1 Overview on the lecture

The aim of this first lecture is to introduce on an intuitive level the notion of a function\(^1\) which is basic for all of calculus and some concepts associated with it. As illustrative examples we will consider cost functions which are needed in microeconomics to discuss the behaviour of firms. At the end of this lecture you should have a basic idea of the following concepts:

- functions and their domains, intervals
- the independent and the dependent variable
- the graph of a function
- linear and quadratic functions, polynomial functions
- the difference quotient
- the tangent and the slope
- increasing and decreasing functions
- convex and concave functions (upward and downward bowed)
- the first and the second derivative

It is important that you memorize these concepts and their meaning because we will expand and build on them in the lectures to follow.

2 Examples of cost functions

A function describes how one quantity changes in response to another quantity. An example is the total cost function of a firm. Consider, for instance, a publisher selling a particular newspaper. His production costs depend on the number of newspapers he prints. This information – together with information on the demand side – will be important if the publisher tries to make a profit out of his business.

In order to maximize profits the publisher must know the relation between the following two variables:

1. the number of newspapers he wants to produce, the quantity of output. This is the independent variable in this example, the producer can choose it freely.

\[^1\text{To be precise we discuss functions with one dependent and one independent variable. In later lectures we will consider functions with several independent and also with several dependent variables.}\]
2. the total costs of producing a given amount of newspapers. This is the dependent variable in our example. It’s value depends on how many newspapers the publisher decides to produce.

There are three ways to describe the relation between production costs and the number of newspapers produced:

1. by a table,
2. by a graph,
3. using an algebraic expression to describe the relationship.

The first two ways appear natural, but it is the third, most compact, way of describing the relationship on which we concentrate in this course. Here are three examples of types of cost functions frequently used in microeconomics. The terminology used will become clear during the lecture.

### 2.1 Example 1: Constant marginal costs

In tabular form:

<table>
<thead>
<tr>
<th>quantity (in 100.000)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>total costs (in 1000£)</td>
<td>90</td>
<td>110</td>
<td>130</td>
<td>150</td>
<td>170</td>
<td>190</td>
<td>210</td>
<td>230</td>
</tr>
</tbody>
</table>

With the aid of a graph:

In algebraic form:

\[
TC(Q) = 90 + 20Q
\]

### 2.2 Example 2: Increasing marginal costs

In tabular form:

<table>
<thead>
<tr>
<th>quantity (in 100.000)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>total costs (in 1000£)</td>
<td>110</td>
<td>135</td>
<td>170</td>
<td>215</td>
<td>270</td>
<td>335</td>
<td>410</td>
<td>495</td>
</tr>
</tbody>
</table>
With the aid of a graph:

\[ TC(Q) = 5Q^2 + 20Q + 110 \]

**2.3 Example 3: U-shaped marginal costs**

In tabular form:

<table>
<thead>
<tr>
<th>quantity (in 100,000)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>total costs (in 1000£)</td>
<td>50</td>
<td>94</td>
<td>114</td>
<td>122</td>
<td>130</td>
<td>150</td>
<td>194</td>
<td>274</td>
</tr>
</tbody>
</table>

With the aid of a graph:

\[ TC(Q) = 2Q^3 - 18Q^2 + 60Q + 50 \]

**3 Functions**

**3.1 Concept and Notation**

A *function* is a rule which specifies for each object in a set $A$ exactly one object in a the set $B$. The set $A$ is called the *domain* and the set $B$ the *co-domain*. 
In this course $A$ and $B$ are mostly subsets of the number line. For a costs function domain and co-domain are the set of non-negative numbers because neither quantities nor costs can be negative numbers. It is important to understand that a function is never completely described just by a formula like $y = f(x) = x^2 + 1$. One has to name the domain and co-domain as well. However, what is the domain or co-domain is often implicitly clear and hence not mentioned.

