BEE1024 – Mathematics for Economists Week 14, Lecture Thursday 01.02.07	Dieter Balkenborg Department of Economics
Functions in two variables	University of Exeter

1 Objective

This lecture has the purpose to make you familiar with *functions in two variables*. *Partial differentiation* is the basic technique to analyze such functions. We will illustrate such functions using computer graphics. Of particular importance are *level curves*, which occur as indifference curves or isoquants in economics. Marginal rates of substitution correspond to the slopes of tangents to such curves. The lecture should enable you to calculate partial derivatives and marginal rates of substitution.

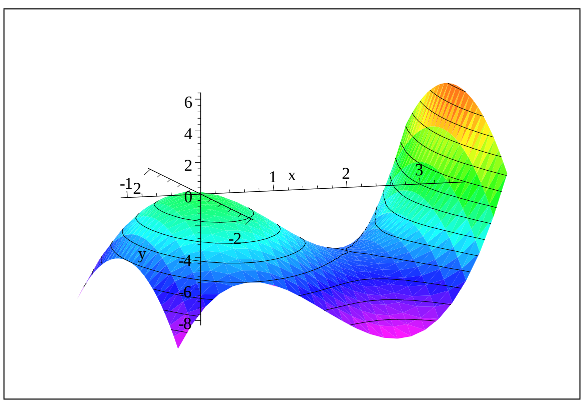
2 Functions in two variables

A *function* $z = f(x, y)$ or simply $z(x, y)$ *in two independent variables with one dependent variable* assigns to each pair (x, y) of (decimal) numbers from a certain domain D in the two-dimensional plane a number $z = f(x, y)$.[1] x and y are hereby the *independent variables* and z is the *dependent variable*. The *graph* of f is the surface in 3-dimensional space consisting of all points $(x, y, f(x, y))$ with (x, y) in D.

Example 1 The cubic polynomial

$$z = f(x, y) = x^3 - 3x^2 - y^2$$

is defined on the entire plane. It has the graph:



The vertical line is here the z-axis, the x- and the y- axis span a horizontal plane. Thus the height of the graph indicates the values of the function. Lighter colours correspond to higher values of the function. For instance, $f(0, 0) = 0$, so the origin of the coordinate system $(0, 0, 0)$ belongs to the graph and is a peak. The function has a saddle point at $(x, y) = (2, 0)$ where $z = f(x, y) = 8 - 3 \times 4 - 0 = -4$.

[1] The two-dimensional plane consisting of all pairs of numbers (x, y) is usually denoted by $\mathbb{R}^2$.

Exercise 1 Evaluate

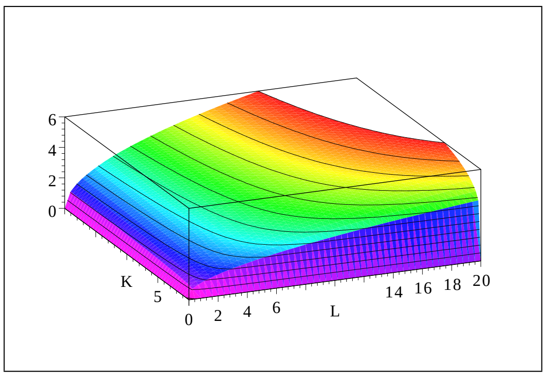
$$\begin{aligned} z &= f(2,1) \\ z &= f(3,0) \\ z &= f(4,-4) \\ z &= f(4,4) \end{aligned}$$

Could the information fit with the graph?

Example 2 A **production function** like

$$Q = \sqrt[6]{K}\sqrt{L} = K^{\frac{1}{6}} L^{\frac{1}{2}}$$

of a firm describes for each combination of capital ($K \geq 0$) and labour ($L \geq 0$) the total amount $Q \geq 0$ of output which the firm can produce with these inputs. The specified function has the graph:



