instance,

$$\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix} = 5 \times (-6) - 4 \times 7 = -30 - 28 = -58$$

Cramer's rule uses determinants to give an explicit formula for the solution of a linear simultaneous system of equations of the form

$$\begin{array}{rcl} ax + by &=& e \\ cx + dy &=& f \end{array}$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

Namely,

$$x^* = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc} \qquad y^* = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$

Thus x^* and y^* are calculated as the quotients of two determinants. In both cases we divide by the determinant of the matrix of the coefficients. For x^* the determinant in the numerator is the determinant of the matrix obtained by replacing in the matrix of coefficients the coefficients a and c of x by the constant terms e and f, i.e., the first column is replaced by the column vector with the constant terms. Similarly, for y^* the determinant in the numerator is the determinant of the matrix obtained by replacing in the matrix of coefficients the coefficients b and d of y by the constant terms e and f, i.e., the second column is replaced by the column vector with the constant terms.

The method works only if the determinant in the denominator is not zero.

In our example,

$$x^* = \frac{\begin{vmatrix} 50 & 7 \\ -18 & -6 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix}} = \frac{50 \times (-6) - (-18) \times 7}{5 \times (-6) - 4 \times 7} = \frac{-300 + 126}{-30 - 28} = \frac{-174}{-58} = 3$$

$$y^* = \frac{\begin{vmatrix} 5 & 50 \\ 4 & -18 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 4 & -6 \end{vmatrix}} = \frac{5 \times (-18) - 4 \times 50}{5 \times (-6) - 4 \times 7} = \frac{-90 - 200}{-58} = \frac{-290}{-58} = 5$$

Exercise 1 Use the above methods to find the optimum in Example 8

3.1.5 Existence and uniqueness.

A linear simultaneous system of equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

can have zero, exactly one or infinitely many solution. To see that only these cases can arise, bring both equations into slope-intercept form[1]

$$\begin{aligned} y &= \frac{e}{b} - \frac{a}{b}x \\ y &= \frac{f}{d} - \frac{c}{d}x \end{aligned}$$

If the slopes of these two linear functions are the same, they describe identical or parallel lines. The slopes are identical when $\frac{a}{b} = \frac{c}{d}$, i.e., when the determinant of the matrix of

[1]We assume here for simplicity that the coefficients b and d are not zero. The arguments can be easily extended to these cases, except when all four coefficients are zero. The latter case is trivial - either there is no solution or all pairs of numbers (x, y) are solutions.

coefficients $ad - cb$ is zero. If in addition the intercepts are equal, both equations describe the same line. In this case all points (x, y) on the line are solutions. If the intercepts are different the two equations describe parallel lines. These do not intersect and hence there is no solution.

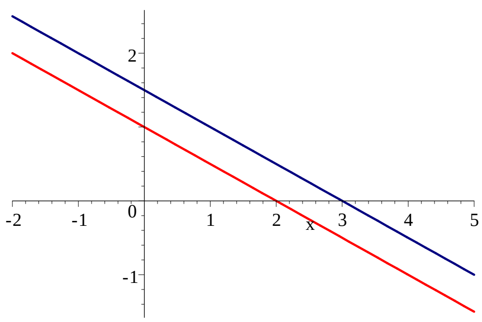
Example 4

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 4 \end{aligned}$$

has no solution: The two lines

$$\begin{aligned} y &= \frac{3}{2} - \frac{1}{2}x \\ y &= 1 - \frac{1}{2}x \end{aligned}$$

are parallel



There is no common solution. Trying to find one yields a contradiction

$$\begin{aligned} \frac{3}{2} - \frac{1}{2}x &= y = 1 - \frac{1}{2}x \qquad | + \frac{1}{2}x \\ \frac{3}{2} &= 1 \end{aligned}$$

3.2 One equation non-linear, one linear

Consider, for instance,

$$\begin{aligned} y^2 + x - 1 &= 0 \\ y + \frac{1}{2}x &= 1 \end{aligned}$$

In this case we solve the linear equation for one of the variables and substitute the result into the non-linear equation. As a result we obtain one non-linear equation in a single unknown. This makes the problem simpler, but admittedly some luck is needed to solve the non-linear equation in one variable. Solving the linear equation for x can sometimes give a simpler non-linear equation than solving for y and vice versa. One may have to try both possibilities. In our example it is convenient to solve for x:

$$\begin{aligned}\frac{1}{2}x &= 1-y \\ x &= 2-2y \\ y^2+(2-2y)-1 &= y^2-2y+1=(y-1)^2=0\end{aligned}$$

So the unique solution is $y^*=1$ and $x^*=2-2\times 1=0$.

3.3 Two non-linear equations

There is no general method. One needs to memorize some tricks which work in special cases.

In the example of profit-maximization with a Cobb-Douglas production function we were led to the system

$$\begin{aligned}\frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}} &= \frac{1}{12} \\ \frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}} &= \frac{3}{12}.\end{aligned}$$

This is a simultaneous system of two equations in two unknowns which looks hard to solve. However, as we have seen, division of the two equations shows that $L=K$ must hold in a critical point.

We can substitute this into, say, the first equation and obtain

$$\begin{aligned}\frac{1}{12} &= \frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}}=\frac{1}{6}K^{-\frac{5}{6}}K^{\frac{1}{2}}=\frac{1}{6}K^{-\frac{5}{6}+\frac{3}{6}}=\frac{1}{6}K^{-\frac{2}{6}}=\frac{1}{6}K^{-\frac{1}{3}} \\ K^{-\frac{1}{3}} &= \frac{1}{2} \\ \frac{1}{\sqrt[3]{K}} &= \frac{1}{2} \\ \sqrt[3]{K} &= 2 \\ K &= 2^3=8\end{aligned}$$

Thus the solution to the first order conditions (and, in fact, the optimum) is $K^*=L^*=8$.

Remark 1 This method works with any Cobb-Douglas production function $Q(K,L)=K^aL^b$ where the indices a and b are positive and sum to less than 1.

Remark 2 The first order conditions (6) and (7) form a "hidden" linear system of equations. Namely, if you take the logarithms (introduced later!) you get the system

$$\begin{aligned}\ln\frac{1}{6}-\frac{5}{6}\ln K+\frac{1}{2}\ln L &= \ln\frac{1}{12} \\ \ln\frac{1}{2}+\frac{1}{5}\ln K-\frac{1}{2}\ln L &= \ln\frac{3}{12}\end{aligned}$$

which is linear in $\ln K$ and $\ln L$. One can solve this system for $\ln K$ and $\ln L$, which then gives us the values for L and K.