Three types of notations are common to denote functions:

a) The inventors of calculus – Isaac Newton (1643 – 1727) and Gottfried Wilhelm Leibniz (1646 – 1716) – used the notation $y(x)$ where $y$ is called the dependent variable and $x$ the independent variable. For instance, let

$$y(x) = x^2 + 1.$$  

Then the value of the variable $y$ depends on the value of the variable $x$ according to the formula on the right, so for $x = 1$ we have $y = 2$, for $x = 3$ we have $y = 10$ and so on, which can also be written as $y(1) = 2$ and $y(3) = 10$. We used this notation above to describe the costs functions: The dependent variable $TC$ denoted total costs and the independent variable $Q$ the quantity produced.

b) Slightly more modern and more explicit is the notation

$$y = f(x) = x^2 + 1.$$  

Again, $y$ and $x$ denote the dependent and independent variable and hence represent numbers. The letter $f$ does, however, not represent a number, but a relationship described by a formula.

$$y = \frac{x^2 + 1}{f(x)}$$

This is the most frequently used notation which we will also adapt.

As mentioned, a function is only completely specified if besides the rule its domain and co-domain are fixed. The above notations require us to deduce domain and co-domain from the context. For instance, when

$$y(x) = \sqrt{x - 1}$$

the domain has to be the set of all numbers bigger or equal to 1 because negative numbers have no roots. As a second example, the function

$$y(x) = 2x^3 - 18x^2 + 60x + 50$$

is defined for all numbers, so we should take the whole number line as the domain and the co-domain of the function. However, when we write

$$TC(Q) = 2Q^3 - 18Q^2 + 60Q + 50$$

and deal with total cost functions it is implicit that the domain and the range are the sets of all non-negative numbers.
c) Most modern, and designed for those who demand complete rigour, is the notation

$$f : A \rightarrow B$$

$$x \mapsto f(x)$$

where $f$ is the name of the function, $A$ is the domain and $B$ the co-domain. For instance

$$f : \{x \geq 1\} \rightarrow \{y \geq 0\}$$

$$x \mapsto \sqrt{x-1}$$

specifies the rule, the domain and the co-domain. (Here the curly brackets indicate a set. So $\{x \geq 1\}$ is the set of all numbers not smaller than one.) We will not use this notation.

### 3.2 Graphs of functions

The graph of a function $y = f(x)$ is the curve consisting of all points $(x, y) = (x, f(x))$ drawn in coordinate system with $x$ on the horizontal and $y$ on the vertical axis where $x$ varies over the domain of the function.

Graphs quickly reveal information which is not obvious from a table of the algebraic description of a function.

A curve or merely a collection of dots?

**The Vertical Line Test:** A curve is the graph of a function if and only if no vertical line intersects the curve more than once.

#### 3.2.1 Inverse functions

To illustrate the vertical line test, consider what happens to the graph of the function if we invert the graph in the sense that we interchange the horizontal and the vertical axis. A point $(x, y)$ then becomes the point $(y, x)$, for instance $(-2, 4)$ becomes $(4, -2)$. As the
result, the graph is mirrored at the 45°-line.

Inverting a graph.

The U-shaped curve in this figure on the left is the graph of the _square function_ \( y = x^2 \). The mirrored C-shaped curve is not the graph of a function because it fails the vertical line test. This is so because every positive number \( y \geq 0 \) has two roots \( \pm \sqrt{y} \), for instance the roots of \( y = 4 \) are \( x = \pm 2 \). Hence the points \((-2, 4)\) and \((2, 4)\) are both on the U-shaped curve and so \((4, -2)\) and \((4, 2)\) are on the C-shaped curve which hence violates the vertical line test. If we restrict the function \( y = x^2 \) to the positive numbers, as on the right, we have an invertible function. Its inverse is \( x = \sqrt{y} \), the _square root function_. Notice that the root symbol \( \sqrt{y} \) refers only to the _positive_ root. \( \sqrt{4} = -2 \) is incorrect, while \((-2)^2 = 4\) is correct.

When we invert the graph of the cost function in Example 3 above the vertical line test shows that we obtain again a graph of a function which we call the _inverse_ of the original function.\(^2\)

\[ Q(TC) = \frac{1}{2} \sqrt{-244 + 2TC + 2\sqrt{(14900 - 244TC + TC^2)}} \]

\[ -\frac{2}{3} \sqrt{(-244 + 2TC + 2\sqrt{(14900 - 244TC + TC^2)}) + 3} \]
In contrast, the inverted graph of the function

\[ TC(Q) = 2Q^3 - 18Q^2 + 48Q + 86 \]

is not the graph of a function:

![Graph of TC(Q)](image1)

![Inverted graph](image2)

