BEE1024 MATHEMATICS FOR ECONOMISTS Mock 2008 - Solutions

Section A (Multivariate Functions and Partial Differentiation)

Question A1 (6 marks): A producer has the production function

$$Q = \sqrt[7]{KL^2}$$

1. Calculate the marginal products of labour and capital.
2. What is the marginal rate of substitution?
3. What is the marginal rate of substitution when 4 units of capital and 3 units of labour are used?
4. Describe the isoquant through the point $(4,3)$ as a univariate function $L = f(K)$.
5. Determine the derivative $f'(K)$. Calculate $f'(4)$.

Solution Notice that the production function can be written as

$$Q = K^{\frac{1}{7}} L^{\frac{2}{7}}$$

1.
$$\frac{\partial Q}{\partial K} = \frac{1}{7} K^{-\frac{6}{7}} L^{\frac{2}{7}} \qquad \frac{\partial Q}{\partial L} = \frac{2}{7} K^{\frac{1}{7}} L^{-\frac{5}{7}}$$

2. The marginal rate of substitution is
$$-\frac{dL}{dK} = \frac{\partial Q/\partial K}{\partial Q/\partial L} = \frac{\frac{1}{7}K^{-\frac{6}{7}}L^{\frac{2}{7}}}{\frac{2}{7}K^{\frac{1}{7}}L^{-\frac{5}{7}}} = \frac{1}{2}K^{-\frac{6}{7}}L^{\frac{2}{7}}K^{-\frac{1}{7}}L^{\frac{5}{7}} = \frac{1}{2}\frac{L}{K}$$

3. When 4 units of capital and 3 units of labour are used it is $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.

4.
$$\begin{aligned} \sqrt[7]{KL^2} &= \sqrt[7]{4 \times 3^2} \\ KL^2 &= 4 \times 3^2 = 2^2 \times 3^2 \\ L &= f(K) = (2 \times 3)/K^{1/2} = 6K^{-1/2} \end{aligned}$$

5.
$$f'(K) = -3K^{-3/2} \qquad f'(4) = -3\frac{1}{\sqrt{4}^3} = -\frac{3}{8}$$

Question A2 : (7 marks) Find all partial derivatives of the following functions.

A2.i. $z = 5x^3y^2 - 3x^2y^2 + 10x - 3y^3 + 5$

Solution:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(5x^3y^2 - 3x^2y^2 + 10x - 3y^3 + 5\right) = 15x^2y^2 - 6xy^2 + 10 \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(5x^3y^2 - 3x^2y^2 + 10x - 3y^3 + 5\right) = 10x^3y - 6x^2y - 9y^2\end{aligned}$$

A2.ii. $z = \frac{x+1}{x-1} - y\left(\ln(y) - 1\right)$

Solution:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x+1}{x-1} - y\left(\ln(y) - 1\right)\right) = \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2} \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+1}{x-1} - y\left(\ln(y) - 1\right)\right) = -\left(\ln(y) - 1\right) - y\left(\frac{1}{y}\right) = -\ln y\end{aligned}$$

A2.iii. $z = x\sqrt{1-y^2}$

Solution:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(x\sqrt{1-y^2}\right) = \sqrt{1-y^2} \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(x\left(1-y^2\right)^{\frac{1}{2}}\right) = x\left(\frac{1}{2}\left(1-y^2\right)^{-\frac{1}{2}}\right)(-2y) = -\frac{xy}{\sqrt{(1-y^2)}}\end{aligned}$$

Question A3 (6 marks): Calculate the Hessian matrix of the function

$$f(x,y) = \ln\left(2^x + y\right).$$

Is the function convex or concave near $x = y = 0$?