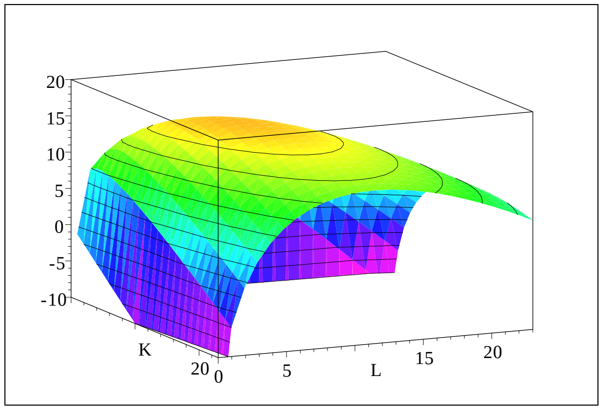
Example 3 Consider a firm with the above production function and assume that the firm is a price taker in the product market and in both factor markets. If P is the price of the commodity produced, r the interest rate (= the price of capital) and w the wage rate (= the price of labour) then the profit of this firm is

$$\begin{aligned} \Pi(K,L) &= PQ - rK - wL \\ &= PK^{\frac{1}{6}} L^{\frac{1}{2}} - rK - wL \end{aligned}$$

if it employs K units of capital and L units of labour. For the prices $P = 12$, $r = 1$, $w = 3$ we obtain the function

$$\Pi(K,L) = 12K^{\frac{1}{6}} L^{\frac{1}{2}} - K - 3L$$

with the graph:

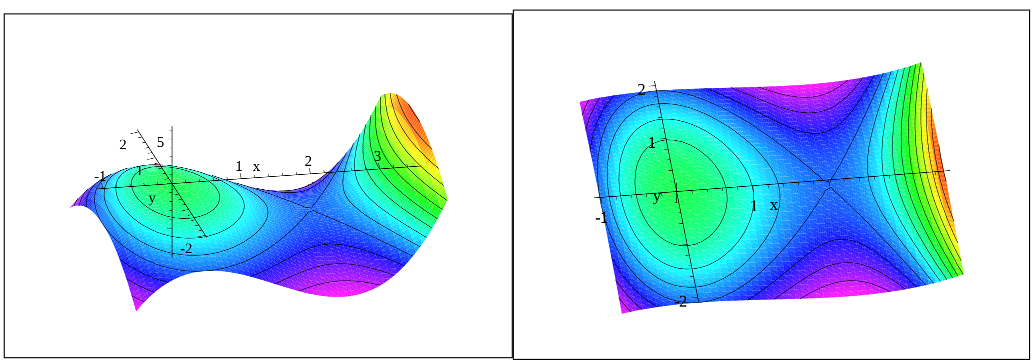


One can show that profits is maximized for $K = L = 8$.

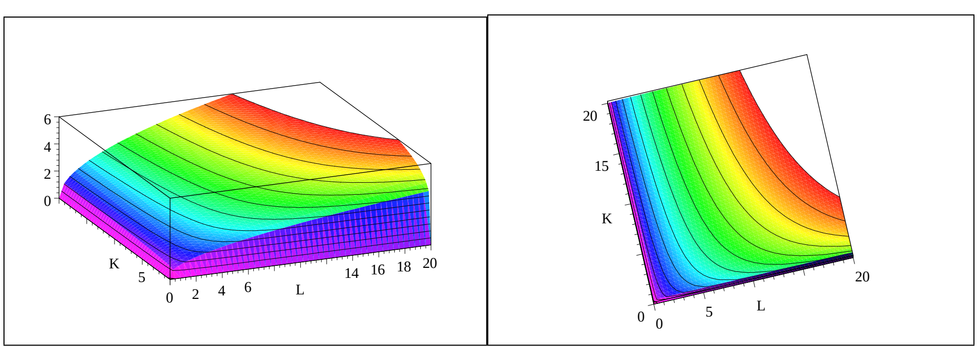
The *level curve* of the function $z = f(x, y)$ for the level c is the solution set to the equation

$$f(x, y) = c$$

where c is a given constant. Geometrically, a level curve is obtained by intersecting the graph of f with a horizontal plane $z = c$ and then projecting into the (x, y)-plane. This is illustrated here for the cubic polynomial discussed above:



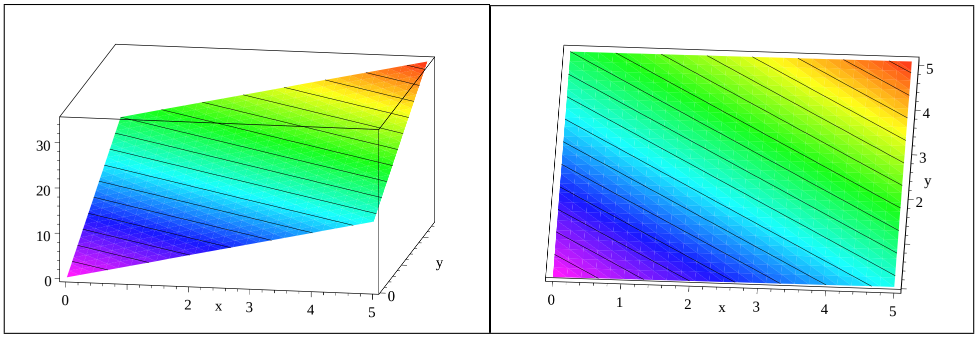
In the case of a production function the level curves are called *isoquants*. An isoquant shows for a given output level capital-labour combinations which yield the same output.



Finally, the linear function

$$z = 3x + 4y$$

has the graph and the level curves:



The level curves of a linear function form a family of parallel lines. This can be seen algebraically as follows: For a given level c the level curve is given by

$$\begin{aligned} c &= 3x + 4y \\ 4y &= c - 3x \\ y &= \frac{c}{4} - \frac{3}{4}x \end{aligned}$$

The level curves are therefore all lines with slope $-\frac{3}{4}$. They differ only in the intercept $\frac{c}{4}$. Hence they are all parallel lines.

Exercise 2 Describe the isoquant of the production function

$$Q = KL$$

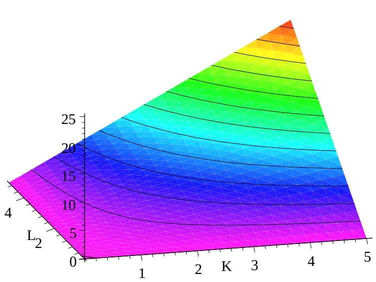
for the quantity $Q = 2$.

Exercise 3 Describe the isoquant of the production function

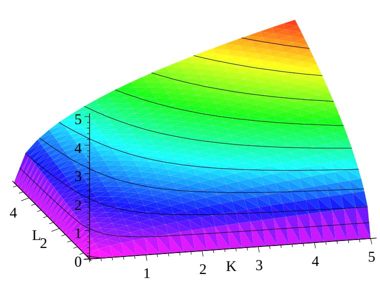
$$Q = \sqrt{KL}$$

for the quantity $Q = 4$.

Remark 1 The exercises illustrate the following general principle: If $h(z)$ is an increasing (or decreasing) function in one variable, then the composite function $h(f(x, y))$ has the same level curves as the given function $f(x, y)$. However, they correspond to different levels.



$Q = KL$



$Q = \sqrt{KL}$

3 Partial derivatives

Exercise 4 What is the derivative of

$$z(x) = a^3 x^2$$

with respect to x when a is a given constant?

Exercise 5 What is the derivative of

$$z(y) = y^3 b^2$$

with respect to y when b is a given constant?

Consider now a function in two variables $z = f(x, y)$. If we fix y at a value $y = y_0$ and vary only x we obtain a function in one variable $z = F(x) = f(x, y_0)$. The derivative of this function $F(x)$ at $x = x_0$ is called the *partial derivative* of f with respect to x and denoted by $\frac{\partial z}{\partial x}_{|x=x_0,y=y_0} = \frac{dF}{dx}_{|x=x_0} = \frac{dF}{dx}(x_0)$.[2] The partial derivative $\frac{\partial z}{\partial y}_{|x=x_0,y=y_0}$ is defined symmetrically. (Notation: "d"="dee", "δ"="delta", "∂"="del")

We do not have to specify y_0 numerically. To partially differentiate in the direction of x it suffices to *think* of y and all expressions containing only y as exogenously fixed constants. We can then use the familiar rules for differentiating functions in one variable in order to obtain $\frac{\partial z}{\partial x}$.

Other common notations for partial derivatives are $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ or f_x, f_y.