### 3.3 Continuous and differentiable functions

Calculus is the method to study differentiable functions. Therefore we will primarily deal with functions of this type. All differentiable functions are continuous. Roughly speaking, a function is *continuous* if its graph can be drawn in a single stroke, without ever lifting the pen. There should be no “jumps”. This must at least hold over all *intervals* where the function is defined. An interval is a part of the number line with no “holes” in it. All examples of functions above were continuous. The function \( y = f(x) = \frac{1}{x} \)

![Graph of f(x)](image3)

is an example of a function with a ‘hole’ in the domain because \( \frac{1}{x} \) is defined for all numbers except zero.\(^3\) \( y = \frac{1}{x} \) is a continuous function because you can draw the graph in one stroke for the negative and for the positive numbers.

An example of a function which is not continuous at \( x = 0 \) is the *sign function* defined by

\[
\text{sign}(x) = \begin{cases} 
+1 & \text{for } x > 0 \\
0 & \text{for } x = 0 \\
-1 & \text{for } x < 0 
\end{cases}
\]

\(^3\)See the appendix of (?) for a detailed explanation of the term.
which has the graph

An important property of continuous functions is known as the theorem of Bolzano: Suppose that the function \( y = f(x) \) is defined and continuous on the interval \( a \leq x \leq b \) and that \( f(a) < 0 \) and \( f(b) > 0 \). Then there exists a root between \( a \) and \( b \), i.e., a number \( c \) with \( a < c < b \) and \( f(c) = 0 \). (The intermediate value theorem discussed in (?), Chapter 1, is a generalization of this theorem.)

Intuitively, a function is differentiable if its graph has no kinks. A function with a kink (or cornerpoint) at \( x = 0 \) is the absolute value function

\[
|x| = \begin{cases} 
  x & \text{for } x > 0 \\
  0 & \text{for } x = 0 \\
  -x & \text{for } x < 0 
\end{cases} = x \cdot \text{sign}(x)
\]

which has the graph
At a kink the graph can have several tangents, i.e., several lines which touch the graph in this point.

For a function to be differentiable there has to be a unique tangent at each point of the graph.\(^4\)

To summarize, a curve is the graph of a function if it passes the vertical line test. The function is continuous if its graph can be drawn in one stroke and it is differentiable if the graph has no kinks.

4 Fixed costs and variable costs

Returning to our three leading examples we notice first that all graphs intersect the vertical axis at a positive level, for instance \(TC(0) = 50\) in the third example. The value of the cost function at zero gives the set-up costs or fixed costs of running the enterprise which do not depend on the number of newspapers actually printed. For instance, in order to guarantee a certain quality of the newspaper the publisher has to hire a number of journalists regardless of how many copies are sold. In contrast, the variable part of costs are paper and ink etc. which increase with output. One defines the fixed costs as

\[ FC = TC(0) \]

\(^4\)In addition we need that the tangents are not vertical lines, so that their slopes are not infinite.
and the variable cost function as

\[ V_C(Q) = TC(Q) - FC \]

In Example 3 one has

\[ FC = 50 \]
\[ V_C(Q) = 2Q^3 - 18Q^2 + 60Q \]

5 Costs are positive and increasing

Obviously, costs are always positive numbers. The graphs show that all cost functions considered above are positively-valued.

It is also intuitive that cost functions should be increasing functions in the sense that higher output means higher costs: \( Q_1 < Q_2 \) implies \( TC(Q_1) < TC(Q_2) \). At least they should be non-decreasing in the sense that \( Q_1 < Q_2 \) implies \( TC(Q_1) \leq TC(Q_2) \).

We see immediately from the graphs which move upward from left to right that this is the case in our examples. We also see it from the tables. However, how can we deduce directly from the algebraic description of a cost function that it is positive and increasing? Here the main problem is to show that a function is increasing because by the definition of an increasing function:

**Theorem 1** Suppose a given function \( TC(Q) \) has non-negative fixed costs \( TC(0) \) and is increasing. Then the costs \( TC(Q) \) are positive for all \( Q > 0 \).

6 Linear functions

The total cost function in Example 1 is an example of a linear function, i.e., a function who’s graph is a (non-vertical) straight line. Let us look at the cost increases \( \Delta TC \) in this example (we use the greek letter “capital delta” to indicate differences).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( TC )</td>
<td>90</td>
<td>110</td>
<td>130</td>
<td>150</td>
<td>170</td>
<td>190</td>
<td>210</td>
<td>230</td>
</tr>
<tr>
<td>( \Delta TC )</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>
We see that the cost increases are constant, regardless of how many newspapers are currently printed, it costs £2,000 more to print 100,000 newspapers more.