Solution: The rules $\frac{d\ln(x)}{dx} = \frac{1}{x}$ and $\frac{da^x}{dx} = \ln(a)\,a^x$ together with the chain rule yield

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{2^x + y} \times \frac{\partial}{\partial x}\left(2^x + y\right) = \frac{(\ln 2)\,2^x}{2^x + y} \\ \frac{\partial f}{\partial y} &= \frac{1}{2^x + y} \times \frac{\partial}{\partial y}\left(2^x + y\right) = \frac{1}{2^x + y}\end{aligned}$$

Using the quotient rule in addition we obtain

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\frac{\partial}{\partial x}\left(\ln(2)\,2^x\right) \times \left(2^x + y\right) - \ln(2)\,2^x \frac{\partial}{\partial x}\left(2^x + y\right)}{\left(2^x + y\right)^2} \\ &= \frac{(\ln 2)^2\,2^x\left(2^x + y\right) - (\ln 2)^2\left(2^x\right)^2}{\left(2^x + y\right)^2} = \frac{(\ln 2)^2\,2^x y}{\left(2^x + y\right)^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\frac{\partial}{\partial y}\left(\ln(2)\,2^x\right) \times \left(2^x + y\right) - \ln(2)\,2^x \frac{\partial}{\partial y}\left(2^x + y\right)}{\left(2^x + y\right)^2} = -\frac{(\ln 2)\,2^x}{\left(2^x + y\right)^2} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\frac{\partial}{\partial y}(1) \times \left(2^x + y\right) - 1 \times \frac{\partial}{\partial y}\left(2^x + y\right)}{\left(2^x + y\right)^2} = -\frac{1}{\left(2^x + y\right)^2}\end{aligned}$$

Evaluated at $x = y = 0$ we obtain (since $2^0 = 1$)

$$\frac{\partial^2 f}{\partial x^2}_{|(0,0)} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y}_{|(0,0)} = -\ln 2 \qquad \frac{\partial^2 f}{\partial y^2}_{|(0,0)} = -1$$

Hence the determinant of the Hessian matrix at $(0,0)$,

$$H = \begin{bmatrix} 0 & -\ln 2 \\ -\ln 2 & -1 \end{bmatrix},$$

is $-(\ln 2)^2$, which is negative. Therefore f is saddle-shaped, neither convex nor concave near $(0,0)$.

Question A4 (6 marks): Demand Q^d for apples depends as follows on the price of apples p, oranges p^* and income b of consumers:

$$Q^d = 500 - 2p^2 + 3p^* + 5b^3$$

Suppose $p = 4$, $p^* = 20$ and $b = 2$.

A4.i. (*3 marks*) Determine demand at the given prices and income.

Solution: $Q^d = 500 - 2 \times 4^2 + 3 \times 20 + 5 \times 2^3 = 568$

A4.ii. (*3 marks*) Determine own-price elasticity, cross price elasticity and income elasticities.

Solution:

$$\begin{aligned}
\frac{dQ^d}{dp} \times \frac{p}{Q^d} &= -4p \times \frac{p}{Q^d} = -16 \times \frac{4}{568} = -\frac{8}{71} \\
\frac{dQ^d}{dp^*} \times \frac{p^*}{Q^d} &= 3 \times \frac{20}{568} = \frac{15}{142} \\
\frac{dQ^d}{db} \times \frac{b}{Q^d} &= 15b^2 \times \frac{b}{Q^d} = 15 \times \frac{8}{568} = \frac{15}{71}
\end{aligned}$$

A4.iii. (*2 mark*) Which of the following terms are appropriate to describe the situation: complement / substitute, necessity/ luxury, elastic / inelastic demand, inferior / normal?

Solution: substitute, necessity, inelastic demand, normal good,

Section B (Optimization)

Question B1 : (10 marks) A monopolist sells the same commodity in two countries, A and B. When he sells the commodity at the price p in country A the quantity demanded in country A will be

$$x = 40 - p$$

When he sells the commodity at the price q in country B the quantity demanded in country B will be

$$y = 60 - 2q$$

When he produces on total $x + y$ units for the two countries his total production costs will be

$$\frac{1}{2}(x+y)^2$$

B1.i. Express the total revenue from sales in the two countries in terms of the variables x, y, p, q. Then, using the demand functions express total revenue as a function of the prices only.

Solution:

$$\begin{aligned} TR &= px + qy \\ TR(p,q) &= p(40-p) + q(60-2q) \end{aligned}$$

B1.ii. Assuming that he sells what he is producing, express total production costs as a function $TC(p,q)$ of the prices only.

Solution:

$$TC(p,q) = \frac{1}{2}(40 - p + 60 - 2q)^2 = \frac{1}{2}(100 - p - 2q)^2$$

B1.iii. Find a critical point of the profit function.