Example 4 Let

$$z(x, y) = y^3 x^2$$

Then

$$\frac{\partial z}{\partial x} = 2y^3 x \qquad \frac{\partial z}{\partial y} = 3y^2 x^2$$

Example 5 Let

$$z = x^3 + x^2 y^2 + y^4.$$

Setting e.g. $y = 1$ we obtain $z = x^3 + x^2 + 1$ and hence

$$\begin{aligned} \frac{\partial z}{\partial x}_{|y=1} &= 3x^2 + 2x \\ \frac{\partial z}{\partial x}_{|x=1,y=1} &= 5 \end{aligned}$$

For fixed, but arbitrary, y we obtain

$$\frac{\partial z}{\partial x} = 3x^2 + 2xy^2$$

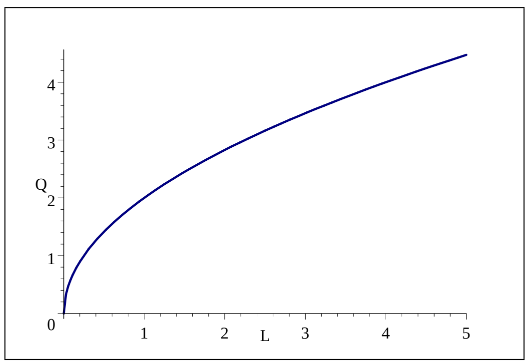
[2] From now on I will often write $\frac{dz}{dx}_{|x=x_0}$ instead of $\frac{dz}{dx}(x_0)$ to denote the derivative evaluated at a special value of x.

as follows: We can differentiate the sum $x^3 + x^2y^2 + y^4$ with respect to x term-by-term. Differentiating x^3 yields $2x^2$, differentiating x^2y^2 yields $2xy^2$ because we think now of y^2 as a constant and $\frac{d(ax^2)}{dx} = 2ax$ holds for any constant a. Finally, the derivative of any constant term is zero, so the derivative of y^4 with respect to x is zero.

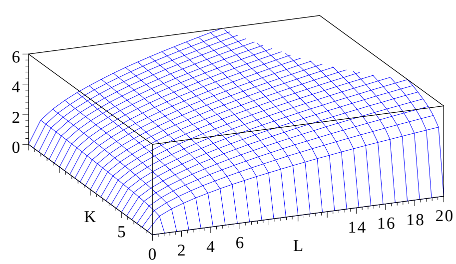
Similarly considering x as fixed and y variable we obtain

$$\frac{\partial z}{\partial y} = 2x^2y + 4x^3$$

Example 6 The partial derivatives $\frac{\partial q}{\partial K}$ and $\frac{\partial q}{\partial L}$ of a production function $q = f(K, L)$ are called the **marginal product of capital** and (respectively) **labour.** They describe approximately by how much output increases if the input of capital (respectively labour) is increased by a small unit. If one fixes for the production function in Example 2 capital input at $K = 64$ one obtains $q = K^{\frac{1}{6}}L^{\frac{1}{2}} = 2L^{\frac{1}{2}}$ which has, as a function in the one variable L, the graph



This graph is obtained from the graph of the function in two variables by intersecting the latter with a vertical plane parallel to L-q-axes.



thus the partial derivatives $\frac{\partial q}{\partial K}$ and $\frac{\partial q}{\partial L}$ describe geometrically the slope of the function in the K- and, respectively, the L- direction. Notice that for any given level of capital there is **diminishing productivity** of labour: The more labour is used, the less is the increase in output when one more unit of labour is employed. Algebraically this can be verified as follows:

$$\frac{\partial q}{\partial L} = \frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}} = \frac{1}{2}\frac{\sqrt[6]{K}}{\sqrt[2]{L}} > 0,$$

the marginal product of labour is always positive. Differentiating a second time we obtain

$$\frac{\partial^2 q}{\partial L^2} = \frac{\partial}{\partial L}\left(\frac{\partial q}{\partial L}\right) = -\frac{1}{4}K^{\frac{1}{6}}L^{-\frac{3}{2}} = -\frac{1}{4}\frac{\sqrt[6]{K}}{\sqrt[2]{L^3}} < 0,$$

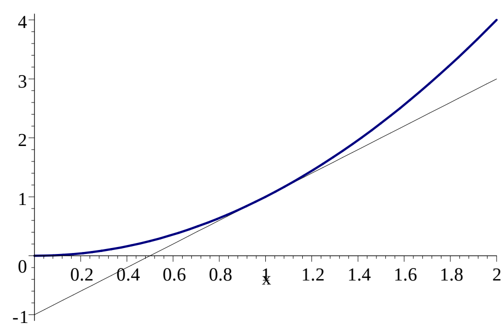
so the marginal product of labour is decreasing. Verify similarly that there is diminishing productivity of capital!

