1 U-shaped average variable costs

The third example of a total cost function discussed in the first handout, week 6, was

\[ TC(Q) = 2Q^3 - 18Q^2 + 60Q + 50 \]

We want to know which quantity a profit-maximizing firm with this cost function should produce when the market is perfectly competitive and the given market price is \( P \).

It turns out that the answer to this question depends on the average variable costs (AVC) and the marginal costs (MC). Hence, we must first discuss how the average variable costs curve looks and how it relates to the marginal costs curve.

In our example, the fixed costs are \( FC = 50 \) and the variable costs are hence

\[ VC(Q) = 2Q^3 - 18Q^2 + 60Q. \]

Average costs are generally costs per item produced, so the average variable cost function is in our example

\[ AVC(Q) = \frac{VC(Q)}{Q} = 2Q^2 - 18Q + 60. \]

As the graph indicates, the AVC curve is U-shaped, i.e., it is strictly convex and has a unique absolute minimum at \( Q^{Min} = 4.5 \). The minimum average variable costs are calculated as

\[ AVC^{Min} = AVC(4.5) = 19.5 \]

To see algebraically that the AVC curve is indeed U-shaped with the describe properties we a) differentiate

\[ AVC'(Q) = 4Q - 18, \]

b) solve the first order condition

\[ AVC'(Q) = 4Q - 18 = 0 \quad \text{or} \quad Q = \frac{18}{4} = 4.5, \]
c) observe that there is a unique solution at 4.5,
d) differentiate again

\[ AVC''(Q) = 4 > 0 \]

and observe hence that our function is indeed strictly convex. In particular, \( Q^{Min} = 4.5 \) is the absolute minimum.

Recall that the marginal costs are the derivative of the total or variable costs (the latter two differ only by a constant term). They are

\[ MC(Q) = \frac{dT}{dQ} = \frac{dV}{dQ} = 6Q^2 - 36Q + 60 \]

2 The relation between AVC, MC and supply

Whenever the AVC curve is U-shaped, i.e., strictly convex with a unique absolute minimum, the following applies:

1) The AVC curve and the MC curve intersect in two points, once on the vertical axis and one in the minimum of the AVC curve.
2) In the downward-sloping part of the AVC curve the MC curve is below the AVC curve, in the upward-sloping part it is above.
3) Above the AVC curve marginal costs are strictly increasing.

The following picture illustrates these facts in our example:

Moreover,

4) The individual supply curve is given by the part of the MC curve above the AVC curve. More precisely:
A) When the price is below the minimum average variable costs, it is optimal for the firm not to produce any output.
B) When the price is above the minimum average variable costs, it is optimal for the firm to produce a positive amount of output. Namely, it is optimal to produce the largest quantity for which the price equals the marginal costs.
C) When the price is exactly equal to the minimum average variable costs, two quantities are optimal to produce, namely zero and the quantity which minimizes AVC.

Applied to our example this means the following:
At prices below 19.5 it is optimal to produce zero.
When the price is exactly 19.5, both \( Q = 0 \) and \( Q = 4.5 \) are optimal. When the price is, for instance, \( P = 30 \) we must first solve the equation \( P = MC(Q) \) or

\[
30 = 6Q^2 - 36Q + 60 \\
0 = 6Q^2 - 36Q + 30 = 6(Q^2 - 6Q + 5) = 6(Q - 1)(Q - 2)
\]

Here both \( Q = 1 \) and \( Q = 5 \) solve this equation. The larger of the two, \( Q = 5 \), is the profit maximizing quantity.

Using the general formula to solve quadratic equations one can obtain the supply function explicitly as follows:

\[
P = 6Q^2 - 36Q + 60 \\
0 = 6Q^2 - 36Q + 60 - P \\
0 = Q^2 - 6Q + 10 - \frac{P}{6}
\]

\[
Q_{1/2} = \frac{-6 \pm \sqrt{36 - 4\left(10 - \frac{P}{6}\right)}}{2} = -3\sqrt{\frac{36 - 4\left(10 - \frac{P}{6}\right)}{4}} \\
= -3 \pm \sqrt{9 - 10 + \frac{P}{6}}
\]

and, by taking the larger root, one obtains the supply function

\[
Q^S = 3 + \sqrt{\frac{P}{6} - 1}
\]

valid for prices above 19.5.