Solution:

$$\begin{aligned} \Pi(p,q) &= p(40-p) + q(60-2q) - \frac{1}{2}(100-p-2q)^2 \\ \frac{\partial \Pi}{\partial p} &= 40 - p - p - (100-p-2q)(-1) = 140 - 3p - 2q = 0 \\ \frac{\partial \Pi}{\partial q} &= 60 - 2q - 2q + 2(100-p-2q) = 260 - 8q - 2p = 0 \end{aligned}$$

The critical point solves the simultaneous system of equations

$$\begin{aligned} 3p + 2q &= 140 \\ 2p + 8q &= 260 \end{aligned}$$

This system has the solution $p = 30$, $q = 25$. (Test: $90 + 50 = 140$, $60 + 200 = 260$)

B1.iv. Use second partial derivatives and the Hessian matrix to show that the critical point is a relative profit maximum.

Solution:

$$\begin{bmatrix} \frac{\partial^2 \Pi}{\partial p^2} & \frac{\partial^2 \Pi}{\partial q \partial p} \\ \frac{\partial^2 \Pi}{\partial q \partial p} & \frac{\partial^2 \Pi}{\partial q^2} \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -2 & -8 \end{bmatrix}$$

The determinant is $24 - 4 = 20$ and $\frac{\partial^2 \Pi}{\partial p^2} < 0$. Hence the quadratic profit function is concave and has a profit maximum at the critical point.

B1.v. Calculate the demand elasticities $\frac{\partial x}{\partial p}\frac{p}{x}$ and $\frac{\partial y}{\partial q}\frac{q}{y}$ in the two countries at the prices set in the profit optimum. Is demand there elastic of inelastic or elastic?

Solution: Demand in country A is 10 at the price $p = 30$. Since $\frac{dx}{dp} = -1$ demand elasticity is therefore $\frac{dx}{dp}\frac{p}{x} = -\frac{30}{10} = -3$ and hence elastic. Demand in country B at the price $q = 25$ is $y = 10$. Since $\frac{dy}{dq} = -2$ the demand elasticity is $\frac{dy}{dq}\frac{q}{y} = -2 \times \frac{25}{10} = -5$ and also elastic.

Question B2 : (10 marks) A consumer has the following utility when he consumes x units of apples and y units of oranges:

$$u(x, y) = -x^2 + 4x - y^2 + 16y$$

Suppose the consumer has a budget of £3.20 to be spend on oranges and apples. Each apple and each orange costs £0.40. Use the method of Lagrange to find the optimal consumption bundle:

B2.i. Write down the budget constraint and the Lagrangian.

Solution: The budget equation is $0.4x + 0.4y = 3.2$. The Lagrangian is therefore

$$\mathcal{L}(x, y) = -x^2 + 4x - y^2 + 16y + \lambda_1 (3.2 - 0.4x - 0.4y) + \lambda_2 x + \lambda_3 y$$

B2.ii. Write down the first-order conditions for a critical point for the Lagrangian. Find a condition for a critical point that does not involve the Lagrange multiplier.

Solution: We assume that none of the non-negativity constraints is binding and hence $\lambda_2 = \lambda_3 = 0$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -2x + 4 - 0.4\lambda_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= -2y + 16 - 0.4\lambda_1 = 0 \\ -2x + 4 &= -2y + 16 \\ -2x + 2y &= 12 \\ -x + y &= 6 \end{aligned}$$

B2.iii. Use the latter condition and the budget constraint to find a candidate for the optimum. (One can show that it is indeed the optimum.)

Solution:

$$\begin{aligned} -x+y &= 6 \\ x+y &= 8 \\ 2y &= 14 \qquad y=7 \qquad x=1 \end{aligned}$$

Question B3 : (5 marks) A firm has the revenue function has the revenue function $TR(Q)$ where Q is the quantity of output produced. The total costs are given by $TC(Q,c) = cQ$ where $c > 0$ is the constant marginal cost. Assuming that the profit optimum is uniquely characterized by the first order condition, show that the optimal profit

$$\Pi^*(c) = \max_Q [TR(Q) - TC(Q,c)]$$

is decreasing in c.