That we have a linear cost function is less obvious when the output levels in the table are not equidistant:

<table>
<thead>
<tr>
<th>Q</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>11</th>
<th>12</th>
<th>17</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>TC</td>
<td>90</td>
<td>150</td>
<td>170</td>
<td>230</td>
<td>310</td>
<td>330</td>
<td>390</td>
<td>490</td>
</tr>
<tr>
<td>ΔTC</td>
<td>60</td>
<td>20</td>
<td>60</td>
<td>80</td>
<td>20</td>
<td>100</td>
<td>60</td>
<td></td>
</tr>
</tbody>
</table>

In this case we have to look at the rates of change or the difference quotients

\[
\frac{\Delta TC}{\Delta Q} = \frac{TC(Q_1) - TC(Q_0)}{Q_1 - Q_0}
\]

where \(Q_0\) and \(Q_1\) are distinct quantities:

<table>
<thead>
<tr>
<th>Q</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>11</th>
<th>12</th>
<th>17</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔQ</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TC</td>
<td>90</td>
<td>150</td>
<td>170</td>
<td>230</td>
<td>310</td>
<td>330</td>
<td>430</td>
<td>490</td>
</tr>
<tr>
<td>ΔTC</td>
<td>60</td>
<td>20</td>
<td>60</td>
<td>80</td>
<td>20</td>
<td>100</td>
<td>60</td>
<td></td>
</tr>
</tbody>
</table>

The main characteristic of a linear function is that the rate of change is the same, whatever two quantities \(Q_0\) and \(Q_1\) we compare. This rate is called the slope or gradient of the line. Economists speak of constant marginal costs: The cost of producing one more unit of output is always the same, regardless of what is already produced. In our example the marginal costs are

\[
\frac{\Delta TC}{\Delta Q} = 20 \left( \times \frac{\text{£1,000}}{100,000} \right) = 20 \times (1\text{p})
\]

so, printing an additional newspaper always costs 20p more. Consequently, printing 100 newspapers more costs £20 more etc.

Generally, for a linear function it is easy to decide whether it is increasing or not:

**Theorem 2** A linear function is increasing if and only if its slope is positive.

Recall from geometry that there is a unique line passing through two distinct points. Correspondingly, we can deduce all there is to know about a linear cost function once we know the total costs at just two distinct quantities \(Q_0\) and \(Q_1\):

1. We can calculate the marginal costs as the ratio between the induced change in costs and the change in the quantity produced

\[
m = \frac{\Delta TC}{\Delta Q} = \frac{TC(Q_1) - TC(Q_0)}{Q_1 - Q_0}
\]

2. Because the rate of the change is the same, regardless of which two quantities we use to calculate it, we have for a fixed quantity \(Q_0\) and any other quantity \(Q\)

\[
\frac{TC(Q) - TC(Q_0)}{Q - Q_0} = m
\]

11
or
\[ TC(Q) = TC(Q_0) + m(Q - Q_0). \]

We can now calculate total costs for any quantity. In general, this description of a linear function is called the point-slope form.

3. In particular, we can calculate the fixed costs as
\[ FC = TC(0) = TC(Q_0) - mQ_0. \]

For any quantity \( Q \) we obtain
\[ TC(Q) = TC(Q_0) - mQ_0 + mQ = FC + mQ \]
which is called the slope-intercept form of a linear function.\(^5\) The variable costs are simply
\[ VC(Q) = mQ. \]

**Exercise 1** The total costs are £1600 for producing 300 CDs and £2000 for producing 500 CDs. Assuming a linear cost function, determine the marginal costs and the fixed costs.

## 7 Non-linear cost functions

Also the cost functions in Example 2 and 3 are increasing. Correspondingly, the cost increases \( \Delta TC \) are always positive in Example 2 and 3, as shown in the following tables.