### 3 Sketch of arguments

Read this part only if you like math!

Finally we indicate why the four facts stated above hold. For a more verbal presentation see Begg, Fischer, and Dornbusch (2000).

Variable costs are, by definition, the product of quantity and average variable costs:

\[
VC(Q) = Q \times AVC(Q)
\]

We can differentiate this equation using the product rule and obtain

\[
MC(Q) = AVC(Q) + Q\frac{dAVC}{dQ}
\]

From this equation we see that marginal costs are equal to average variable costs at the minimum of the AVC curve (since there \( \frac{dAVC}{dQ} = 0 \)), they are below the AVC curve when
the latter is downward-sloped \((\frac{dAVC}{dq} < 0)\) and above when the latter is upward sloped \((\frac{dAVC}{dq} > 0)\).\(^1\)

Differentiating again gives

\[
\frac{dMC}{dq} = \frac{dAVC}{dq} + Q\frac{d^2AVC}{dq^2} = 2\frac{dAVC}{dq} + Q\frac{d^2AVC}{dq^2}
\]

We have \(Q > 0\) and, since the AVC curve is strictly convex, \(\frac{d^2AVC}{dq^2} \geq 0\). In the upward-sloping part of the AVC curve we have \(\frac{dAVC}{dq} > 0\) and get overall \(\frac{dMC}{dq} > 0\), i.e., the marginal cost curve is increasing above the AVC curve.

To see that the AVC curve and the MC curve meet on the vertical axis one has to know the definition of the derivative as a limit of difference quotient (or “rates of change”) (see Hoffmann and Bradley (2000), Chapter 2, Section 1). Actually, \(AVC (Q) = \frac{VC(Q)}{Q} = \frac{VC(Q) - VC(0)}{Q - 0}\) is a difference quotient at zero and therefore

\[
MC (0) = \frac{dVC}{dq} (0) = \lim_{Q \to 0} \frac{VC (Q) - VC (0)}{Q - 0} = \lim_{Q \to 0} AVC (Q)
\]

\((AVC (0)\) is, of course, not defined.\)

We have shown the statements 1 - 3 above.

Concerning statement 4 I skip the very technical argument why an absolute profit maximum always exists when the AVC curve is U-shaped. (Essentially one can show that the profit function must be decreasing for very large quantities.) Assuming it exists, it can either be at \(Q = 0\) or it can be at a positive quantity. In the latter case it must be a “peak” and hence the first order condition \(P = MC (Q)\) must be satisfied. It follows that the part of a supply curve where a strictly positive quantity is produced must be a part of the marginal cost curve.

When zero output is produced, only the fixed costs are to be paid: \(\Pi (0) = -FC\). For \(Q > 0\) we can rewrite the profit function as follows:

\[
\Pi (Q) = PQ - VC(Q) - FC = PQ - Q \times AVC(Q) - FC
\]

\[
= Q (P - AVC(Q)) - FC
\]

For prices below the minimum average variable costs \(P - AVC(Q)\) is negative for all quantities \(Q > 0\). Therefore \(\Pi (Q) < -FC = \Pi (0)\) and it it is optimal to produce zero. In words: one loses on average more on variable costs per item produced than one gains in revenues and hence it is better to produce nothing. (The fixed costs must be paid anyway.)

For prices \(P > AVC_{Min}\) only the largest solution to the equation \(P = MC (Q)\) gives a point on the MC curve which is above the AVC curve. For this solution \(P = MC (Q) > AVC (Q)\) is satisfied and hence \(\Pi (Q) > \Pi (0)\). For all other solutions \(\Pi (Q) < \Pi (0)\). Hence this solution is the only candidate for the profit maximum. Since we assumed one, this must be it.

When \(P = AVC_{Min}\) one has \(P = MC (Q_{Min}) = AVC (Q_{Min})\). Hence \(\Pi (0) = \Pi (Q_{Min})\). All other critical points of the profit function can be ruled out, so these two quantities must be optimal.

\(^1\)The AVC curve cannot have saddle points since it is assumed to be strictly convex. This rules out \(\frac{dAVC}{dq} = 0\) except for the minimum.
References