Answer B3 : In optimum $\frac{\partial \Pi}{\partial Q} = 0$ where $\Pi(Q,c) = TR(Q) - TC(Q,c)$ and hence by the chain rule it holds at the optimal $Q^*(c)$ that

$$\frac{d\Pi^*}{dc} = \frac{\partial \Pi}{\partial Q} \times \frac{dQ^*}{dc} + \frac{\partial \Pi}{\partial c} \times \frac{dc}{dc} = \frac{\partial \Pi}{\partial c} = -\frac{dTC}{dc} = -Q < 0$$

(The identity of the first and third term is the envelope theorem.). Thus $\Pi^*(c)$ is decreasing in c.

Section C (Logarithms, Exponentials and Integrals)

Question C1 : (5 marks) How quickly will money triple if it is invested at a nominal annual interest rate of 3 percent when interest is compounded (a) continuously, (b) quarterly, (c) annually?

Solution:

$$\begin{aligned} \text{a)} \quad 3 &= e^{0.03t} \Leftrightarrow t = \frac{\ln 3}{0.03} \approx 36.52 \\ \text{b)} \quad 3 &= \left(1 + \frac{0.03}{4}\right)^{4t} \Leftrightarrow t = \frac{\ln 3}{4 \times \ln\left(1 + \frac{0.03}{4}\right)} \approx 36.758 \\ \text{c)} \quad 3 &= (1+0.03)^t \Leftrightarrow t = \frac{\ln 3}{\ln 1.03} \approx 37.167 \end{aligned}$$

Question C2 : (5 marks) Use logarithmic differentiation to find the derivative of the function

$$y = (2x+2)^{-5}(x+2)^{\frac{7}{2}}$$

Solution :

$$\begin{aligned} \ln y &= \ln\left((2x+2)^{-5}(x+2)^{\frac{7}{2}}\right) = -5\ln(2(x+1)) + \frac{7}{2}\ln(x+2) \\ \frac{d\ln y}{dx} &= -5\frac{2}{2x+2} + \frac{7}{2}\frac{1}{x+2} \\ \frac{dy}{dx} &= y(x)\left(\frac{d\ln y}{dx}\right) = (2x+2)^{-5}(x+2)^{\frac{7}{2}}\left[-5\frac{1}{x+1} + \frac{7}{2}\frac{1}{x+2}\right] \end{aligned}$$

At no extra marks (!) you may simplify this to

$$\frac{dy}{dx} = -\frac{1}{64(x+1)^6}(x+2)^{\frac{5}{2}}(3x+13)$$

Question C3 : (5 marks) An epidemic spreads throughout a community so that t weeks after its outbreak, the number of residents who have been infected is given by a function of the form

$$f(t) = \frac{A}{1+Ce^{-kt}}$$

where A is the total number of susceptible residents. Show that the epidemic is spreading most rapidly when half of the residents have been infected.

Solution : We have to show that the first derivative

$$f'(t) = \frac{ACke^{-kt}}{(1+Ce^{-kt})^2}$$

is maximized where $f(t) = A/2$, i.e. where $Ce^{-kt} = 1$ or $t = \frac{1}{k}\ln C$. The second derivative is

$$\begin{aligned} f''(t) &= \frac{-ACk^2e^{-kt}\left(1+Ce^{-kt}\right)^2 + 2AC^2k^2e^{-2kt}\left(1+Ce^{-kt}\right)}{(1+Ce^{-kt})^4} \\ &= \frac{ACk^2e^{-kt}}{(1+Ce^{-kt})^3}\left(-1-Ce^{-kt}+2Ce^{-kt}\right) = \frac{ACk^2e^{-kt}}{(1+Ce^{-kt})^3}\left(Ce^{-kt}-1\right) \end{aligned}$$

For $t < \frac{1}{k}\ln C$ we have $Ce^{-kt} > 1$, so $f''(t) > 0$ and hence $f'(t)$ is increasing in this range. For $t > \frac{1}{k}\ln C$ we have $Ce^{-kt} < 1$, so $f''(t) > 0$ and hence $f'(t)$ is decreasing in this range. Thus $f'(t)$ has a maximum at $t = \frac{1}{k}\ln C$ where $Ce^{-kt} = 1$ and hence $f''(t) = 0$.