**Example 2:**

\[
\begin{array}{cccccccc}
Q & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
TC & 110 & 135 & 170 & 215 & 270 & 335 & 410 & 495 \\
\Delta TC & 25 & 35 & 45 & 55 & 65 & 75 & 85 \\
\end{array}
\]

**Example 3:**

\[
\begin{array}{cccccccc}
Q & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
TC & 50 & 94 & 114 & 122 & 130 & 150 & 194 & 274 \\
\Delta TC & 44 & 20 & 8 & 8 & 20 & 44 & 80 \\
\end{array}
\]

However, the cost increases are no longer constant because the cost functions are no longer linear. Similarly, the rates of change \( \frac{\Delta TC}{\Delta Q} \) are no longer constant.

To deal with such cases one uses tangents to approximate the graph near a point. The rates of change can then be approximated by the slope of a tangent, at least for small changes of the quantity produced. The following graph indicates that the tangent at

\(^5\)Because the marginal cost \( m \) is the slope of the line and the fixed costs \( FC \) give the intercept of the line with the vertical axis.
(3, TC(3)) = (3, 215) is indeed a pretty good approximation of the correct cost function in Example 2 for quantities between 2 (×100,000) and 4 (×100,000):

With the methods introduced below the equation for the tangent is calculated as:

\[ t(Q) = 215 + 50(Q - 3) \]

where 50(×1p) is the slope of the tangent. So the cost of an additional newspaper is roughly 50p more, additional 1,000 copies cost roughly £50 more etc. In economics the slope of the tangent 50(×1p) is called the marginal costs because it is approximately the cost of producing a ‘small’ unit more. In our example the exact cost of producing an additional newspaper is

\[
(\text{TC}(3,000,01) - \text{TC}(3)) \times (\text{£}1000) = (215.0005000005 - 215) \times (\text{£}1000) = 0.0005000005 \times (\text{£}1000) = 50.0005p
\]

### 7.1 The first derivative

The gradient of a function \( y = f(x) \) at a value \( x_0 \) of the independent variable is the slope of the tangent to the graph of \( f(x) \) at the point \( (x_0, f(x_0)) \). It is written as \( y'(x_0) \) or \( f'(x_0) \) (Newton) or as \( \frac{dy}{dx}(x_0) \) or as \( \frac{df}{dx}(x_0) \) (the differential quotient, Leibniz).

Consequently, the tangent is the graph of the linear function

\[ t(x) = f(x_0) + f'(x_0)(x - x_0) \]

in point-slope form.

The new function which assigns to each value of the independent variable \( x \) the slope of the corresponding tangent is called the (first) derivative of \( y = f(x) \).\(^6\) It is denoted by \( f'(x) \) (Newton) or \( \frac{df}{dx} \) or \( \frac{dy}{dx} \) (Leibniz). The method to calculate derivatives is called differentiation.

\(^6\)Because it is a new function derived from the old function \( y(x) \).
7.2 Polynomials

A polynomial of degree $n$ is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

with constants $a_n, a_{n-1}, \ldots, a_0$ where the leading coefficient $a_n$ is not zero. $a_n x^n$ is called the leading term and $a_0$ the constant term. Roughly speaking, a polynomial is a sum of powers $x^k$ of the independent variable which are called monomials (mono = single, poly = many). Special cases are the constant functions $f(x) = a_0$, the linear functions $f(x) = a_1 x + a_0$, the quadratic functions $f(x) = a_2 x^2 + a_1 x + a_0$ and the cubic functions $y(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$.

The derivative of a power function $y = x^k$ is

$$y' = k x^{k-1}$$

The derivative of a polynomial function $f(x)$ is

$$f'(x) = na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + 2a_2 x^2 + a_1 x + 0a_0 x^0 - 1$$

In particular, the derivative of a cubic function is quadratic, the derivative of a quadratic function is linear, the derivative of a linear function $a_1 x + a_0$ is constant (because the slope is constant) and the derivative of a constant function is zero.

7.3 Marginal costs in Examples 2 and 3

Notice first that for the linear total costs function $TC(Q) = 90 + 20Q$ that the marginal costs are indeed $MC(Q) = \frac{dTC}{dQ} = 20$ in accordance with the above rule for differentiating.