Exercise 6 Find the partial derivatives of

$$z = \left(x^2 + 2x\right)\left(y^3 - y^2\right) + 10x + 3y$$

4 The tangent plane

For a function $y = f(x)$ in one variable the derivative $\frac{df}{dx}_{|x=x_0}$ at x_0 is the slope of the tangent to the graph of f through the point $(x_0, f(x_0))$.



The tangent of $f(x) = x^2$ at $x = 1$.

This tangent is the graph of the linear function

$$y = l(x) = f(x_0) + \frac{df}{dx}_{|x=x_0}(x - x_0).$$

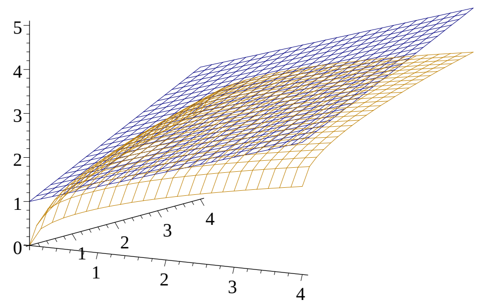
This is so because $l(x_0) = f(x_0)$, so the graph of l goes through the point $(x_0, f(x_0))$. Moreover,

$$y = l(x) = \left(f(x_0) - \frac{df}{dx_0}_{|x=x_0} x_0\right) + \frac{df}{dx}_{|x=x_0} x,$$

so $l(x)$ is indeed a linear function in x with intercept $\left(f(x_0) - \frac{df}{dx_0}_{|x=x_0} x_0\right)$ and the "right" slope $\frac{df}{dx}_{|x=x_0}$.

The graph of a function in one variable is a curve and its tangent is a line. By analogy, the graph of a function $z = f(x, y)$ in two variables is a surface and its tangent at a point

(x_0, y_0) is a *plane*.



Non-vertical planes are the graphs of linear functions $z = ax + by + c$. The tangent plane of the function $z = f(x, y)$ in (x_0, y_0) is, correspondingly, the graph of the linear function

$$z = l(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}_{|x=x_0, y=y_0} (x - x_0) + \frac{\partial f}{\partial y}_{|x=x_0, y=y_0} (y - y_0) \tag{1}$$

since the graph of this linear function contains the point $(x_0, y_0, f(x_0, y_0))$ and has the right slopes in the x- and y-directions.

Example 7 This is the example shown in the above graph. The tangent plane of $z = f(x, y) = \sqrt{x} + \sqrt{y}$ at the point $(1, 1)$ is obtained as follows: We have

$$\frac{\partial z}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x}} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{y}}$$

and hence

$$\frac{\partial z}{\partial x}_{|x=1, y=1} = \frac{1}{2} \qquad \text{and} \qquad \frac{\partial z}{\partial y}_{|x=1, y=1} = \frac{1}{2}.$$

Since $f(1, 1) = 2$ the tangent plane is the graph of the linear function

$$z = l(x, y) = 2 + \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) = 1 + \frac{1}{2}x + \frac{1}{2}y$$

Using the *abbreviations* $dx = x - x_0$, $dy = y - y_0$ and $dz = z - z_0$ the formula for the tangent can be neatly rewritten as

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{2}$$

an expression which is called the **total differential**.

5 The slope of level curves

Suppose (x_0, y_0) is a point on the level curve $f(x, y) = c$. Then the slope of tangent to the level curve in this point in the x-y-plane is

$$\frac{dy}{dx}_{|x=x_0,y=y_0} = -\left.\frac{\partial z}{\partial x}_{|x=x_0,y=y_0}\right/ \frac{\partial z}{\partial y}_{|x=x_0,y=y_0} \tag{3}$$

provided $\frac{\partial z}{\partial y}_{|x=x_0,y=y_0} \neq 0$. This is known as the *rule for implicit differentiation*: If the function $y = y(x)$ is implicitly defined by the equation $f(x, y(x)) = c$, then its derivative $\frac{dy}{dx}$ is given by the above formula.

Example 8 A linear function

$$z = ax + by + c$$

has the partial derivatives $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$. According to the above formula its level curves have the slope

$$-\left.\frac{\partial z}{\partial x}\right/ \frac{\partial z}{\partial y} = -\frac{a}{b}.$$

Indeed, its level curves are the solutions to the equations

$$\bar{c} = ax + by + c$$

with some constant $\bar{c}$. Solving the equation for y we obtain

$$y = -\frac{a}{b}x + \frac{\bar{c} - c}{b}$$

which is a linear function with slope $-\frac{a}{b}$.