Question C4 : (5 marks) Evaluate the integrals

C4.i. $\int\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}\right)dx$

Solution:

$$\begin{aligned} \int\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}\right)dx &= \int\left(x^{-3}+x^{-2}+x^{-1}\right)dx = -\frac{1}{2}x^{-2} + (-1)x^{-1} + \ln|x| + C \\ &= -\frac{1}{2x^2} - \frac{1}{x} + \ln|x| + C \end{aligned}$$

C4.ii. $\int\left(x^3-4\right)dx$

Solution:

$$\int \left(x^3 - 4\right) dx = \frac{1}{4}x^4 - 4x + C$$

C4.iii. $\int_{-1}^{3} \left(x^3 - 4\right) dx$

Solution:

$$\int_{-1}^{3} \left(x^3 - 4\right) dx = \left[\frac{1}{4}x^4 - 4x\right]_{-1}^{3} = \left(\frac{81}{4} - 12\right) - \left(\frac{1}{4} + 4\right) = 4$$

Question C5 : (5 marks)

C5.i. (*2 marks*) Use the substitution $u = 1 - 2x$ to integrate $\int 6e^{1-2x} dx$. Use the substitution $u = (x+2)^2$ to integrate $\int \frac{e^{(x+2)^2}}{4} (x+2)\, dx$.

Solution:

$$\begin{aligned} u &= 1 - 2x \\ du &= -2dx \\ \int 6e^{1-2x} dx &= -\frac{6}{2} \int e^u du = -3e^u + C = -3e^{1-2x} + C \end{aligned}$$

$$\begin{aligned} u &= (x+2)^2 \\ du &= 2(x+2)\, dx \\ \int \frac{e^{(x+2)^2}}{4} (x+2)\, dx &= \int \frac{e^u}{4} (x+2) \frac{du}{2(x+2)} = \frac{1}{8} \int e^u du = \frac{1}{8} e^{(x+2)^2} + C \end{aligned}$$

C5.ii. Use integration by parts to evaluate $\int (1 \times \ln x)\, dx$. Use integration by parts to evaluate $\int \ln(x+2)\, dx$

Solution:

$$\begin{aligned} u = x \quad & v = \ln x \\ u' = 1 \quad & v' = \tfrac{1}{x} \end{aligned}$$

$$\begin{aligned} \int (1 \times \ln x) &= x \ln x - \int x \times \frac{1}{x} dx = x \ln x - \int 1 dx \\ &= x \ln x - x + C = x(\ln x - 1) + C \end{aligned}$$

$$\begin{aligned} u = x + 2 \quad & v = \ln(x+2) \\ u' = 1 \quad & v' = \tfrac{1}{x+2} \end{aligned}$$

$$\begin{aligned} \int (1 \times \ln(x+2)) &= (x+2)\ln(x+2) - \int (x+2) \times \frac{1}{x+2} dx = (x+2)\ln(x+2) - \int 1 dx \\ &= (x+2)\ln(x+2) - (x+2) + C = (x+2)(\ln(x+2) - 1) + C \end{aligned}$$

Section D (Linear Algebra)

Question D1. (*6 marks*) Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

D1.i. Which of the following products exist? ABC, $AB'C$, $AB'C'$, $B'CA$

Solution:

ABC	$AB'C$	$AB'C'$	$B'CA$
$3\times 2, 4\times 2, 4\times 3$	$3\times 2, 2\times 4, 4\times 3$	$3\times 2, 2\times 4, 3\times 4$	$2\times 4, 4\times 3, 3\times 2$
no	yes	no	yes

D1.ii. Evaluate the products in a) when they exist.

Solution:

$$\begin{aligned} AB'C &= \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 1 \\ 8 & 7 & 4 \\ 10 & 11 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B'CA &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 0 & 0 \\ 10 & 11 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 13 \\ 4 & 3 \end{bmatrix} \end{aligned}$$

Question D2 : (6 marks) Use matrix algebra to find the solution of the simultaneous system of equations

$$\begin{aligned} 3x + 5y &= 7 \\ 7x - 13y &= 2 \end{aligned}$$

In doing so, find the inverse of the coefficient matrix. Express your solution using fractions, not decimal numbers.