We can now calculate the marginal cost function in Example 2 as:

$$TC(Q) = 5Q^2 + 20Q + 110$$

$$MC(Q) = \frac{dTC}{dQ} = 2 \times 5Q + 20Q = 10Q + 20$$

In particular, $MC(3) = 30 + 20 = 50$, as claimed above. In Example 3:

$$TC(Q) = 2Q^3 - 18Q^2 + 60Q + 50$$

$$MC(Q) = \frac{dTC}{dQ} = 3 \times 2Q^2 - 2 \times 18Q + 60 = 6Q^2 - 36Q + 60$$

The following tables compare the cost increases from the above tables with the marginal costs. Example 2:

<table>
<thead>
<tr>
<th>$Q$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TC$</td>
<td>110</td>
<td>135</td>
<td>170</td>
<td>215</td>
<td>270</td>
<td>335</td>
<td>410</td>
<td>495</td>
</tr>
<tr>
<td>$\Delta TC$</td>
<td>25</td>
<td>35</td>
<td>45</td>
<td>55</td>
<td>65</td>
<td>75</td>
<td>85</td>
<td></td>
</tr>
<tr>
<td>$MC$</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
</tr>
</tbody>
</table>

\(^7\)The constant function $y(x) = 0$ is considered as a polynomial “of degree $-\infty$.”
Example 3:

<table>
<thead>
<tr>
<th>Q</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>TC</td>
<td>50</td>
<td>94</td>
<td>114</td>
<td>122</td>
<td>130</td>
<td>150</td>
<td>194</td>
<td>274</td>
</tr>
<tr>
<td>ΔTC</td>
<td>44</td>
<td>20</td>
<td>8</td>
<td>8</td>
<td>20</td>
<td>44</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>MC</td>
<td>60</td>
<td>30</td>
<td>12</td>
<td>4</td>
<td>12</td>
<td>30</td>
<td>60</td>
<td>102</td>
</tr>
</tbody>
</table>

7.4 Increasing functions and upward-slopedness

We can now give a very simple algebraic argument to show that the cost function $TC(Q)$ in Example 2 is upward-sloped for positive $Q$ in the sense that all tangents have positive slope: Namely, the marginal costs $MC(Q) = 10Q + 20$ are always bigger than 20 and hence positive.\(^8\)

Geometrically, the following conjecture now suggests itself:

**Conjecture 1** A function is increasing if and only if all its tangents are upward-sloped, i.e., have positive slope.

It turns out that this conjecture is ‘almost’ correct. However, the following two qualifications have to be made:

a) In the following example all tangents to the graph have positive slope, but the function is not increasing.

![Graph](image)

Upward-sloped, but not increasing.

However, if we restrict attention to one of the intervals $x < -1$, $-1 < x < 1$ or $1 < x$, our conjecture holds.

\(^8\)An algebraic argument for Example 3 is more tricky, involving the infamous “quadratic extension”:

\[
MC(Q) = 6(Q^2 - 6Q + 10) = 6((Q^2 - 6Q + 9) + 1) = 6((Q - 3)^2 + 1) \geq 0
\]

For any $Q$ we know that $(Q - 3)^2$ is non-negative, hence $(Q - 3)^2 + 1$ and $6((Q - 3)^2 + 1)$ are positive numbers.
b) In the following example the tangent to the graph at (0, 1) is horizontal, i.e., it has slope zero. Nonetheless, the function is strictly increasing:

![Graph showing horizontal tangent]

Increasing with a horizontal tangent.

If the derivative is occasionally zero but otherwise positive, the function is still increasing. Generally, the following can be shown:

**Theorem 3** A continuously differentiable function\(^9\) is increasing on an interval if and only if its first derivative is non-negative in the interval and not constantly zero on any subinterval.

**Theorem 4** A continuously differentiable function\(^\text{10}\) is decreasing on an interval if and only if its first derivative is non-positive in the interval and not constantly zero on any subinterval.

The following example of a non-decreasing, but not increasing function is ruled out by the conditions of the theorem:

![Graph showing non-decreasing, but not increasing]

Non-decreasing, but not increasing.

Notice that a horizontal line never intersects the graph of function twice if and only if the function is increasing or decreasing. Therefore the vertical line test yields:

\(^9\)“continuously differentiable” means that the first derivative exists and is a continuous functions.

\(^\text{10}\)“continuously twice differentiable” means that the first and the second derivative exist and are continuous functions.
Theorem 5  A function is invertible if and only if it is increasing or decreasing.

Summary: The first derivative measures how steeply a function increases. Increasing functions have positive derivatives, decreasing functions have negative derivatives.