Example 9 An isoquant of the production function $Q = K^{\frac{1}{6}}L^{\frac{1}{2}}$ has according to the above formula the slope

$$\begin{aligned} \frac{dK}{dL} &= -\left.\frac{\partial Q}{\partial L}\right/ \frac{\partial Q}{\partial K} = -\left.\left(\frac{1}{2}K^{\frac{1}{6}}L^{-\frac{1}{2}}\right)\right/ \left(\frac{1}{6}K^{-\frac{5}{6}}L^{\frac{1}{2}}\right) \\ &= -\frac{6}{2}K^{\frac{1}{6}}K^{\frac{5}{6}}L^{-\frac{1}{2}}L^{-\frac{1}{2}} = -3\frac{K}{L}. \end{aligned}$$

in the point (K, L).

Indeed, an isoquant is the set of solutions to the equation $\bar{q} = K^{\frac{1}{6}}L^{\frac{1}{2}}$ for fixed $\bar{q}$. Solving for K we see that the isoquant is the graph of the function

$$K = \left(\bar{q}L^{-\frac{1}{2}}\right)^6 = \bar{q}^6 L^{-3}.$$

Differentiation yields

$$\frac{dK}{dL} = -3\bar{q}^6 L^{-4}$$

and since (K, L) is on the isoquant

$$\frac{dK}{dL} = -3\left(K^{\frac{1}{6}}L^{\frac{1}{2}}\right)^6 L^{-4} = -3\frac{K}{L}.$$

Economically, $-\frac{dK}{dL}$ is the **marginal rate of substitution** of labour for capital: If labour is reduced by one unit, how much more capital is (approximately) needed to produce the same output? Just reducing labour by one unit reduces output by the marginal product of labour $\frac{\partial q}{\partial L}$. Each additional unit of capital produces $\frac{\partial q}{\partial K}$ – the marginal product of capital – more units. Therefore, if now capital input is increased by $\frac{\partial q}{\partial L} \Big/ \frac{\partial q}{\partial K}$ units, output changes overall by

$$-\frac{\partial q}{\partial L} + \left(\frac{\partial q}{\partial L} \Big/ \frac{\partial q}{\partial K}\right)\frac{\partial q}{\partial K} = 0.$$

Remark 2 A quick way to memorize the formula (3) is to use the total differential as follows: In order to stay on the level curve one must have $dz = z - z_0 = 0$, so

$$0 = dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

hence

$$dy = -\left(\frac{\partial z}{\partial x} \Big/ \frac{\partial z}{\partial y}\right)dx$$

or

$$\frac{dy}{dx} = -\frac{\partial z}{\partial x} \Big/ \frac{\partial z}{\partial y}.$$

Remark 3 The *implicit function theorem* states that if $\frac{\partial z}{\partial y}_{|x=x_0,y=y_0} \neq 0$ for some point (x_0, y_0) on a level curve $f(x, y) = c$, then one can find a function $y = g(x)$ defined near $x = x_0$ such that the level curve is locally the graph of this function, i.e. one has $f(x, g(x)) = c$ for x near x_0.

Exercise 7 For the production function

$$Q = \sqrt{KL}$$

calculate the marginal rate of substitution for $K = 3$ and $L = 4$.

6 The chain rule

Suppose that $z = f(x, y)$ is a function in two variables x, y. Suppose that x depends on the variable t via $x = g(t)$ and that y depends on the variable t via $y = h(t)$. Then we can define the *composite* function in one variable $z = F(t) = f(x(t), y(t))$ which has t as the independent and z as the dependent variable. The derivative of this function, if it exists, is calculated as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Notice that this is just the total differential divided by dt.

Example 10 Suppose $z = xy$, $x = t^3 + 2t$, $y = t^2 - t$. Then, according to the chain rule,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = y \times \left(3t^2 + 2\right) + x \times (2t - 1) \\ &= \left(t^2 - t\right)\left(3t^2 + 2\right) + (2t - 1)\left(t^3 + 2t\right)\end{aligned}$$

Since $z = xy = (t^3 + 2t)(t^2 - t)$ we get the same result using the product rule:

$$\frac{dz}{dt} = \left(3t^2 + 2\right)\left(t^2 - t\right) + \left(t^3 + 2t\right)(2t - 1).$$