Solution:

$$
\begin{aligned}
A\vec{x} &= \begin{bmatrix} 3 & 5 \\ 7 & -13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \\
A^{-1} &= \frac{1}{\det A} \operatorname{ad}(A) = \frac{1}{3 \times (-13) - 5 \times 7} \begin{bmatrix} -13 & -5 \\ -7 & 3 \end{bmatrix} \\
&= \frac{1}{-74} \begin{bmatrix} -13 & -5 \\ -7 & 3 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{-74} \begin{bmatrix} -13 & -5 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \frac{1}{-74} \begin{bmatrix} -101 \\ -43 \end{bmatrix} = \begin{bmatrix} \frac{101}{74} \\ \frac{43}{74} \end{bmatrix}
\end{aligned}
$$

Question D3. (7 marks) Calculate the inverse of the matrix

$$
A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}
$$

and use this information to solve the linear simultaneous system of equations

$$
\begin{aligned}
y + 2z &= 4 \\
x + z &= 4 \\
2x + y &= 8
\end{aligned}
$$

Solution: The matrices of 2×2-minors, cofactors and the adjoint are

$$
M = \begin{bmatrix} -1 & -2 & 1 \\ -2 & -4 & -2 \\ 1 & -2 & -1 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \ \operatorname{ad} A = C' = C
$$

Since

$$
A\,(\operatorname{ad} A) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}
$$

the determinant of A is 4 and the inverse is

$$
A^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}
$$

The system of equations has the solution

$$
\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}
\end{aligned}
$$

Question D4. (12 marks) Consider the matrix

$$P = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

D4.i. Calculate P^2. What is the inverse of P? What is the determinant?

Solution:

$$\begin{aligned} P^2 &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Id_2 \\ \det P &= -\frac{9}{25} - \frac{16}{25} = -1 \end{aligned}$$

Since $PP = Id_2$, $P^{-1} = P$ by definition.

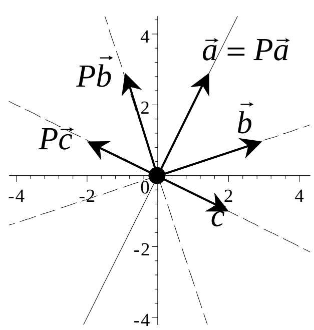
D4.ii. Draw the six vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ \vec{c} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$P\vec{a}$, $P\vec{b}$ and $P\vec{c}$ on graph paper. What is the geometric interpretation of the mapping from any vector $\vec{x}$ to the vector $P\vec{x}$?

Solution:

$$\begin{aligned} P\vec{a} &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ P\vec{b} &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ P\vec{c} &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{aligned}$$



Every vector is mirrored at the line $y = 2x$

D4.iii. Show that for any vector two vectors

$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \ \vec{y} = \begin{bmatrix} c \\ d \end{bmatrix}$$

that $(P\vec{x})' \, (P\vec{y}) = \vec{x}'\vec{y}$.

Solution:

$$\begin{aligned}
(P\vec{x})'(P\vec{y}) &= \left(\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right)'\left(\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}\right) \\
&= \begin{bmatrix} -\frac{3}{5}a + \frac{4}{5}b \\ \frac{4}{5}a + \frac{3}{5}b \end{bmatrix}'\begin{bmatrix} -\frac{3}{5}c + \frac{4}{5}d \\ \frac{4}{5}c + \frac{3}{5}d \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{5}a + \frac{4}{5}b & \frac{4}{5}a + \frac{3}{5}b \end{bmatrix}\begin{bmatrix} -\frac{3}{5}c + \frac{4}{5}d \\ \frac{4}{5}c + \frac{3}{5}d \end{bmatrix} \\
&= \left(-\frac{3}{5}a + \frac{4}{5}b\right)\left(-\frac{3}{5}c + \frac{4}{5}d\right) + \left(\frac{4}{5}a + \frac{3}{5}b\right)\left(\frac{4}{5}c + \frac{3}{5}d\right) \\
&= \left(\frac{9}{25}ac - \frac{12}{25}ad - \frac{12}{25}bc + \frac{16}{25}bd\right) + \left(\frac{16}{25}ac + \frac{12}{25}ad + \frac{12}{25}bc + \frac{9}{25}bd\right) \\
&= ac + bd \\
\vec{x}'\vec{y} &= \begin{bmatrix} a \\ b \end{bmatrix}'\begin{bmatrix} c \\ d \end{bmatrix} = ac + bd
\end{aligned}$$